

GREEN CHICKEN PROBLEMS - OCTOBER 2010

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Question 1: Consider a 67×67 chessboard; there are thus 67 rows, each row containing 67 squares. Using only the edges of the squares on the board, how many different squares can you make? For example, if we had a 2×2 board there would be 5 squares, the four small squares and then the giant square that is everything. You may leave your answer as a product of explicit factors (if it were 12321, you could leave it as $9 \cdot 1369$).

Question 2: 2010 people are waiting to board a plane. The first person's ticket says Seat 1; the second person in line has a ticket that says Seat 2, and so on until the 2010th person, whose ticket says Seat 2010. The first person is from Wossamotta University and cannot read, and randomly chooses one of the 2010 seats (note: he might randomly choose to sit in Seat 1). From this point on, the next 2009 people will always sit in their assigned seats if possible; if their seat is taken, they will randomly choose one of the remaining seats (after the first person, the second person takes a seat; after the second person, the third person takes a seat, and so on). What is the probability the 2010th person sits in Seat 2010?

Question 3: Pythagoras knew that $x^2 + y^2 = z^2$ has non-zero integer solutions (such as $3^2 + 4^2 = 5^2$). The great Pierre de Fermat conjectured that for any integer $n \geq 3$, there are no integer solutions to $x^n + y^n = z^n$ without $xyz = 0$ (in other words, at least one of x, y and z is zero); thanks to Taylor and Wiles we now have a proof that Fermat was correct. The set of integer solutions of these two equations have very different behavior; to make peace between them we consider the new equation $x^2 + y^2 = z^n$ for integer $n \geq 3$. Find a solution when $n = 2010$ with x, y and z all positive integers, or prove that there are no such solutions.

Question 4: Instead of taking a math contest, Middlebury and Williams decide to settle who gets the Green Chicken by playing the following game. Consider the first one million positive integers. Player A's goal is to choose 10000 of these numbers such that at the end of the choosing procedure there are at least 20 pairs of chosen integers with the same positive difference (for example, (12,39), (39,66) and (101,128) count as three pairs with a difference of 27). A turn consists of Player A choosing 10 new numbers, and then Player B replacing up to 10 of *any* number currently chosen with new numbers not currently chosen (thus B may replace a number A has just chosen, or a number A or B chose earlier, with a new number). We keep playing until A has chosen 10000 numbers, allowing B to get her final turn. Determine which player has a winning strategy, and prove your claim.

Question 5: Let $\{a_n\}_{n=1}^{\infty}$ be *any* sequence of positive numbers. Consider the new series $\{b_n\}_{n=1}^{\infty}$ where

$$b_n = \frac{a_n}{2010 + n^3 a_n^2}.$$

Either prove that the sum $\sum_{n=1}^{\infty} b_n$ always converges, or find a sequence a_n such that the sum $\sum_{n=1}^{\infty} b_n$ diverges.

Question 6: Let $n \geq 1$ be a positive integer. Prove

$$\int_{y=0}^1 \int_{x=0}^1 n(1 - xy)^{n-1} dx dy = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Here are the problems and solutions.

Question 1: Consider a 67×67 chessboard; there are thus 67 rows, each row containing 67 squares. Using only the edges of the squares on the board, how many different squares can you make? For example, if we had a 2×2 board there would be 5 squares, the four small squares and then the giant square that is everything. You may leave your answer as a product of explicit factors (if it were 12321, you could leave it as $9 \cdot 1369$).

