The Undecidability of the Domino Problem and the Creation of the First Aperiodic Tiling

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Part 1:
The Formalist tradition, Hilbert and the Entscheidungsproblem

- David Hilbert was a German mathematician who is recognized as one of the most influential and universal mathematicians of the 19th and early 20th centuries
- He famously put forth a influential list of 23 unsolved mathematical and metamathamatical problems at the International Congress of Mathematicians in Paris in 1900.
- Contributed greatly to the Formalist tradition in mathematics in the early 20th century.

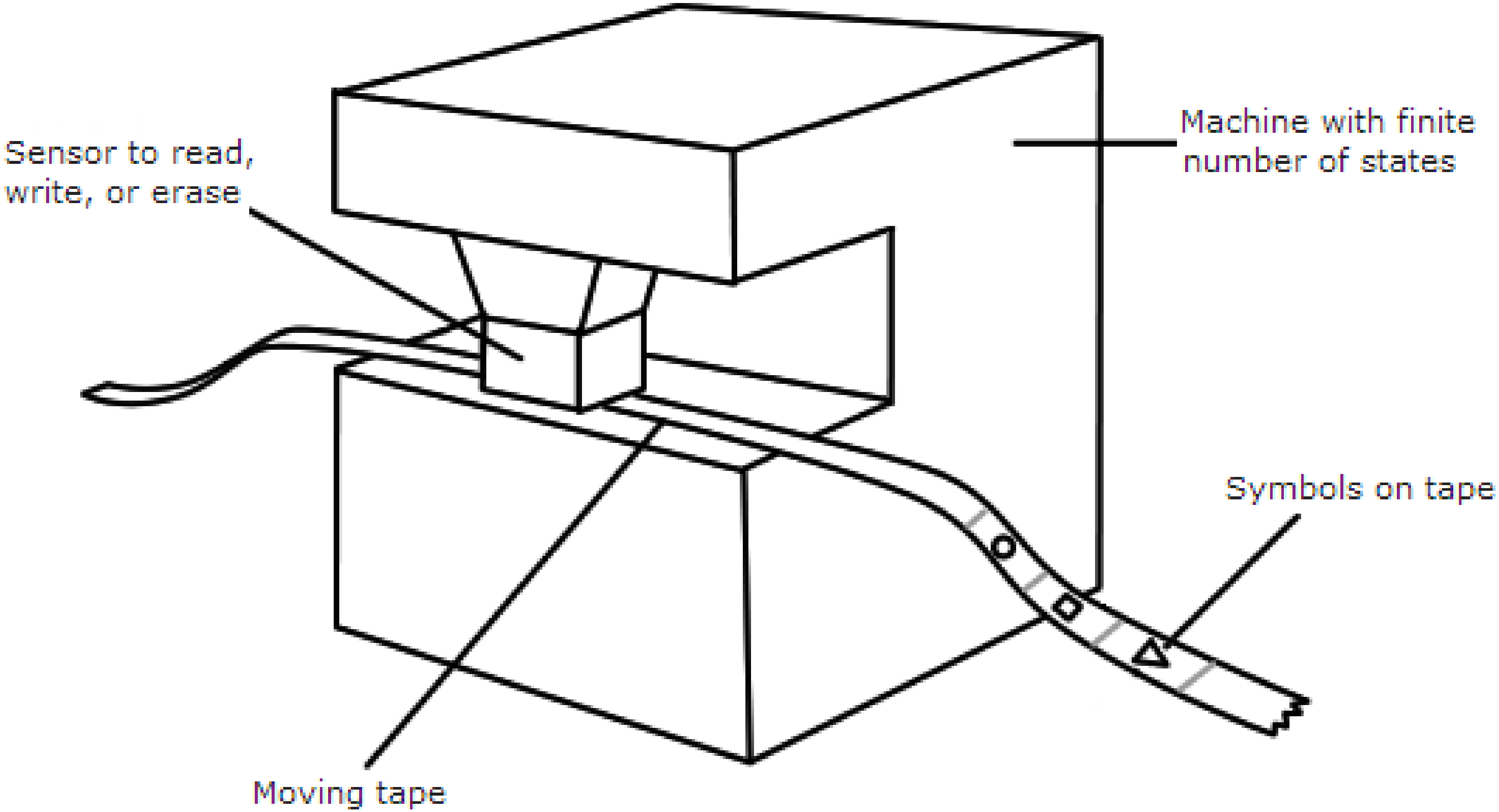
- The Entscheidungsproblem (German for 'decision problem') was a challenge posed by David Hilbert in 1928.
- It asked for an algorithm that takes as input a statement of a first-order logic and answers "Yes" or "No" according to whether the statement is universally valid.
- The Entscheidungsproblem can also be viewed as asking for an algorithm to decide whether a given statement is provable from the axioms using the rules of logic.
- Before the Entscheidungsproblem could be answered, the notion of "algorithm" had to be formally defined. This was done by Alonzo Church in 1936 with the concept of "effective calculability" based on his λ calculus and by Alan Turing in the same year with his concept of Turing machines.

