The Williams College Math-Stats Bulletin

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The Green Chicken returned safely to Williams last Saturday after a rousing victory by Williams over Middlebury, led by top scorers Edward Newkirk, Natee Pitiwan, Kefei Lei, and Yang Du, and coach Professor Stoiciu. Competition and solutions appended, along with November Conundrum.

Monday, November 5 COLLOQUIUM, 1–1:45 pm, Bronfman 106 (*Refreshments to follow*)
Michelle Donnelly '08, Williams College

Primes in Arithmetic Progressions

We can easily verify that there are an infinite number of primes such that the remainder is 1 when the prime is divided by 2. How many primes are there for which the remainder is 1 when divided by 5 or 67? We will prove that there are infinitely many primes that when divided by an integer m have a remainder of 1. This is an important special case of Dirichlet's theorem on arithmetic progressions.

Wednesday, November 7 COLLOQUIUM, 1–1:45 pm, Bronfman 106 (*Refreshments to follow*)
Anna Ferguson '08, Williams College

Decision Theory: What to do when Faced with Uncertainty

A population parameter represents a particular population characteristic, such as the mean. Population parameters are typically estimated with data. These estimators are also referred to as decisions, or decision functions. Associated with a decision function is a risk; this risk is also a function of the population parameter of interest. We will see how Bayesian analysis helps us choose a decision function (estimator) with a small amount of risk.

Wednesday, November 7 MATH PUZZLE NIGHT, 7–8 pm, Math/Stats Library

Thursday (not Friday), November 8 FACULTY SEMINAR, 1–1:50 pm, Bronfman 106 (*Refreshments to follow*)
Professor Genevieve Walsh, Tufts University

Commensurability of Hyperbolic Knot Complements

Two 3-manifolds are commensurable if they have homeomorphic finite-sheeted covers. I will discuss this relation for knot complements and present evidence towards the conjecture that there are at most 3 hyperbolic knot complements in a given commensurability class. I will also argue that commensurability classes are the best thing since sliced bread. This is work in progress.

☙ THE NOVEMBER CONUNDRUM ❧

Let n and m be positive integers. How many positive integer solutions does the equation below have?

$$X_1 + X_2 + \ldots + X_m = n$$

Impress your friends and family with the prestige and prizes you'll receive for a socko solution! (Submit said solutions to Professor Rafalski.)

Thirtieth Annual Green Chicken Contest 10/27/07

1.Show that every set of ten distinct integers chosen from the set {1,2 ,. ..,1 00} contains two disjoint subsets with the same sum.

2. The New England Tennis Club invites 64 players of equal ability to compete in a single elimination tournament (a player losing a match is eliminated). What is the chance that a particular pair of players will be paired together at some point during the tournament?

3. In $\triangle ABC$, $AB = 6$, $AC = 5$, and $BC = 4$. Determine the ratio $\angle BCA / \angle CAB$.

4. Let U_n be the integer consisting of n 1's. Note that 3 divides $U_3 = 111$,(i.e. U_3 is a multiple of 3). Show that there are infinitely many n for which n divides U_n.

5. Let $a_1 = 1$ and $a_{n+1} = \cos(\arctan a_n)$ for $n \geq 1$.
(a) What is a_{12}?
(b) What is $\lim_{n\to\infty} a_n$?

6.(a) Find $T > 0$ for which $\int_0^T x^{-\ln x}\,dx = \int_T^\infty x^{-\ln x}\,dx$.

(b) Evaluate the integrals above for appropriate T given that $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.

Solutions for the 30th Annual Green Chicken Contest

1. If S is a subset of 10 integers from the set {1, 2, …, 100}, then the smallest possible sum of subsets from S is 1 and the largest is 91 + 92 + … + 100 = 955. Let S be any subset of 10 integers from {1, 2, …, 100}. Then there are $2^{10}-1 = 1023$ nonempty subsets of S. Since 1023 > 955, by the pigeonhole principle, there are at least two subsets A and B of S with a common sum. Of course A and B might not be disjoint. But then $A-(A\cap B)$ and $B-(A\cap B)$ are two nonempty disjoint subsets of S with the same sum.

2. There are 63 = 64 – 1 matches in all since each match eliminates one player. The number of possible pairings of the 64 players is $\binom{64}{2}$ = (64)(63)/2. So the probability of a particular pairing occurring during the tournament is just their quotient, namely 1/32.

3. Let a be the side opposite angle A, b be the side opposite angle B, and c be the side opposite angle C. By hypothesis, a = 4, b = 5, and c = 6. Now by the law of cosines: $a^2 = b^2 + c^2 - 2bc\cos A$. Hence 16 = 61 – 60 cos A. Thus cos A = -45/-60 = ¾. Similarly, $c^2 = a^2 + b^2 - 2ab\cos C$. Hence 36 = 41 – 40 cos C. Thus cos C = -5/-40 = 1/8.

