On the Limiting Distribution of Eigenvalues of Large Random Regular Graphs with Weighted Edges

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Abstract

In 1981, B. D. McKay proved some fundamental results on the limiting distribution of eigenvalues of large random regular graphs [M]. Here we explore the analogous problem for a large random regular graph with random weights on its edges. We also examine how the distribution of weights and the distribution of eigenvalues are related, and put forward several conjectures.

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1 Review of Graph Theory

A graph is a set of points ("vertices") connected by lines ("edges"). For the purposes of this paper, we will assume there are no multiple edges (ie given two vertices, there is always at most one edge between them) and that there are no loops (ie there is no edge from a vertex to itself). We will further assume that there are no directed edges (ie an edge does not indicate a direction of travel). Here are some typical graphs:

We now introduce some terminology we will need.
A graph is said to be d-*regular* if every vertex meets exactly d edges. For example, graph (b) above is 3-regular.
A *path* is an ordered n-tuple of vertices $< v_1, v_2, \ldots, v_n >$ s.t. $\forall i \in \{1, \ldots, n-1\}$, there is an edge between v_i and v_{i+1}.
A *closed walk* is a path $< v_1, v_2, \ldots, v_n >$ s.t. $v_1 = v_n$.
A *proper cycle* is a closed walk $< v_1, v_2, \ldots, v_{n-1}, v_1 >$ s.t. $\forall i, j \in \{1, \ldots, n-1\}$, $v_i = v_j \iff i = j$.
A *cycle* is any closed walk $< v_1, v_2, \ldots, v_n >$ where $\exists 1 \leq i_1 < i_2 < \cdots < i_k \leq n$ s.t. $< v_{i_1}, v_{i_2}, \ldots, v_{i_k} >$ is a proper cycle.
A *tree* is a graph containing no cycles. (see (c) above)

If G is a graph with N vertices, we say G has size N, or more concisely $|G| = N$.

Given a graph G of size N, we can construct an associated $N \times N$ matrix (the *adjacency matrix*) as follows. Label the vertices of G as $1, 2, \ldots, N$. For all $i, j \in \{1, 2, \ldots, N\}$ define

$$a_{ij} = \begin{cases} 0 & \text{if there is no edge between } i \text{ and } j \\ 1 & \text{otherwise} \end{cases}$$

Then the adjacency matrix is the $N \times N$ matrix whose ij^{th} entry is a_{ij}. For example, the adjacency matrix of (b) is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Note that for any graph, the adjacency matrix is real symmetric. For d-regular graphs, each row and column has precisely d 1's (and all the other entries are 0). Henceforth, if we refer to the eigenvalues of a graph, we really mean the eigenvalues of the associated adjacency matrix.

1.1 Weighted Graphs

Take some graph G of size N; call its adjacency matrix A_G. Choose some probability distribution $\mathbb{W}(x)$ (the *weight distribution*). For all pairs i, j s.t. $1 \leq i < j \leq N$, draw a random w_{ij} (the ij^{th} *weight*) from our distribution $\mathbb{W}(x)$. We define the *weight vector* to be the collection of all the weights, in the dictionary order, ie

$$\overrightarrow{w} = (w_{12}, w_{13}, \ldots, w_{1N}, w_{23}, \ldots, w_{2N}, \ldots, w_{(N-1)N})$$

Note that $\overrightarrow{w}$ has $\frac{N(N-1)}{2}$ entries.

For all i, j s.t. $1 \leq i, j \leq N$ we now define

$$b_{ij} = \begin{cases} w_{ij}a_{ij} & \text{if } i < j \\ w_{ji}a_{ji} & \text{if } i > j \\ 0 & \text{if } i = j \end{cases}$$

Let $A_{G,\overrightarrow{w}} = (b_{ij})$ be the $N \times N$ matrix whose ij^{th} entry is b_{ij}. We can look at $A_{G,\overrightarrow{w}}$ as the adjacency matrix of the "weighted graph" $(G, \overrightarrow{w})$, ie the graph G with weight w_{ij} attached to the edge from i to j.