Part 2:
Effective procedures and Turing Machines

- An effective procedure is a procedure which takes some class of problems and reduces the solution to a set of steps which:
  - Always give some answer rather than ever give no answer
  - Always give the right answer and never give a wrong answer
  - Always be completed in a finite number of steps, rather than in an infinite number
  - Work for all instances of problems of the class
- An effective method for calculating the values of a function is an algorithm; functions with an effective method are sometimes called effectively calculable.

- Several independent efforts to give a formal characterization of effective calculability led to a variety of proposed definitions (general recursion, Turing machines, λ-calculus) that later were shown to be equivalent.
- A Turing machine is a hypothetical device that manipulates symbols on a strip of tape according to a table of rules. Despite its simplicity, a Turing machine can be adapted to simulate the logic of any computer algorithm.
- Turing wrote that the Turing machine, here called a Logical Computing Machine, consisted of:

...an unlimited memory capacity obtained in the form of an infinite tape marked out into squares, on each of which a symbol could be printed. At any moment there is one symbol in the machine; it is called the scanned symbol. The machine can alter the scanned symbol and its behavior is in part determined by that symbol, but the symbols on the tape elsewhere do not affect the behavior of the machine. However, the tape can be moved back and forth through the machine, this being one of the elementary operations of the machine. Any symbol on the tape may therefore eventually have an innings. (Turing 1948, p. 61)



A Turing Machine

http://commonsenseatheism.com/wp-content/uploads/2011/12/turing_machine.gif

Part 3:

Hao Wang and the Domino Problem

- Hao Wang was a Chinese American logician and mathematician who also wrote extensively on the philosophy of mathematics. In a 1961 paper Wang proposed a method for deciding an important case of David Hilbert's Entscheidungsproblem
- He discussed tiling the plane by equally-sized square plates with edges marked by specific symbols or colors, now called Wang tiles or Wang dominoes.
- A set dominos is “solvable” if an infinite plane, ruled by squares of the same size as the dominoes, can be covered by copies of dominoes in the set, with a domino on each square, in such a way that symbols on adjacent domino edges match.
- According to Wang's student, Robert Berger, The Domino Problem deals with the class of all domino sets and It consists in deciding, for each domino set, whether or not it is solvable.

- We say that the Domino Problem is decidable or undecidable according to whether there exists or does not exist an algorithm which, given the specifications of an arbitary domino set, will decide whether nor not the set is solvable.
- In other words, is there an effective procedure for settling the problem for any given domino set?
- Wang made the conjecture that a finite set of plates is solvable if and only if it has at least one periodic solution. He conjectured that there is no aperiodic domino set and observed that if this conjecture is true, then the Domino Problem is decidable. If every domino set either does not admit a tiling, or admits a periodic tiling, then there is an algorithm for deciding which is the case.
- One of the most important contributions of Wang was the invention of Wang tiles who also showed that any Turing machine, as a set of computations, can be translated into a particular set of Wang tiles, a complex result used by Berger in his proof.

Part 4:
Preliminaries of Berger's proof of the Undecidability of the Domino Problem.

- A domino set is a finite set of square plates, whose dominoes are the same size with edges uniquely marked with symbols. Using unlimited copies one seeks to assemble the copies on an infinite plane ruled into squares according to the following rules:
  - No dominoe may be rotated or reflected.
  - A domino must be placed exactly over a ruled square.
  - The symbols on adjacent domino edges must match.
  - Every square must be covered with a domino.
- The domino set is called solvable if and only if the dominoes can be so assembled.