But $\cos C = 1/8 = 2(3/4)^2 - 1 = 2(\cos A)^2 - 1 = \cos 2A$. Since $0 < 2A < \pi$ and $0 < C < \pi$, it follows that 2A = C. So the ratio A/C or ∠BCA to ∠CAB is 2/1.

4. Notice that $3|U_3$ and that the next example is $9|U_9$. You might conjecture that $n|U_n$ whenever $n = 3^k$ for some $k \geq 1$. This is correct as we now demonstrate: Let $n|U_n$ for some n. So there is an integer m for which $U_n = mn$.

$$\text{Then } U_{3n} = \frac{10^{3n}-1}{9} = \frac{(10^n-1)(10^{2n}+10^n+1)}{9}$$
$$= U_n(10^{2n}+10^n+1) = mn\,(10^{2n}+10^n+1). \text{ So } n|\,U_{3n}.$$

Further, $10^{2n}+10^n+1 \equiv 1+1+1 \equiv 0 \pmod 3$. Thus $3n|\,U_{3n}$. The result follows.

5. (a) Notice that $a_{n+1} = \cos(\arctan a_n) = \dfrac{1}{\sqrt{1+a_n^{\,2}}}$ for all $n \geq 1$. So $a_2 = \dfrac{1}{\sqrt{2}}$, $a_3 = \dfrac{\sqrt{2}}{\sqrt{3}}$, $a_4 = \dfrac{\sqrt{3}}{\sqrt{5}}$, etc. Let $f_1 = 1$, $f_2 = 1$, and $f_{n+2} = f_n + f_{n+1}$ for all $n \geq 1$ be the Fibonacci sequence. We will show by induction that $a_n = \dfrac{\sqrt{f_n}}{\sqrt{f_{n+1}}}$ for all $n \geq 1$:

The result checks for the case n = 1. Now assume it's true for some arbitrary n. Then

$$a_{n+1} = \frac{1}{\sqrt{1+a_n^2}} = 1\Big/\sqrt{1+f_n/f_{n+1}} = \frac{\sqrt{f_{n+1}}}{\sqrt{f_{n+1}+f_n}} = \frac{\sqrt{f_{n+1}}}{\sqrt{f_{n+2}}} \text{ as required.}$$

Hence $a_{12} = \dfrac{\sqrt{144}}{\sqrt{233}}$.

(b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{\sqrt{f_n}}{\sqrt{f_{n+1}}}$. But $\lim_{n\to\infty} \dfrac{f_{n+1}}{f_n} = \dfrac{1+\sqrt{5}}{2}$, the "golden ratio". So $\lim_{n\to\infty} a_n$ is the square root of the reciprocal of the golden ratio, i.e. $\lim_{n\to\infty} a_n = \sqrt{\dfrac{2}{1+\sqrt{5}}} = \sqrt{\dfrac{\sqrt{5}-1}{2}}$.

6. (a) Let y = ln x. So $dy = \frac{1}{x}dx$, $x = e^y$ and $dx = e^y dy$.

Then $\int_0^T x^{-\ln x}dx = \int_{-\infty}^{\ln T} e^{-y^2}e^y dy = \int_{-\infty}^{\ln T} e^{y-y^2}dy$ and $\int_T^\infty x^{-\ln x}dx = \int_{\ln T}^\infty e^{y-y^2}dy$.

Now let z = y – ½ ; so dz = dy. Then $z^2 = y^2 - y + \frac{1}{4}$ and $y - y^2 = \frac{1}{4} - z^2$. Hence $e^{y-y^2} = e^{1/4}e^{-z^2}$. Notice that the improper integrals $e^{1/4}\int_{-\infty}^{\ln T - 1/2} e^{-z^2}dz$ and $e^{1/4}\int_{\ln T - 1/2}^{\infty} e^{-z^2}dz$ both converge. But e^{-z^2} is a positive, even function. So the integrals are equal if and only if ln T – ½ = 0. Thus $T = e^{1/2}$.

(b) Since $\int_{-\infty}^\infty e^{-z^2}dz = \sqrt{\pi}$, it follows that $\int_0^\infty e^{-z^2}dz = \sqrt{\pi}/2$. We get that $\int_0^{\sqrt{e}} x^{-\ln x}dx = \int_{\sqrt{e}}^\infty x^{-\ln x}dx = e^{1/4}\int_0^\infty e^{-z^2}dz = \dfrac{e^{1/4}\sqrt{\pi}}{2}$.