Solution 1: Consider more generally an $N \times N$ board. The number of squares of size 1 is just N^2 , the number of squares of size 2 is just $(N - 1)^2$, and in general the number of squares of size n is just $(N - (n - 1))^2$. Thus the total number of squares is simply

$$\sum_{n=1}^N (N - (n - 1))^2 = \sum_{m=1}^N m^2 = \frac{N(N + 1)(2N + 1)}{6};$$

taking $N = 67$ (the largest prime factor of 2010, of course) we get 102510. If you don't remember the formula for sums of squares, you can rederive it by induction if you remember it is a cubic polynomial. You guess the sum is $aN^3 + bN^2 + cN + d$, but you can figure out the values when $N = 0, 1, 2$ and 3 (which are 0, 1, 5 and 14, respectively). This allows you to figure out the coefficients a, b, c . If you remember calculus, you know that as $N \rightarrow \infty$ the main term must look like $N^3/3$ (which comes from replacing $\sum_{m \leq N} m^2$ with $\int_{t \leq N} t^2 dt$). This gives you that $a = 1/3$. If you happen to remember that it is N times something, you can then see that $d = 0$.

Question 2: 2010 people are waiting to board a plane. The first person's ticket says Seat 1; the second person in line has a ticket that says Seat 2, and so on until the 2010th person, whose ticket says Seat 2010. The first person is from Wossamotta University and cannot read, and randomly chooses one of the 2010 seats (note: he might randomly choose to sit in Seat 1). From this point on, the next 2009 people will always sit in their assigned seats if possible; if their seat is taken, they will randomly choose one of the remaining seats (after the first person, the second person takes a seat; after the second person, the third person takes a seat, and so on). What is the probability the 2010th person sits in Seat 2010?

Solution 2: The answer is 50%. If the first person takes seat 1, the 2010th person always takes the right seat; if the first person takes the 2010th seat then clearly the 2010th person never takes that seat. Each of these events happen with equal probability (specifically, $1/2010$). We are left with the case when the first person takes one of the middle seats (seats 2 through 2009). If the person takes seat k , then people 2, 3, ..., $k - 1$ all take the right seat, and then person k comes and finds seat k taken, and seats 1, $k + 1, \dots, 2010$ available. Note that it is exactly like we have started playing the game with a smaller number of people, in particular, at least 2 people. We are now done by induction, as each of these cases has probability $1/2$. (Technically we need to prove the case of just two people is 50%, but this is clear as either the first person takes the first seat or the second seat.)

Question 3: Pythagoras knew that $x^2 + y^2 = z^2$ has non-zero integer solutions (such as $3^2 + 4^2 = 5^2$). The great Pierre de Fermat conjectured that for any integer $n \geq 3$, there are no integer solutions to $x^n + y^n = z^n$ without $xyz = 0$ (in other words, at least one of x, y and z is zero); thanks to Taylor and Wiles we now have a proof that Fermat was correct. The set of integer solutions of these two equations have very different behavior; to make peace between them we consider the new equation $x^2 + y^2 = z^n$ for integer $n \geq 3$. Find a solution when $n = 2010$ with x, y and z all positive integers, or prove that there are no such solutions.

Solution 3: This problem is adapted from a problem posted at

<http://server.math.uoc.gr/~tzanakis/Courses/NumberTheory/MathInduction.pdf>

We proceed by induction, showing how to find a solution for n if we know a solution for $n - 2$. We have a solution when $n = 1$ (take the triple $(3,4,25)$, and a solution when $n = 2$ (take $(3,4,5)$). For general n , we proceed by induction. Assume for an $n \geq 2$ we have a solution (x_n, y_n, z_n) . We claim this yields a solution for $n + 2$; simply take the triple

$$(x_{n+2}, y_{n+2}, z_{n+2}) = (z_n x_n, z_n y_n, z_n).$$

As (x_n, y_n, z_n) is a solution for exponent n , we have $x_n^2 + y_n^2 = z_n^2$. Multiplying both sides by z_n^2 gives $(x_n z_n)^2 + (y_n z_n)^2 = z_n^{n+2}$, which is the desired solution. The proof is completed by induction. What is nice here is that we have to break into two cases, whether or not n is even or odd. Fortunately it is easy to find a solution in each basis case.