Suppose G is a d-regular graph of size N. Since G has $\frac{Nd}{2}$ edges, $\overrightarrow{w}$ will have at most $\frac{Nd}{2}$ nonzero entries. Thus, all the other $\frac{N(N-1)}{2} - \frac{Nd}{2}$ entries are *forced* to be 0. This turns out to be a useful property for us. Thus, we will be dealing with d-regular graphs from now on. For brevity, define

$$R_{N,d} = \{G : G \text{ is } d\text{-regular and } |G| = N\}.$$

1.2 Moments of a Graph

We begin by recalling the standard definition of moments of a probability distribution $p(x)$: we define the k^{th} moment of the distribution to be

$$\mathbb{E}[x^k] = \int_{\mathbb{R}} x^k p(x)dx$$

For the case of a matrix, we have a discrete distribution of eigenvalues. Suppose A is an $N \times N$ matrix with N (not necessarily distinct) eigenvalues $\{\lambda_{i,A}\}$. We define a probability "measure"

$$\mu_A(x) = \frac{1}{N}\sum_{i=1}^{N} \delta(x - \lambda_{i,A})$$

where δ is the Dirac delta functional. Essentially, we are putting an equal weight (of $\frac{1}{N}$) on each eigenvalue. Now from definition of the k^{th} moment we have

$$\begin{aligned}
\mathbb{E}_A[x^k] &= \int_{\mathbb{R}} x^k \mu_A(x)dx \\
&= \int_{\mathbb{R}} x^k \cdot \frac{1}{N}\sum_{i=1}^{N} \delta(x - \lambda_{i,A})dx \\
&= \frac{1}{N}\sum_{i=1}^{N} \int_{\mathbb{R}} x^k \delta(x - \lambda_{i,A})dx \\
&= \frac{1}{N}\sum_{i=1}^{N} \lambda_{i,A}^k \\
&= \frac{1}{N} Tr(A^k)
\end{aligned}$$

If the ij^{th} entry of the $N \times N$ matrix A is a_{ij}, it can be easily proved by induction that

$$Tr(A^k) = \sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\cdots\sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}$$

Thus we obtain the formula

$$\mathbb{E}[x^k] = \frac{1}{N}\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\cdots\sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1} \qquad (1)$$

Suppose we take a weighted graph $(G, \overrightarrow{w})$, where $G \in R_{N,d}$ and the entries of $\overrightarrow{w}$ are drawn from the weight distribution $\mathbb{W}$. Let its adjacency matrix $A_{G,\overrightarrow{w}}$ have entries b_{ij}. Then we can use equation (1) to calculate the moments of the distribution of eigenvalues of $(G, \overrightarrow{w})$. The first moment is trivial:

$$\mathbb{E}[x] = \frac{1}{N} \sum_{i=1}^{N} b_{ii} = 0,$$

since no edge goes from a vertex to itself (hence $b_{ii} = 0, \forall i$). The second moment is only slightly more involved:

$$\begin{aligned} \mathbb{E}_{G,\overrightarrow{w}}[x^2] &= \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} b_{ij} b_{ji} = \frac{1}{N} \sum_{1 \leq i,j \leq N} b_{ij}^2 \\ &= \frac{1}{N} \sum_{i=1}^{N} b_{ii}^2 + 2 \left(\frac{1}{N} \sum_{1 \leq i < j \leq N} b_{ij}^2 \right) \\ &= \frac{2}{N} \sum_{1 \leq i < j \leq N} w_{ij}^2 \end{aligned}$$

where the subscripts on $\mathbb{E}$ indicate dependence. Denote the m^{th} moment of $\mathbb{W}$ by σ_m. Recall that the assumption that G is d-regular forces all but $\frac{Nd}{2}$ of the entries of $\overrightarrow{w}$ to be 0. Thus, averaging all possible $\overrightarrow{w}$'s yields

$$\begin{aligned} \mathbb{E}_G[x^2] &= \int_{\mathbb{R}} \left(\frac{2}{N} \sum_{i=1}^{\frac{Nd}{2}} w_i^2 \right) \mathbb{W}(x) dx \\ &= \frac{2}{N} \sum_{i=1}^{\frac{Nd}{2}} \int_{\mathbb{R}} w_i^2 \mathbb{W}(x) dx \\ &= \frac{2}{N} \sum_{i=1}^{\frac{Nd}{2}} \sigma_2 \\ &= d\sigma_2 \end{aligned}$$

which is independent of G (and even N). So in fact we can write

$$\mathbb{E}[x^2] = d\sigma_2 \tag{2}$$

Before proceeding to more moment calculations, we pause to develop the theory a little further.