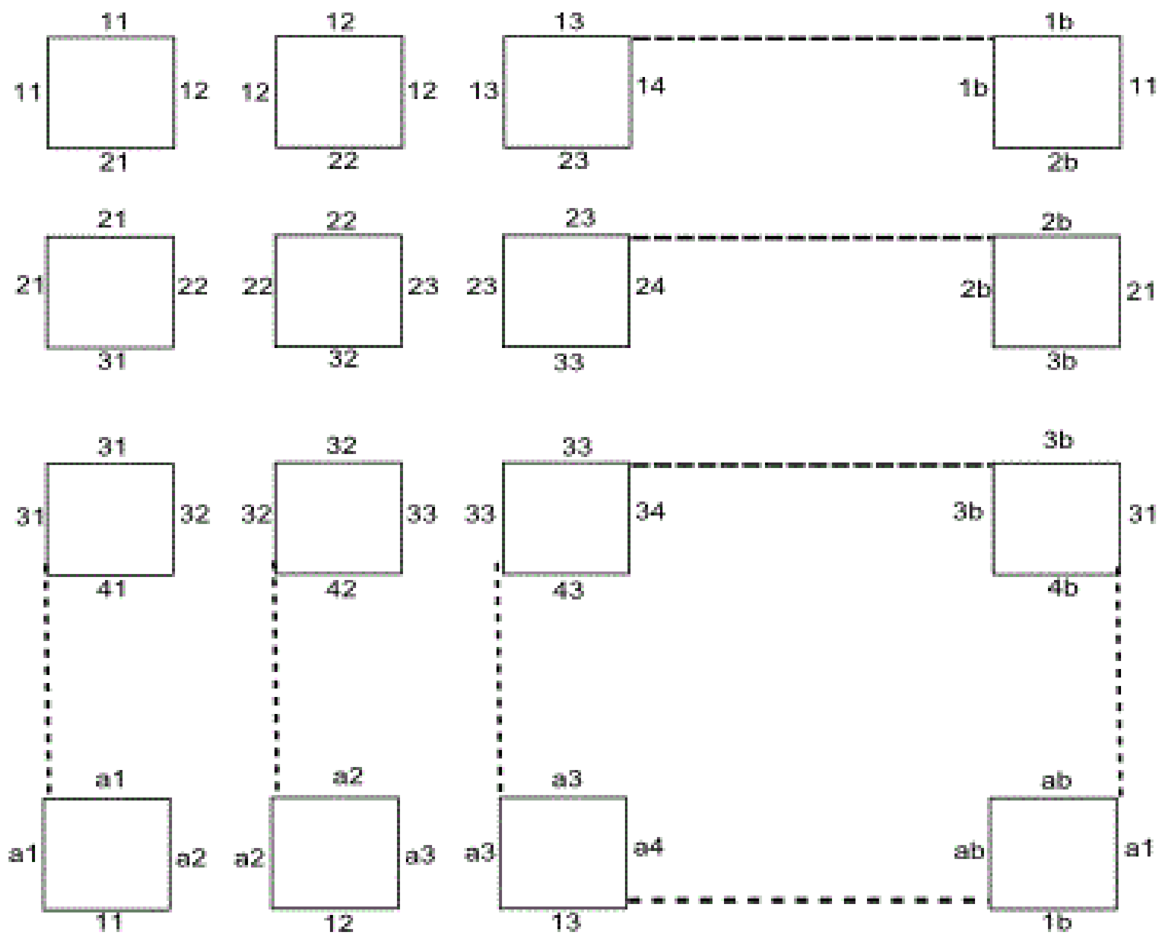