For the desired solution, we start with $n = 2$ and the solution $(3,4,5)$. Every time we increment n by 2 we multiply the x and y components by 5. We do this $\frac{2010-2}{2} = 1004$ times, meaning we multiply by 5^{1004} . Thus a solution is

$$(x, y, z) = (3 \cdot 5^{1004}, 4 \cdot 5^{1004}, 5).$$

Question 4: Instead of taking a math contest, Middlebury and Williams decide to settle who gets the Green Chicken by playing the following game. Consider the first one million positive integers. Player A's goal is to choose 10000 of these numbers such that at the end of the choosing procedure there are at least 20 pairs of chosen integers with the same positive difference (for example, $(12,39)$, $(39,66)$ and $(101,128)$ count as three pairs with a difference of 27). A turn consists of Player A choosing 10 new numbers, and then Player B replacing up to 10 of *any* number currently chosen with new numbers not currently chosen (thus B may replace a number A has just chosen, or a number A or B chose earlier, with a new number). We keep playing until A has chosen 10000 numbers, allowing B to get her final turn. Determine which player has a winning strategy, and prove your claim.

Solution 4: An important clue in solving this problem is to note that B can always move every new choice of A. In other words, B can force the final, chosen numbers to be whatever she desires! Another way of putting this is that A can do whatever it wants, but B can always force the outcome to be whatever she wants provided that 10,000 of the 1,000,000 integers are chosen. How many pairs are there? There are

$$\binom{10000}{2} = \frac{10000 \cdot 9999}{2}$$

pairs among the 10,000 chosen objects. There are 999,999 positive differences from subtracting elements in $\{1, \dots, 1,000,000\}$ from each other. By the Pidgeonhole Principle (or Dirichlet's Box Principle), at least one of these differences occurs at least the average number of times, which is

$$\frac{\binom{10000}{2}}{1000000} = \frac{10000 \cdot 9999}{2 \cdot 999999} > \frac{10^4 \cdot 10^4}{4 \cdot 10^6} > 25,$$

and thus there must be at least 25 pairs with a common positive difference. Note if we did a bit more algebra we'd get a lower bound of 49.995, which would allow us to prove there are at least 20 pairs so that no element of one pair is in another.

Finally, to answer the question: whatever team goes first has a winning strategy – just play the game! There is *nothing* the second player can do to win! An interesting question to ponder is the following: would the problem have felt harder if you were told B could move at most 9 numbers on a turn? Or at most 50% of the numbers on the board? Or any complicated function?

Question 5: Let $\{a_n\}_{n=1}^{\infty}$ be any sequence of positive numbers. Consider the new series $\{b_n\}_{n=1}^{\infty}$ where

$$b_n = \frac{a_n}{2010 + n^3 a_n^2}.$$

Either prove that the sum $\sum_{n=1}^{\infty} b_n$ always converges, or find a sequence a_n such that the sum $\sum_{n=1}^{\infty} b_n$ diverges.

Solution 5: It's always a good idea to test some simple cases. The most natural sequence to try is $a_n = 1/n$. In this case, we get $b_n = \frac{1}{n(n+2010)}$, and the sum $\sum_{n=1}^{\infty} b_n$ converges. If $a_n = n$ then $b_n = \frac{n}{2010+n^5}$, which converges very rapidly. If instead $a_n = 1/n^2$ then $b_n = \frac{1/n^2}{2010+1/n}$ which converges. It is thus looking like the series always converges.

The simplest way to attack this problem is to note that, for any n , we have

$$0 \leq b_n \leq \min\left(a_n, \frac{1}{n^3 a_n}\right).$$

If a_n is small (relative to n) then the first term is small and the second term is large; however, if a_n is large then the first term is large and the second is small. Thus, depending on the size of a_n , we use either the first or the second term.