2 Some Theory

2.1 McKay's Paper

In this section we highlight a few results contained in McKay's paper (keeping the enumerations the same as those in [M], though slightly modifying his notation).

Lemma 2.1
Given G a d-regular graph. Fix a vertex v_0 in G, and suppose that for some $r \in \mathbb{N}$ the subgraph of G induced by all vertices at most a distance of $r/2$ away from v_0 contains no cycles. Then there are exactly $\theta(r)$ closed walks of length r starting at v_0, where

$$\theta(r) = \begin{cases} d \sum_{k=0}^{\frac{r}{2}-1} \binom{r}{k} \frac{r-2k}{r} (d-1)^k & \text{if } r \text{ is even} \\ 0 & \text{if } r \text{ is odd} \end{cases} \tag{3}$$

Lemma 2.3
Given a collection $\{G_i\}$ of d-regular graphs such that:
(i) $|G_i| \to \infty$ as $i \to \infty$; and
(ii) $\forall k \geq 3$, $\frac{C_k(G_i)}{|G_i|} \to 0$ as $i \to \infty$, where $C_k(G)$ denotes the number of k-cycles in a graph G.
Further, define

$$F(G, x) = \frac{\#\{\lambda : \lambda \leq x \text{ and is an eigenvalue of } G\}}{|G|}$$

Then $\forall r \geq 0$,

$$\int x^r dF(G_i, x) \to \theta(r) \text{ as } i \to \infty \tag{4}$$

Theorem 1.1
Given a sequence of d-regular graphs $\{G_i\}$, s.t. $|G_i| \to \infty$ as $i \to \infty$. Define

$$F(x) = \begin{cases} 0 & \text{if } x \leq -2\sqrt{d-1} \\ \int_{-2\sqrt{d-1}}^{x} \frac{d\sqrt{4(d-1)-u^2}}{2\pi(d^2-u^2)} du & \text{if } |x| < 2\sqrt{d-1} \\ 1 & \text{if } x \geq 2\sqrt{d-1} \end{cases} \tag{5}$$

Then $\frac{C_k(G_i)}{|G_i|} \to 0 \; \forall k \geq 3$ iff $F(G_i, x) \to F(x) \; \forall x$.

Recall that $R_{N,d}$ is the set of all d-regular graphs on N vertices.
Lemma 4.1
$\forall k \geq 3$, the average number of k-cycles in the members of $R_{N,d}$ is

$$C_{k,N,d} \to \frac{(d-1)^k}{2k} \text{ as } N \to \infty \tag{6}$$

Theorem 4.2
Let $F_{N,d}(x)$ be the average of $F(G, x)$ over all $G \in R_{N,d}$. Then $F_{N,d}(x) \to F(x)$ as $N \to \infty$, for every x.

2.2 More Results

We now use the results from [M] to prove a few of our own. But first, a bit more notation.
Let $C_{G,\overrightarrow{w},k} = \sum' b_{i_1 i_2} b_{i_2 i_3} \cdots b_{i_k i_1}$, where the sum is over all cycles $< i_1, i_2, \ldots, i_k, i_1 >$.
Let $T_{G,\overrightarrow{w},k} = \sum' b_{i_1 i_2} b_{i_2 i_3} \cdots b_{i_k i_1}$, where the sum is over all closed walks $< i_1, i_2, \ldots, i_k, i_1 >$ which are not cycles. Thus from equation (1) we have

$$\mathbb{E}_{G,\overrightarrow{w}}[x^k] = \frac{1}{N} C_{G,\overrightarrow{w},k} + \frac{1}{N} T_{G,\overrightarrow{w},k} \tag{7}$$

since any closed walk $< i_1, i_2, \ldots, i_k, i_1 >$ is either a cycle or not.

Theorem 2.1. *For any fixed $k \geq 3$,*

$$\lim_{N \to \infty} \frac{1}{N} \mathbb{E}_{N,d}[C_{G,\overrightarrow{w},k}] = 0$$

where we use $\mathbb{E}_{N,d}$ to signify that we average over all $G \in R_{N,d}$ and over all possible $\overrightarrow{w}$ (our weight distribution $\mathbb{W}$ is assumed to be fixed).