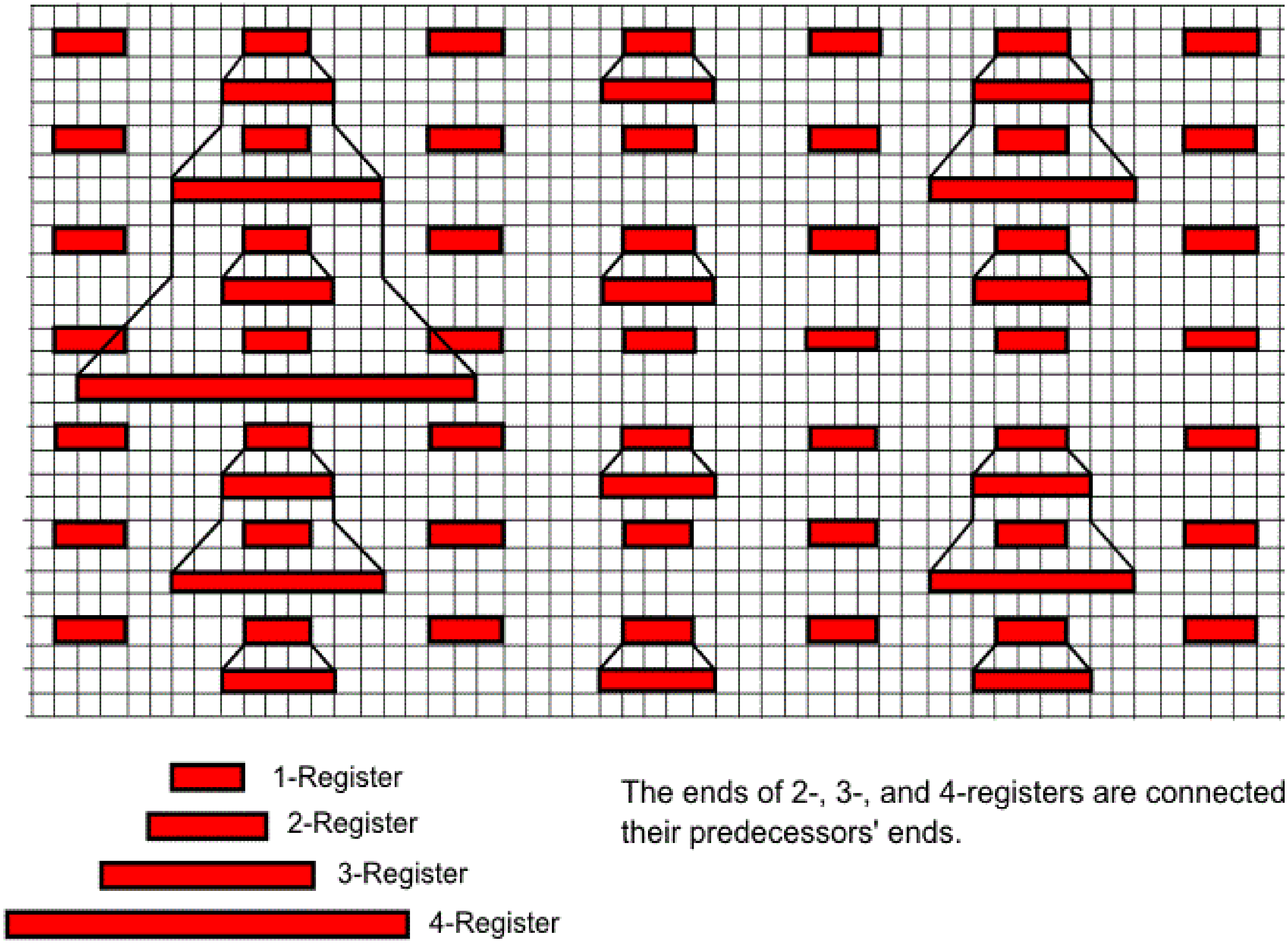