What we'll do is split our sum into two sums. One part we'll estimate by bounding b_n by a_n , and the other by bounding b_n with $1/n^3 a_n$. We want the sum over the a_n 's we keep to converge; if we only keep terms where $a_n \leq 1/n^{5/4}$ then that sum converge. We find

$$\begin{aligned} \sum_{n=1}^{\infty} b_n &= \sum_{n=1}^{\infty} \frac{a_n}{2010 + n^3 a_n^2} \\ &= \sum_{\substack{n=1 \\ a_n \leq 1/n^{5/4}}} \frac{a_n}{2010 + n^3 a_n^2} + \sum_{\substack{n=1 \\ a_n \geq 1/n^{5/4}}} \frac{a_n}{2010 + n^3 a_n^2} \\ &= \sum_{\substack{n=1 \\ a_n \leq 1/n^{5/4}}} a_n + \sum_{\substack{n=1 \\ a_n \geq 1/n^{5/4}}} \frac{1}{n^3 a_n} \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n^{5/4}} + \sum_{n=1}^{\infty} \frac{1}{n^3/n^{5/4}} \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n^{5/4}} + \sum_{n=1}^{\infty} \frac{1}{n^{7/4}}. \end{aligned}$$

We have two p -series with p 's greater than 1, and thus both sums converge (the first sum has $p = 5/4$, while the second has $p = 7/4$). Thus, no matter what sequence a_n we choose, the sum of the b_n 's converges.

The following is an alternate solution given by several students. Instead of writing b_n in terms of a_n , write a_n in terms of b_n . This leads to the quadratic equation

$$n^3 b_n a_n^2 - a_n + 2010 b_n = 0.$$

Solving, we find

$$a_n = \frac{1 \pm \sqrt{1 - 8040 n^3 b_n^2}}{2 n^3 b_n}.$$

As the a_n 's are positive, the discriminant must be non-negative, thus

$$1 - 8040 n^3 b_n^2 \geq 0,$$

which implies

$$b_n \leq \left(\frac{1}{8040n^3} \right)^{1/2} \leq \frac{1}{n^{3/2}};$$

as the sum of $1/n^{3/2}$ converges, we see the sum of the b_n 's converges.

One more solution found by students: use the arithmetic / geometric mean. Note that for $x, y > 0$ we have $\sqrt{xy} \leq (x + y)/2$. We have

$$\begin{aligned} b_n &= \frac{a_n}{2010 + n^3 a_n^2} \\ &= \frac{1}{2010/a_n + n^3 a_n} \\ &\leq \left(\frac{2010}{a_n} \cdot n^3 a_n \right)^{-1/2} \\ (1) \quad &\leq \frac{1}{2010n^{3/2}}, \end{aligned}$$

and again we see the series converges.

Question 6: Let $n \geq 1$ be a positive integer. Prove

$$\int_{y=0}^1 \int_{x=0}^1 n(1-xy)^{n-1} dx dy = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Solution 6: We first solve it using the 'Bring it over' method'. The remaining solutions first integrate with respect to x . Note

$$(2) \quad \int_{x=0}^1 n(1-xy)^{n-1} dx = \frac{n}{y} \int_{x=0}^1 (1-xy)^{n-1} y dx = \frac{n}{y} (1 - (1-y)^n);$$

we do not need to worry about convergence issues when $y = 0$ (note that $1 - (1-y)^n = O(y)$).

Zeroth solution: Bring it over: Let

$$I_n = \int_{y=0}^1 \int_{x=0}^1 n(1-xy)^{n-1} dx dy.$$

Then

$$\begin{aligned} I_{n+1} &= \int_{y=0}^1 \int_{x=0}^1 (n+1)(1-xy)^n dx dy \\ &= \int_{y=0}^1 \int_{x=0}^1 (n+1)(1-xy)^{n-1} dx dy - \int_{y=0}^1 \int_{x=0}^1 (n+1)(1-xy)^{n-1} xy dx dy \\ &= \frac{n+1}{n} \int_{y=0}^1 \int_{x=0}^1 n(1-xy)^{n-1} dx dy - \int_{y=0}^1 \left[\int_{x=0}^1 (n+1)x(1-xy)^{n-1} y dx \right] dy \\ (3) \quad &= \frac{n+1}{n} I_n - \int_{y=0}^1 \left[\int_{x=0}^1 (n+1)x(1-xy)^{n-1} y dx \right] dy. \end{aligned}$$