Proof. Fix some $G \in R_{N,d}$, and choose some weight vector $\overrightarrow{w}$. All but $\frac{Nd}{2}$ of the entries of $\overrightarrow{w}$ must be 0; label those weights which aren't necessarily 0, $\{w_1, w_2, \ldots, w_s\}$ (where $s = \frac{Nd}{2}$).

Choose some k-cycle in G; it traverses some of these weighted edges, so its contribution is $w_1^{r_1} w_2^{r_2} \cdots w_s^{r_s}$, where $r_i \geq 0$ and $\sum r_i = k$ (r_i represents the number of times total our k-cycle has traversed the edge with weight w_i). Averaging over $\overrightarrow{w}$, we have that the expected contribution of a k-cycle is thus

$$\begin{aligned}
\mathbb{E}[w_1^{r_1} \cdots w_s^{r_s}] &= \mathbb{E}[w_1^{r_1}] \cdots \mathbb{E}[w_s^{r_s}] \\
&= \sigma_{r_1} \sigma_{r_2} \cdots \sigma_{r_s} \\
&= \sigma_1^{\alpha_1} \sigma_2^{\alpha_2} \cdots \sigma_s^{\alpha_s}
\end{aligned}$$

where the α_i's satisfy $\alpha_i \geq 0, \sum i\alpha_i = k$. The latter condition implies that

$$\alpha_{k+1} = \alpha_{k+2} = \cdots = \alpha_s = 0$$

So in fact,

$$\mathbb{E}[w_1^{r_1} \cdots w_s^{r_s}] = \sigma_1^{\alpha_1} \cdots \sigma_k^{\alpha_k}$$

where

$$\alpha_i \geq 0, \; \sum_{i=1}^{k} i\alpha_i = k$$

Let $M = \max\{\sigma_1^{\alpha_1} \cdots \sigma_k^{\alpha_k} : \alpha_i \geq 0, \sum_{i=1}^k i\alpha_i = k\}$. Note that M depends only $\{\sigma_i\}$ and k; in particular, it does not depend on N. We highlight this fact by writing $M = M(\mathbb{W}, k)$. Define $C_{G,k}$ to be the total number of k-cycles in G. Then

$$\begin{aligned}
\frac{1}{N}\mathbb{E}_{N,d}[C_{G,\overrightarrow{w},k}] &= \frac{1}{|R_{N,d}|} \sum_{G \in R_{N,d}} \left(\frac{1}{N} \mathbb{E}_{\overrightarrow{w}}[C_{G,\overrightarrow{w},k}] \right) \\
&\leq \frac{1}{|R_{N,d}|} \sum_{G \in R_{N,d}} \left(\frac{1}{N} C_{G,k} M(\mathbb{W}, k) \right) \\
&= \frac{M(\mathbb{W}, k)}{N} \cdot \frac{1}{|R_{N,d}|} \sum_{G \in R_{N,d}} C_{G,k} \\
&= \frac{M(\mathbb{W}, k)}{N} \cdot C_{k,N,d}
\end{aligned}$$

which tends to 0 as N tends to infinity, by Lemma 4.1 of McKay (see equation (6) above), as well as by the fact that $M(\mathbb{W}, k)$ does not depend on N. $\square$

Corollary 2.2.

$$\lim_{N \to \infty} \mathbb{E}_{N,d}[x^k] = \lim_{N \to \infty} \frac{1}{N} \mathbb{E}_{N,d}[T_{G,\overrightarrow{w},k}] \tag{8}$$

Proof. This follows directly from Theorem 2.1 and equation (7). $\square$

Corollary 2.3. *Whenever k is odd,*

$$\lim_{N\to\infty} \mathbb{E}_{N,d}[x^k] = 0 \tag{9}$$

Proof. It is well-known that all closed walks of odd length are cycles. Thus for odd k, $T_{G,\overrightarrow{w},k} = 0$; we conclude by Corollary 2.2. $\square$

3 Higher Moments

3.1 The Fourth Moment

Now we know (by corollary 2.3) that we need not bother with the odd moments. The problem of finding even moments is simplified (since we only need to consider d-regular graphs which are locally trees, by corollary 2.2). To gain intuition, we begin by explicitly calculating the 4^{th} moment.