Table 1

The (a,b)-Periodic Domino Set

- We require that the symbols on domino edges be chosen from a countable set, which we can be done without restricting the class of domino sets so far as solvability is concerned. Then we can create a 1-to-1 mapping taking each domino set D into a non-negative integer N(D). Let P(x) be a function over the non-negative intergers such that:
  - P(x) = 1 if, for some domino set D, x=N(D) and D is solvable.
  - = 0 otherwise.
- Then the Domino Problem is decidable or undecidable according to whether there exists or does not exist a Turing machine which computes P(x).

- Specifically, we can use domino sets D_Z such that for any Turing machine Z, D_Z is solvable if and only if Z, starting with a blank tape in its initial state, eventually halts. Thus we have
  - $P(N(D_Z)) = 1$ if Z eventually halts
  - $= 0$ if Z never halts
- Assuming the Turing mahcine Z eventually halts, the solution of the domino set D_Z will model the operation of Z by containing, at specified places, rows of dominoes called registers, whose symbols represent complete descriptions of each operation of Z. If we number the operations starting with 1, we can have rows of dominoes represent any n number of operations, where the (n-1)-th operation will be $2^n + 1$ dominoes long and be called n-registers for $n>1$. Each n-register will have a specific (n-1)-register as its predecessor.



The ends of 2-, 3-, and 4-registers are connected to their predecessors' ends.

Figure 1 Distribution of Registers in the Solution Plane

- Berger used the notion of channels to represent the symbols on the edges of the dominos, so that two dominoes may attach only if the symbols in all the channels on their common edge match. The function of the channels are to propogate lines called signals.
- Because of the 1-to-1 correspondence between signal lines and channel symbols, a well-signaled domino set is completely specified by its signal lines alone.

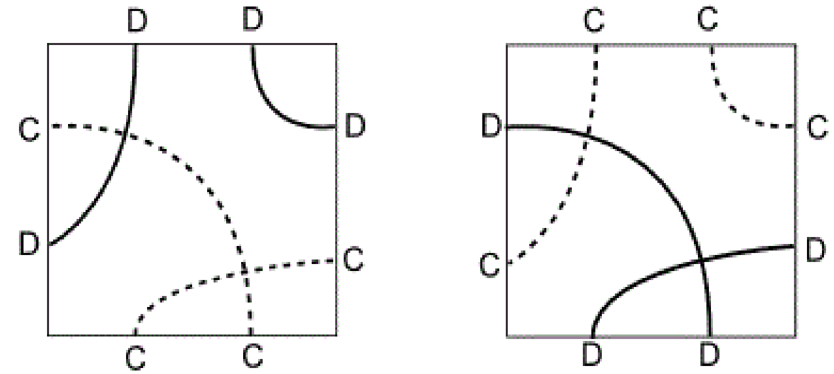


Figure 2a The Well-Signaled Domino Set S

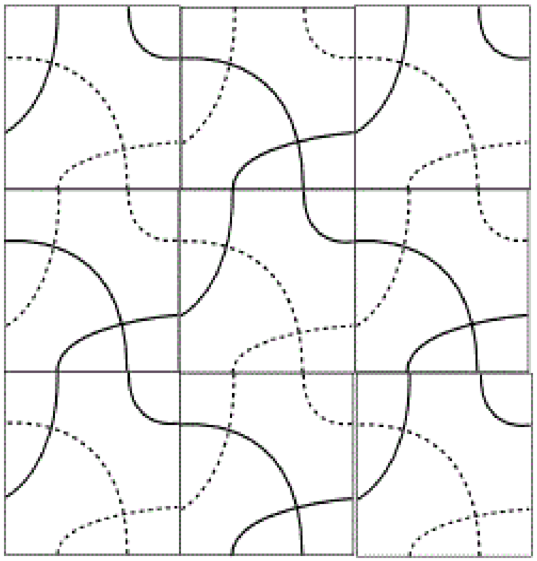


Figure 2b Part of the Solution of S

Part 5:
The Skeleton Set K.

- The skeleton domino set was designed as a solvable set having no torus, and whose solutions consisted of empty registers. The empty registers are simply rows of dominoes having a common feature, here having a special signal called an R-signal running the length of the row.
- Registers are located relative to their predecessor registers by means of certain signals called B-, C- and D-signals. In solutions of K, the figure formed by a succession of registers connected by signals are called skeletons, which start with 1-registers and include any successors.

- K has special dominoes called <u>embryo dominoes</u> that are used only for 1-registers, or <u>embryos</u>. The spacing of the embryos is gaurunteed by a base set of (4, 8)-periodic dominoes.
- Each domino is given a <u>base number</u> that acts as the symbol of the top channel of the left edge of the domino. Placement of one domino of K on the solution plane will determine for each square of the plane the base number of the domino to be placed on it.

