Leonhard Euler and The Basel Problem

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Outline

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Historical Context

- Born on April 15, 1707 in Basel, Switzerland.
- Lived during the "Age of Reason"
- Euler dedicated his life to the study of mathematics, language, philosophy, and theology.
- Described as *"a precocious youth, blessed with a gift for languages and an extraordinary memory,"* and *"he was also a fabulous mental calculator, able to perform intricate arithmetical computations without benefit of paper and pencil."* (Dunham vix.)
- After living 76 years, he eventually passed away on September 18, 1783.

Euler's Education & Influences

- First educated by his father, then attended the University of Basel
- By 1723, he had received his Bachelor of Arts degree and a Master's degree in philosophy
- Decided he was interested in studying math instead of theology
- Received private math lessons from Johann Bernoulli
- Published several articles on different mathematical topics, including isochronous curves and reciprocal trajectories
- When Euler was writing about his career later in life he said, *"I was given permission to visit [Johann Bernoulli] freely every Saturday afternoon and he kindly explained to me everything I could not understand."* (Dunham xx)
- Bernoulli and Euler were able to work collaboratively on many projects.
- Based on Johann Bernoulli's advice, he studied the work of "Varignon, Descartes, Newton, Galileo, van Schooten, Herman, Taylor, and Willis"
- Taught his two sons Johann Albrecht Euler and Christopher Euler

What is the Basel Problem?

- One of the most famous problems he solved in the early 1700s was the Basel Problem.
- Named after the city of Basel in Switzerland, where Euler lived as a child, and where he went to university.
- The Basel Problem deals with summing the infinite series of reciprocals of integers squared.
- This problem looks at summing the following series to infinity:

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots \qquad or \qquad 1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\cdots$$

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \ ?$$

In 1735, Euler proved that this series sums to an exact number:

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$$

Pietro Mengoli

- First considered by the Italian mathematician Pietro Mengoli in 1644.
- In 1650, Mengoli "included the problem in *Novae quadraturae arithmetica*, a book on the summation of series." (Benko 244)
- Jacob Bernoulli, Johann Bernoulli, Daniel Bernoulli, Leibniz, Stirling, and de Moivre all attempted to solve the problem.
- Hard to calculate an exact solution because the series converges very slowly.

Bernoulli's Approximation

- He started by looking at the inequality $2k^2 \geq k(k+1)$
- From this equation he recognized that $\frac{1}{k^2} \leq \frac{1}{\frac{k(k+1)}{2}}$
- So $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{k^2} + \cdots \leq 1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \cdots + \frac{1}{\frac{k(k+1)}{2}} + \cdots$
- Bernoulli knew $\sum_{k=1}^{\infty} \frac{1}{\frac{k(k+1)}{2}}$ converged to 2 and that it was greater than $\sum_{k=1}^{\infty} \frac{1}{k^2}$
- He proved $\sum_{k=1}^{\infty} \frac{1}{k^2} \leq 2$
- Today we would call this process the "comparison test" for series.
- He gave up and wrote, "If anyone finds and communicates to us that which thus far has eluded our efforts, great will be our gratitude." (Dunham 42)

Euler's Estimation

- Euler first approached this problem in 1731 by looking at approximations of the series:
- 10th partial sum of the series: $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{100} = 1.54977$
- 100th partial sum of the series: $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{10000} = 1.63498$
- 1000th partial sum of the series: $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{1000000} = 1.64393$
- When you take Euler's exact solution of $\frac{\pi^2}{6}$ and look at the decimal, it is equivalent to 1.644934066842…, only accurate to 2 decimal places
- This confirms Bernoulli's approximation

Euler's Proof

- Euler is proving that the Basel problem sums to an exact number:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

- Starts with an nth degree "infinite polynomial" P(x) with nonzero roots $a_1, a_2, a_3, \ldots, a_n$ where P(x) =0.

$$P(x) = \left(1 - \frac{x}{a_1}\right)\left(1 - \frac{x}{a_2}\right)\left(1 - \frac{x}{a_3}\right) \ldots \left(1 - \frac{x}{a_n}\right)$$

- When you substitute x=0 into the factored form of the polynomial above you get P(0)=1.
- "Euler's next claim is "what holds for a finite polynomial holds for an infinite polynomial."

$$P(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

- Second, Euler needed to use the series expansion of $\sin x$ which looks like:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

Euler's Proof

- Euler used the information from above to start the proof by saying:
- $P(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots$
- By looking at this formula it is clear that P(0)=1, just like the factored form of the polynomial from before. Next Euler multiplied P(x) by $\frac{x}{x}$.
- $P(x) = x\left[\frac{1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}-\cdots}{x}\right]$
- $P(x) = \frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots}{x} = \frac{\sin x}{x}$

Euler's Proof

- We recognize that the numerator of this series is the expansion of $\sin x$
- He multiplied by $\frac{x}{x}$ to get this series equivalent to $\frac{\sin x}{x}$. The equation above "has zeroes at x = $\pm k\pi$ for k=1, 2… since these are the zeroes of the function $\sin x$, we can now use the claim above and write P(x) as an infinite product" and factor it, and set P(x) equal to the equations below:
- $P(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$
- $\text{P(x)} = \left(1 - \frac{x}{\pi}\right)\left(1 - \frac{x}{-\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 - \frac{x}{-2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 - \frac{x}{-3\pi}\right) \ldots$
- $\text{P(x)} = \left[1 - \frac{x^2}{\pi^2}\right]\left[1 - \frac{x^2}{4\pi^2}\right]\left[1 - \frac{x^2}{9\pi^2}\right]\left[1 - \frac{x^2}{16\pi^2}\right] \ldots$
- Next Euler expanded the infinite product equation above to get:
- $P(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots$
- $P(x) = 1 - \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \cdots\right)x^2 + \cdots$

Euler's Proof

- Next he took the coefficients from $1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}-\cdots$ and set them equal to the equation above to get:

$$-\frac{1}{3!}=-\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\frac{1}{16\pi^2}+\cdots\right)$$

- Since we know $-\frac{1}{3!}=-\frac{1}{3\times2\times1}=-\frac{1}{6}$ we can simplify $-\frac{1}{3!}$ to $-\frac{1}{6}$.

- Thus, $-\frac{1}{6}=-\frac{1}{\pi^2}\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots\right)$
- Next he divided both sides of the equation by $-\frac{1}{\pi^2}$ to get $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots$ alone.

- $\frac{-\frac{1}{6}}{-\frac{1}{\pi^2}}=-\frac{1}{6}\times-\frac{\pi^2}{1}=\frac{\pi^2}{6}$
- So, $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}=\frac{\pi^2}{6}$ Q.E.D
- From the start we know, $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots = 1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\cdots$
- So Euler has proved,

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$$

Where does this lead us?

- Euler proved other even powered series including: $\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}, \sum_{k=1}^{\infty} \frac{1}{k^6} = \frac{\pi^6}{945}, \sum_{k=1}^{\infty} \frac{1}{k^8} = \frac{\pi^8}{9450},$ $\sum_{k=1}^{\infty} \frac{1}{k^{10}} = \frac{\pi^{10}}{93555}$
- He then looked at series of the form $\sum_{k=1}^{\infty} \frac{1}{k^3}$ but was unable to sum series with odd exponents
- It was not until 1978 that Roger Apéry proved that this series sums to an irrational number (Dunham 60)
- To this day the question of summing p-series with odd exponents is still unknown

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