24th Green Chicken Contest - Solutions

① $10+11+\dots+99 = \frac{90(10+99)}{2} = 4905$
So palindrome must begin with 48 _ _. Hence it's 4884. So the missing number $= 4905-4884 = 21$.

② $p(\text{next two rolls are 4's}) = \frac{1}{6}\cdot\frac{1}{6} = \frac{1}{36}$
To find $p(\text{roll } 4,4,4 \mid \text{given sum } 12)$, count all ways sum could be 12. There is 1 way to get 4,4,4; 6 ways to get 3,4,5; 3 ways to get 3,3,6; 6 ways to get 2,4,6; 3 ways to get 2,5,5; and 6 ways to get 1,5,6. So $p(4,4,4 \mid \text{sum } 12) = \frac{1}{25}$.
So more likely that 1st three rolls were all 4.

③ $f_1(11) = 2$, $f_2(11) = 4$, $f_3(11) = 16$, $f_4(11) = 37$, $f_5(11) = 58$, $f_6(11) = 89$, $f_7(11) = 145$, $f_8(11) = 42$, $f_9(11) = 20$, $f_{10}(11) = 4$, $f_{8k+1}(11) = 20 \ \forall k \geq 1$. Thus $f_{2001}(11) = 20$.

④ $x^3+7^3 = x^3+343 = (x+7)(x^2-7x+49)$ So $(x+7) \mid (x^3+343)$ for all x. If $x+7$ divides x^3+49, then $(x+7) \mid (x^3+343)-(x^3+49) = 294$. The largest such x is $x = 287$.

⑤ Let $(I+2001A)(I+2001B)(I+2001C) = P$

$P = (I+2001A)\left[I+2001(B+C+2001BC)\right]$
$P = (I+2001A)(I+0) = I+2001A$.

But $P = \left[I+2001(A+B+2001AB)\right](I+2001C)$
$P = (I+0)(I+2001C) = I+2001C$
$\Rightarrow A = C$.

Similarly consider $M = (I+2001C)(I+2001A)(I+2001B)$
Get $I+2001C = I+2001B$
$\Rightarrow C = B$

⑥ The discriminant of $f(x)-g(x) = x^2+(2b-2a)x+(1-2ab)$
is $D = B^2-4AC = [2(b-a)]^2 - 4(1-2ab)$
$= 4(b^2-2ab+a^2) - 4 + 8ab$
$= 4(a^2+b^2-1)$.
S is the region where $D<0$.
$\Rightarrow a^2+b^2<1$
So S is the interior of a circle of radius 1
Area of S is π.