An Introduction to Self-Similar and Combinatorial Tiling Part II

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Self-Similar vs. Combinatorial Tiling

- Substitution/Inflate and Subdivide Rule
Self-Similar vs. Combinatorial Tiling

- Substitution/Inflate and Subdivide Rule
- No geometric resemblance to itself
- Substitution of non-constant length
One-Dimensional Symbolic Substitution

- $\mathcal{A}$ is a finite set called an *alphabet* whose elements are *letters*.
- $\mathcal{A}^*$ is the set of all *words* with elements from $\mathcal{A}$.
- A *symbolic substitution* is any map $\sigma : \mathcal{A} \rightarrow \mathcal{A}^*$. 
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$$a \rightarrow ab \rightarrow aba \rightarrow aba \rightarrow aba \rightarrow aba \rightarrow aba \rightarrow \cdots$$
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- The block lengths are Fibonacci numbers $1, 2, 3, 5, 8, 13, \ldots$

- substitution of non-constant length or combinatorial substitution
Substitution Matrix

- The substitution matrix $M$ is the $n \times n$ matrix with entries given by

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  \[ \sigma(a) = ab \]
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- substitution matrix for one-dimensional Fibonacci substitution:

  \[
  M = \begin{pmatrix}
  m_{11} & m_{12} \\
  m_{21} & m_{22}
  \end{pmatrix} = \begin{pmatrix}
  1 & 1 \\
  1 & 0
  \end{pmatrix}
  \]
Eigenvectors and Eigenvalues

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$$A\mathbf{v} = \lambda\mathbf{v}$$

- $\lambda$ is the **eigenvalue** of $A$ corresponding to $\mathbf{v}$.

- This equation has non-trivial solutions if and only if

$$\det(A - \lambda I) = 0$$

- Solve for $\lambda$ to find eigenvalues.
Expansion Constant

- Eigenvalues are the roots of the characteristic polynomial

$$\lambda^2 - \lambda - 1 = 0$$
Expansion Constant

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\[ \lambda^2 - \lambda - 1 = 0 \]

- Perron Eigenvalue - largest positive real valued eigenvalue that is larger in modulus than the other eigenvalues of the matrix
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- **Perron Eigenvalue** - largest positive real valued eigenvalue that is larger in modulus than the other eigenvalues of the matrix

- Perron Eigenvalue of Fibonacci substitution matrix:

\[ \frac{1 + \sqrt{5}}{2} = \gamma \]

the golden mean
The Fibonacci Direct Product Substitution

The direct product of the one-dimensional Fibonacci substitution with itself.

\[ \{a,b\}_x\{a,b\} \]

where \((a,a) = 1\), \((a,b) = 2\), \((b,a) = 3\), \((b,b) = 4\).
The Fibonacci Direct Product Substitution

The direct product of the one-dimensional Fibonacci substitution with itself.

\[ \{a,b\} \times \{a,b\} \]

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where \((a,a) = 1, (a,b) = 2, (b,a) = 3, (b,b) = 4.\]

- Not self-similar
- Not an inflate and subdivide rule
Several Iterations of Tile Types
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Substitution Matrix

1 → 2 4
   1 3
2 → 1 3
3 → 2 1
4 → 1
Substitution Matrix

The substitution matrix for the Fibonacci Direct Product is

$$M = \begin{pmatrix}
    m_{11} & m_{12} & m_{13} & m_{14} \\
    m_{21} & m_{22} & m_{23} & m_{24} \\
    m_{31} & m_{32} & m_{33} & m_{34} \\
    m_{41} & m_{42} & m_{43} & m_{44}
\end{pmatrix} = \begin{pmatrix}
    1 & 1 & 1 & 1 \\
    1 & 0 & 1 & 0 \\
    1 & 1 & 0 & 0 \\
    1 & 0 & 0 & 0
\end{pmatrix}$$
Expansion Constant

- Eigenvalues are the roots of the characteristic polynomial

\[ \det (M - \lambda I) = 0 \]

\[ \lambda^4 - \lambda^3 - 4\lambda^2 - \lambda + 1 = 0 \]
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- Perron Eigenvalue of Fibonacci Direct Product substitution matrix:

\[ \left( \frac{1 + \sqrt{5}}{2} \right)^2 = \gamma^2 \]

the golden mean squared
Replace and Rescale Method
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Rescale volumes by the Perron Eigenvalue raised to the $n^{th}$ power: $1/\gamma^{2n}$ where $n$ corresponds to the $n^{th}$-level of our block.
Replace-and-rescale Method

Rescale volumes by the Perron Eigenvalue raised to the $n^{th}$ power:
$1/\gamma^{2n}$ where $n$ corresponds to the $n^{th}$-level of our block.

This results in a level-0 tile with different lengths that the original. Repeat for other tile types. The substitution rule is now self-similar.
Self-Similar Fibonacci Direct Product

Combinatorial Substitution Rule
Self-Similar Fibonacci Direct Product

Combinatorial Substitution Rule

Self-Similar Inflate and Subdivide Rule
Replace-and-rescale Method

Compare level-5 tiles of the Fibonacci Direct Product (left)

Combinatorial Tiling
Replace-and-rescale Method

Compare level-5 tiles of the Fibonacci Direct Product (left) and the self-similar tiling (right).

Combinatorial Tiling

Self-Similar Tiling
Replace-and-rescale Method

- Not known whether replace-and-rescale method always works
- Replace-and-rescale method works in all known examples

Self-Similar Fibonacci DPV

Self-Similar non-Pisot DPV
References

• N.P. Frank, A primer of substitution tilings of the Euclidean plane, Expositiones Mathematicae, 26 (2008) 4, 295-386

Further Readings