Choose some arbitrary vertex v of a d-regular graph G, and assume that locally G is a tree (ie the subgraph induced by all those vertices of G whose distance from v is less than 4, contains no cycles. Thus we may view v is being at depth 0, all those vertices directly adjacent to v to be at depth 1, all those vertices directly adjacent to a vertex of depth 1 to be at depth 2, etc. We must count the number of closed walks starting (and ending) at v, of length 4. We begin by choosing an edge from v - this will be the first edge we will traverse. Clearly, we can traverse it at most 4 times. If we traverse it exactly 4 times, we finish our walk at v, so such a walk is possible. What if we traverse our first edge exactly 3 times? Then we're stuck: we have only one more move (since the total length of the walk is 4), and we're 1 away from v. To return to v in time, we would have to go back up the same edge we just traversed 3 times. But this contradicts our assumption that we begin by traversing this first edge exactly 3 times. So no legal path begins by crossing some edge exactly 3 times.

What about twice? If we begin our walk by crossing our first edge exactly twice, then we're back at v, and have 2 more moves. We cannot go back along that first edge - that would contradict that we begin by crossing it exactly twice - but we can take any other of the remaining $(d-1)$ edges extending from v. In fact, after choosing this second edge, we are forced to traverse it exactly twice. Finally, what if we begin by crossing our first edge exactly once? We have three moves left, and we are at depth 1 in a tree (locally). We cannot go back along the edge we just crossed, so we must continue getting deeper into the tree. Choose an edge extending from the vertex where we are (other than the edge we just came down). This will be our second edge. We can traverse it at most three times - but that would leave us at depth 2, with no more moves. If we traverse it twice however, we could then take our first edge back up to v, thus making a legal closed walk. The reader may check that if we traverse the second edge exactly once, there is no way to get back to v in time.

Thus, there are three different types of paths: one in which we cross the first edge exactly 4 times; one in which we cross the first edge exactly twice, and then a different edge exactly twice; and one in which we traverse the first edge exactly once, a distinct second edge exactly twice, and the original edge once more. We represent these possibilities more succinctly as (respectively): (4); $(2, 2)$; and $(1, 2, 1)$. Now we count how many of each such path there can be. For paths of the type (4), for the first edge we have d choices (since $G \in R_{N,d}$). Once we choose one of these edges, say it has weight w. So the total contribution is $d \cdot w^4$. For a walk of type $(2, 2)$, say the first edge has weight w_1, and the second has weight w_2. We can choose the first in d ways, and the second in $d - 1$ ways. So altogether the contribution is $d(d - 1)w_1^2 w_2^2$. Finally, the contribution from paths of the type $(1, 2, 1)$ is also $d(d - 1)w_1^2 w_2^2$. Averaging over $\mathbb{W}$ yields a total contribution of

$$d\sigma_4 + d(d-1)\sigma_2\sigma_2 + d(d-1)\sigma_2\sigma_2 = d\sigma_4 + 2d(d-1)\sigma_2^2$$

where σ_m is the m^{th} moment of $\mathbb{W}$.

This is for one of the N vertices v. Thus by corollary 2.2, we obtain (after cancelling the N's) that the 4^{th} moment of the limiting distribution of eigenvalues of a large random d-regular graph is $d\sigma_4 + 2d(d - 1)\sigma_2^2$.

3.2 Higher Moments

Using similar reasoning, we can find all possible path types of length 6. They are:
(6); $(4, 2)$; $(3, 2, 1)$; $(2, 4)$; $(2, 2, 2)$; $(2, 1, 2, 1)$;
$(1, 4, 1)$; $(1, 2, 3)$; $(1, 2, 2, 1)$; $(1, 2, 1, 2)$; $(1, 1, 2, 1, 1)$,
where $(a_1, a_2, \ldots, a_m)$ represents a closed walk where we start at our point v, traverse some edge from v exactly a_1 times, then traverse some edge from wherever we ended up exactly a_2 times, then traverse some edge from wherever we are now exactly a_3 times, etc. Call this collection of all possible path types of length 6, $\mathcal{L}_6$. For each path type in $\mathcal{L}_6$, we figure out as above the number of

