The Twenty-Ninth Annual Green Chicken Contest

October 28, 2006 (at Williams College)

1. Let a and b be two positive rational numbers such that $a^{1/3}+b^{1/3}$ is a rational number. Prove that $a^{1/3}$ and $b^{1/3}$ are rational numbers.

2. Show that there exists a unique real number a such that $\cos a = a$.

3. Prove that

$$\sqrt{1+\sqrt{2+\cdots+\sqrt{n}}} < 3$$

for any integer $n \geq 1$.

4. Show that there are no integers x and y such that

$$x^{2006} + 213! = 1793^y.$$

5. Given n red points and n blue points in the plane (no three collinear) show that they can be matched with n line segments each joining a red point to a blue point and with no segments crossing.

6. Let $f(n)$ be the number of permutations without fixed points of a set with n elements. Show that

$$\underbrace{f \circ f \circ \ldots \circ f}_{\text{2006 times}}(29)$$

is a multiple of 5.