11	12	13	14	15	16	17	18
21	22	23	24	25	26	27	28
31	32	33	34	35	36	37	38
41	42	43	44	45	46	47	48

Figure 3a The Base Numbers

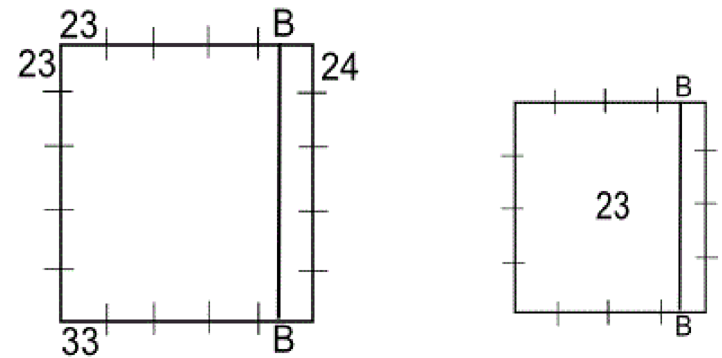


Figure 3b
The Five-Channel and Four-Channel Domino Diagrams

- The torus created by the base set of tiles are not determined by their base numbers, but instead by their channels, four on each side. These channels vary in their function as they channel different signals through the solution set, so the periodic placement of the base set of dominoes does not create a true torus for K.

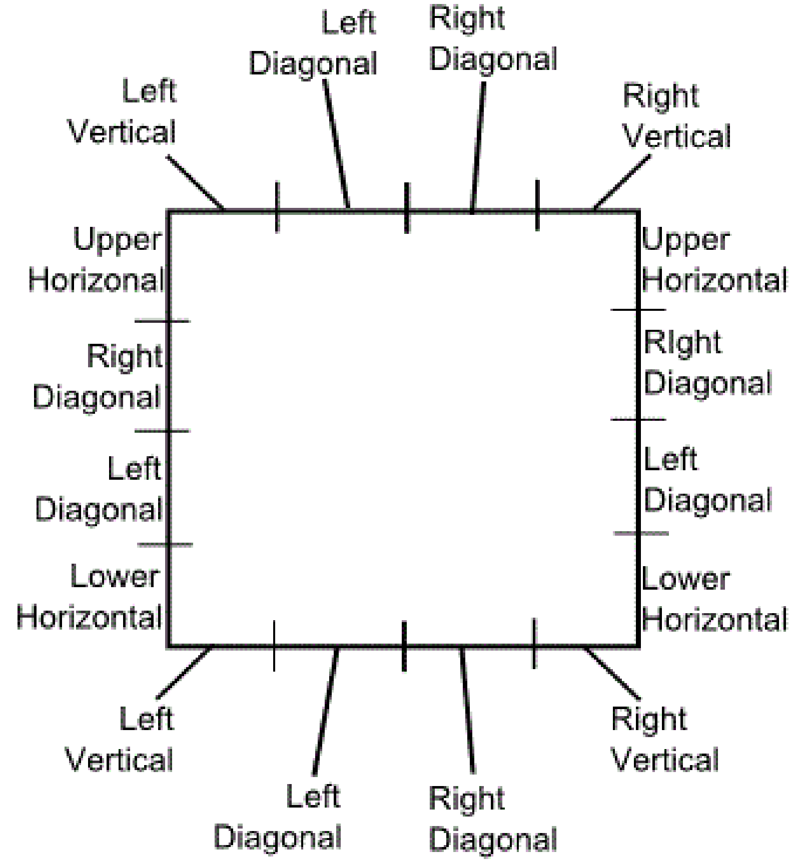


Figure 4 The Channels for K

- The growth or termination of each n-skeleton is assured by parity signals. These signals are placed on the channels of the (2, 2) domino set of 11, 12, 21 and 22; the associated signals are called 11-signals, 12-signals, etc.

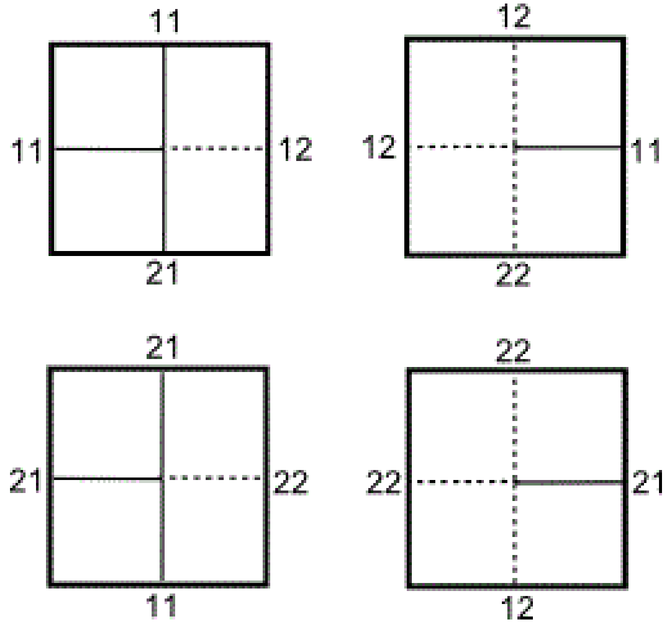


Figure 5
The (2,2)-Periodic Domino Set with Signals

- The signals dictate the parity of the register, determining whether growth or termination will occur. K is constructed so each register has some parity, where only those registers with parity 11 will have successors.

