A Generalization of Pascal’s Triangle

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Pascal Triangles

\begin{align*}
n = 0: & & 1 \\
n = 1: & & 1 & 1 \\
n = 2: & & 1 & 2 & 1 \\
n = 3: & & 1 & 3 & 3 & 1 \\
n = 4: & & 1 & 4 & 6 & 4 & 1
\end{align*}
Pascal Triangles

Any entry in Pascal’s Triangle can be defined by the binomial coefficients.

\[ T_{n,k} = \binom{n}{k} \]
Properties

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n
\]
Generalization

\[ a \]
\[ b \ a \]
\[ b \ a+b \ a \]
\[ b \ a+2b \ 2a+b \ a \]
\[ b \ a+3b \ 3a+3b \ 3a+b \ a \]
In the **blue**, the coefficients of $b = \binom{n}{k}$, shifted down one row.
In the **red**, the coefficients of $a = \binom{n}{k}$, shifted down and to the right one row and column.
Shifting a value down one row is the same as $\binom{n-1}{k}$.
Shifting a value to the right is the same as $\binom{n}{k-1}$. Combining these gets us this:

$$T_{n,k} = a\binom{n-1}{k-1} + b\binom{n-1}{k} \quad \text{for } n, k > 0$$

$$T_{n,0} = b \quad \text{for } n > 0$$

$$T_{0,0} = a$$
a
b a
b a+b a
b a+2b 2a+b a
b a+3b 3a+3b 3a+b a

\[ T_{4,2} = 3a + 3b \]

\[ T_{4,2} = a \binom{4-1}{2-1} + b \binom{4-1}{2} \]

\( \binom{3}{1} = 3 \) and \( \binom{3}{2} = 3 \)

\[ \therefore T_{4,2} = 3a + 3b \]
Testing the general case

When \( a \) and \( b = 1 \), \( T_{n,k} \) should be equal to \( \binom{n}{k} \)

Known:

\[
\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}
\]

by definition of Pascal’s Triangle.

\[
T_{n,k} = a \binom{n-1}{k-1} + b \binom{n-1}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}
\]

for \( a, b = 1 \)

Therefore:

For \( a, b = 1 \), \( T_{n,k} = \binom{n}{k} \)
Mathematica evaluation

\[\begin{align*}
a &= a; \\
b &= b; \\
p[n_,k_] &= a \text{Binomial}[n - 1, k - 1] + b \text{Binomial}[n - 1, k]; \\
p[0,0] &= a; \\
\text{Column[Table[p[n,k],\{n,0,10\},\{k,0,n\}]} &\text{ Left]} \\
\text{Do[Print[Sum[p[n-i,i],0,Floor[n/2]]]} &\text{\{n,0,5\}]} \\
\end{align*}\]

generates the following:

\[
\begin{align*}
\{a\} \\
\{b, a\} \\
\{b, a + b, a\} \\
\{b, a + 2 b, 2 a + b, a\} \\
\{b, a + 3 b, 3 a + 3 b, 3 a + b, a\} \\
\{b, a + 4 b, 4 a + 6 b, 6 a + 4 b, 4 a + b, a\}
\end{align*}
\]

Replacing \(a\) and \(b\) with numbers will generate the appropriate values instead of the generalization.
Generalizations of figurative numbers - Triangular Numbers

Known:

\[ \triangle_{n-1} = \frac{n(n-1)}{2} \]

\[ \triangle_{n-1} = \binom{n}{2} \text{ for } n \geq 2 \]

Generalized:

\[ T_{n,2} = a\left(\binom{n-1}{1}\right) + a\left(\binom{n-1}{2}\right) = a(n-1) + b\left(\binom{n-1}{2}\right) \]

\[ a(n) + b\left(\binom{n}{2}\right) = \triangle_{n,a,b} \text{ for } n \geq 2 \]
Generalizations of figurative numbers - Tetrahedral Numbers

Known: \( h_n = \binom{n+3}{3} \)

\[
h_n = \binom{n+3}{3} \quad \text{for} \quad h_n = \frac{n(n+1)(n+2)}{6}
\]

Generalized:

\[
T_{n+3,3} = a\binom{n+2}{2} + b\binom{n+2}{3}
\]
Future avenues of research

- Diagonal sums
- Properties of generalized figurative numbers
- Generalized proof