ways of choosing edges etc. The only difference is that this time, the path type $(2,2,2)$ has two fundamentally different interpretations: one is where edge w_1 is crossed twice, then edge w_2 is crossed twice, then edge w_1 is crossed twice more (leading to a total contribution of $d(d-1)w_1^4w_2^2$); the other is where edge w_1 is crossed twice, then edge w_2 is crossed twice, then edge w_3 is crossed twice (leading to a contribution of $d(d-1)(d-2)w_1^2w_2^2w_3^2$). Other than this, all the calculations are straightforward, and one finds that the sixth moment is $d\sigma_6 + 6d(d-1)\sigma_4\sigma_2 + 3d(d-1)^2\sigma_2^3 + 2d(d-1)(d-2)\sigma_2^3$.

In the same fashion, one finds $\mathcal{L}_8$ (ie all path types of length 8) to be
$(8); (6,2); (5,2,1); (4,4); (4,2,2); (4,1,2,1); (3,4,1); (3,2,3); (3,2,2,1); (3,2,1,2);$
$(3,1,2,1,1); (2,6); (2,4,2); (2,3,2,1); (2,2,4); (2,2,2,2); (2,2,1,2,1); (2,1,4,1);$
$(2,1,2,3); (2,1,2,2,1); (2,1,2,1,2); (2,1,1,2,1,1); (1,6,1); (1,4,3); (1,4,2,1);$
$(1,4,1,2); (1,3,2,1,1); (1,2,5); (1,2,4,1); (1,2,3,2); (1,2,2,3); (1,2,2,2,1);$
$(1,2,2,1,2); (1,2,1,4); (1,2,1,2,2); (1,2,1,2,1,1); (1,2,1,1,2,1); (1,1,4,1,1);$
$(1,1,2,3,1); (1,1,2,2,1,1); (1,1,2,1,3); (1,1,2,1,2,1); (1,1,2,1,1,2);$
$(1,1,1,2,1,1,1)$
and the 8^{th} moment to be

$$\begin{aligned} d\sigma_8 \;&+\; 8d(d-1)\sigma_6\sigma_2 + 6d(d-1)\sigma_4^2 + 16d(d-1)^2\sigma_4\sigma_2^2 + 12d(d-1)(d-2)\sigma_4\sigma_2^2 + \\ &+\; 4d(d-1)^3\sigma_2^4 + 8d(d-1)^2(d-2)\sigma_2^4 + 2d(d-1)(d-2)(d-3)\sigma_2^4 \end{aligned}$$

(note: the author is not completely sure that all the coefficients are correct here... although he is sure that there are no more path types in $\mathcal{L}_8$.)

One immediate observation is that given $k \in \mathbb{N}^*$, then the $2k^{th}$ moment is of the form

$$\sum_{(s_1,s_2,\ldots,s_j)\in\mathcal{P}_k} \; \sum_{(r_1,r_2,\ldots,r_\ell)\in\mathcal{P}_{(j-1)}} c_i \cdot d(d-1)^{r_1}(d-2)^{r_2}\cdots(d-\ell)^{r_\ell}\sigma_{2s_1}\sigma_{2s_2}\cdots\sigma_{2s_j} \tag{10}$$

where the outer sum runs over all partitions $s_1 > s_2 > \cdots > s_j$ of k, the inner sum runs over all partitions $r_1 > r_2 > \cdots > r_\ell$ of $(j-1)$, and the c_i's are some appropriate constants. In this study, we have yet to find a method of generating these coefficients (beyond the trivial first coefficient, which is always 1).

What we have been able to generate, however, is the sequence of all path types. Before we describe the method, we introduce some more notation. Define

$$\chi_2(n) = \begin{cases} 0 & \text{if } n \equiv 0 \mod 2 \\ 1 & \text{if } n \equiv 1 \mod 2 \end{cases}$$

Next, given $\overrightarrow{a} \in (\mathbb{N}^*)^m$, $\overrightarrow{e} \in \{0,1\}^m$ for some $m \in \mathbb{N}^*$, define

$$S_k(\overrightarrow{a}, \overrightarrow{e}) = \sum_{i=1}^{k} (-1)^{e_i} \chi_2(a_i)$$

(where a_i is the i^{th} entry of $\overrightarrow{a}$, and e_i the i^{th} entry of $\overrightarrow{e}$).