We integrate the last piece by parts. Let $u = x$, $dv = (1 - xy)^{n-1}y dx$. Then $du = dx$, $v = -\frac{1}{n}(1 - xy)^n$ and thus

$$\begin{aligned}
 I_{n+1} &= \frac{n+1}{n}I_n + (n+1) \int_{y=0}^1 \left[\frac{x(1-xy)^n}{n} \Big|_{x=0}^1 - \int_{x=0}^1 \frac{n+1}{n}(1-xy)^n dx \right] dy \\
 &= \frac{n+1}{n}I_n + \int_{y=0}^1 \frac{(n+1)(1-y)^n}{n} - \frac{1}{n}I_{n+1} \\
 \frac{n+1}{n}I_{n+1} &= \frac{n+1}{n}I_n + \frac{1}{n} \\
 (4) \quad I_{n+1} &= I_n + \frac{1}{n+1}.
 \end{aligned}$$

As $I_1 = 1$, the claim readily follows.

First solution: Induction: The integral in (2) is easily evaluated by induction. We do one or two steps, as the pattern is clear and the induction straightforward. Write

$$(1-y)^n = (1-y)(1-y)^{n-1} = (1-y)^{n-1} - y(1-y)^{n-1}.$$

Thus the answer is just

$$\begin{aligned}
 &\int_0^1 \frac{1 - (1-y)^{n-1} + y(1-y)^{n-1}}{y} dy \\
 &= \int_0^1 \frac{1 - (1-y)^{n-1}}{y} dy + \int_0^1 (1-y)^{n-1} dy \\
 &= \int_0^1 \frac{1 - (1-y)^{n-1}}{y} dy + \frac{1}{n}.
 \end{aligned}$$

By a straightforward induction, we have

$$\int_0^1 \frac{1 - (1-y)^{n-1}}{y} dy = \frac{1}{n-1} + \frac{1}{n-2} + \cdots + \frac{1}{1}.$$

Thus the answer is just

$$\int_{y=0}^1 \int_{x=0}^1 n(1-xy)^{n-1} dx dy = \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{1} \approx \log n + \gamma,$$

where $\gamma \approx .5772$ is the Euler-Mascheroni constant.

Second solution: Geometric series: We provide an alternate way to evaluate the integral in (2). We use the finite geometric series formula:

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r};$$

Taking $r = 1 - y$ yields

$$1 + (1-y) + (1-y)^2 + \cdots + (1-y)^{n-1} = \frac{1 - (1-y)^n}{y}.$$

We integrate the above with respect to y from 0 to 1. As trivially $\int_0^1 (1-y)^{k-1} dy = 1/k$, we have

$$\int_0^1 \frac{1 - (1-y)^n}{y} dy = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} \approx \log n + \gamma,$$

providing another method to evaluate the integral.

Third solution: Binomial Theorem and induction:

We want to show

$$\int_0^1 \int_0^1 n(1-xy)^{n-1} dx dy = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n};$$

by induction it suffices to show

$$D(n+1) = \int_0^1 \int_0^1 (n+1)(1-xy)^n dx dy - \int_0^1 \int_0^1 n(1-xy)^{n-1} dx dy = \frac{1}{n+1}$$

(instead of induction one could also proceed by using telescoping series). This approach, of course, is similar to the other methods above. We use the Binomial Theorem to expand the two integrands and then integrate with respect to x and y . We have

$$D(n+1) = \binom{n}{0} \frac{1}{1} - \binom{n}{1} \frac{1}{2} + \binom{n}{2} \frac{1}{3} - \binom{n}{3} \frac{1}{4} + \cdots + (-1)^n \binom{n}{n} \frac{1}{n+1};$$

note, however, that the right hand side above is the same as

$$\int_0^1 (1-t)^n dt$$

(just use the Binomial Theorem again and integrate term by term). This integral is easily seen to be $1/(n+1)$, which implies that $D(n+1) = 1/(n+1)$, or

$$\int_0^1 \int_0^1 n(1-xy)^{n-1} dx dy = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$