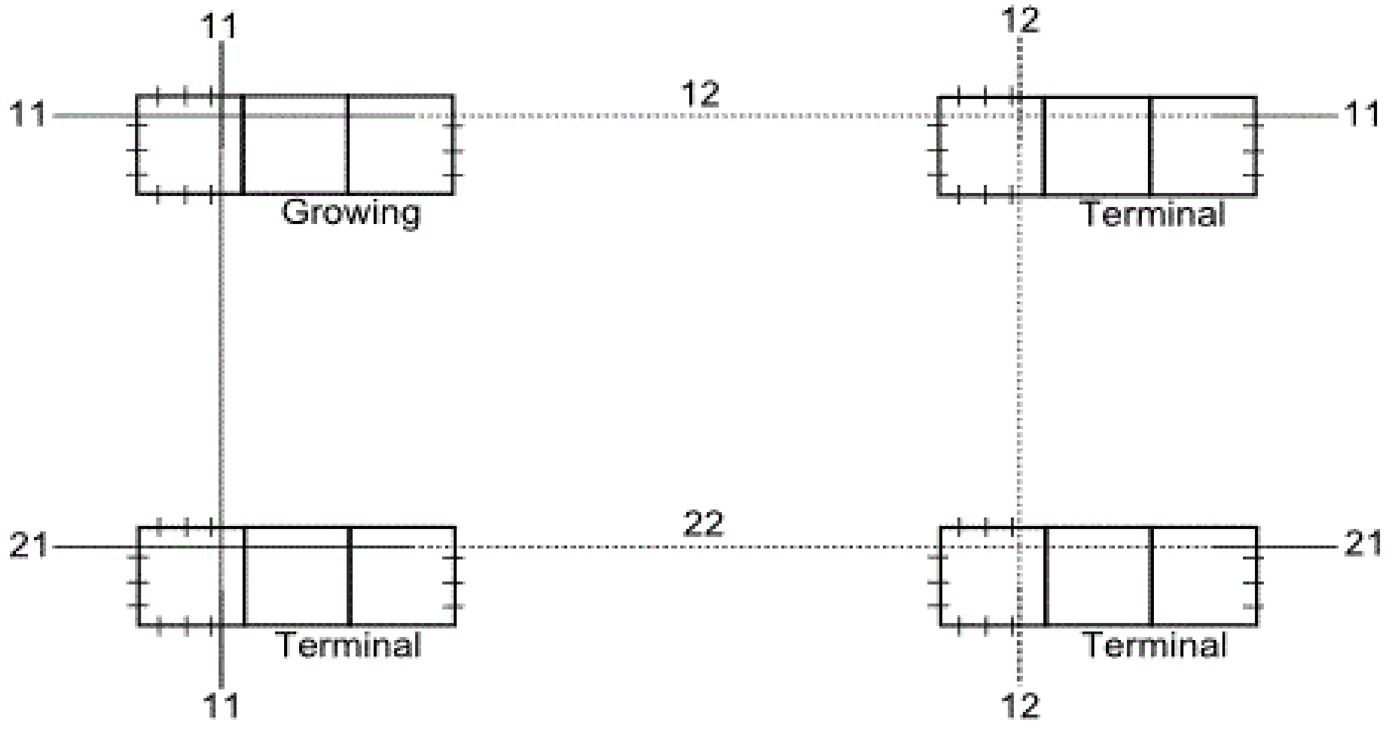


Figure 6
Parity Signals in 1-Registers

- The set K of dominoes creates and uncountably many non-equivalent sloutions, that follow the following sequence of iterative construction:

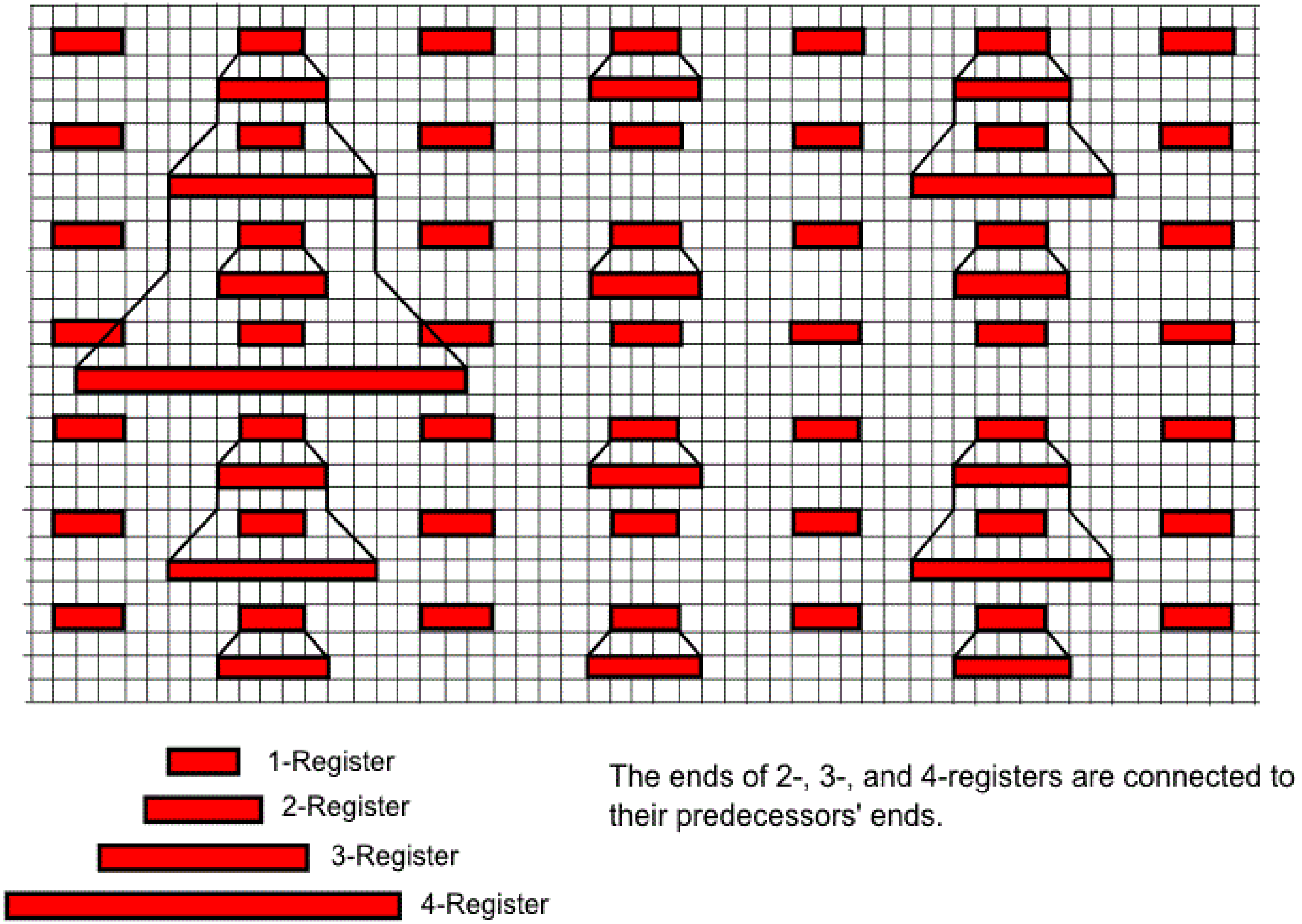
(1) At time t_0 a randomly chosen base number is assigned to a randomly chosen square of the solution plane. This choice determines the base number for each square in the plane and in particular locates the embryos. The signals of the embryos are drawn.

(2) For $n>0$, the following steps take place at time t_n:

(a) A randomly chosen parity is assigned to a randomly chosen n-register. This choice determines the parity signals passing through each n-register in the solution. These signals are drawn.

(b) Those n-registers with parity 11 receive in their left, center and right squares the B- and D-signals of prototypes 11, 13 and 12 respectively. This marks them as growing registers.

(c) From the growing n-registers skeleton signals emanate locating (n+1)-registers.



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