Suppose we are looking at $\mathcal{L}_N$, where N is even. If $\overrightarrow{a} = (a_1, a_2, \ldots, a_m)$ is a path type in $\mathcal{L}_N$, what can we say about $\overrightarrow{a}$? Clearly, $\sum a_i = N$. Also, we start and end at a depth of 0. The depth does not change after an a_i that is even, and the depth changes by 1 (though in which direction is less clear) after an a_i which is odd. Moreover, because we are locally in a tree, if a_{i-1} is odd and moving deeper, then a_i must continue in that direction. Similarly, if a_{i-1} is even and getting shallower, then a_i must be towards the deeper part of the tree. Finally, whenever we are at depth 0, we must move towards the deeper part of the tree on the next step.

A more precise way of writing all this down is as follows:

Definition 3.1 (Precise Definition of $\mathcal{L}_N$)**.**

$$\begin{aligned}
\overrightarrow{a} \in \mathcal{L}_N \quad \Longleftrightarrow \quad & \textit{for some } m \in \mathbb{N}^*, \overrightarrow{a} \in (\mathbb{N}^*)^m, \sum_{i=1}^{m} a_i = N \textit{ and} \\
& \exists \overrightarrow{e} \in \{0,1\}^m \textit{ satisfying} \\
& (1) \quad e_1 = 0 \\
& (2) \quad S_k(\overrightarrow{a}, \overrightarrow{e}) = 0 \Longrightarrow e_{k+1} = 0 \\
& (3) \quad \chi_2(a_k) = 1 \textit{ and } e_k = 0 \Longrightarrow e_{k+1} = 0 \\
& (4) \quad \chi_2(a_k) = 0 \textit{ and } e_k = 1 \Longrightarrow e_{k+1} = 0 \\
& (5) \quad S_m(\overrightarrow{a}, \overrightarrow{e}) = 0
\end{aligned}$$

Conjecture 3.2 (Generating $\mathcal{L}_N$)**.** *Fix an even $N \in \mathbb{N}^*$. We generate the set L_N as follows:*

(1) $(N) \in L_N$

(2) $(N-2k, 2k) \in L_N$ *for all* $k \in \{1, 2, \ldots, \lfloor \frac{N}{4} \rfloor\}$

(3) *Suppose* $(a_1, a_2, \ldots, a_n) \in L_N$ *and for some* j, a_j *and* a_{j+1} *are even. Then* $(a_1, \ldots, a_{j-1}, a_{j+1}, a_j, a_{j+2}, a_{j+3}, \ldots, a_n) \in L_N$.

(4) *Suppose* $(a_1, \ldots, a_n) \in L_N$. *If* $a_j, a_{j+1}, \ldots, a_{j+h}$ *are all even for some* j, h, *then* $(a_1, \ldots, a_{j-1}, a_j - k, a_{j+1}, \ldots, a_{j+i}, k, a_{j+i+1}, \ldots, a_{j+h}, \ldots, a_n) \in L_N$, *for all* $k \in \{1, 2, \ldots, a_j - 1\}$ *and for all* $i \in \{1, 2, \ldots, h\}$.

Then $L_n = \mathcal{L}_N$.

Although the author has not proven this rigorously yet, the weaker statement $L_N \subseteq \mathcal{L}_N$ seems quite reasonable.

Before moving on to other matters, we mention a nice way of checking our moment calculations. Observe that McKay's results can be viewed as being about weighted graphs, where the weight is always 1. Thus if we set $\sigma_m = 1$ for all m, our computed k^{th} moment should collapse to $\theta(k)$ in [M] (see equation (3)).

4 Another Conjecture

We now consider a completely different question. How are the weight distribution $\mathbb{W}(x)$ and the distribution of eigenvalues related? Or a narrower query: if we assume that the moments of the distribution of eigenvalues behave like those of some known distribution, what conditions are forced on the moments of $\mathbb{W}(x)$? Recall that the k^{th} moment of the semicircle is

$$2 \cdot \frac{(k-1)!!}{(k+2)!!}$$

if k is even, and 0 if k is odd. Since all the odd moments of our distribution are 0 as well (independently of $\mathbb{W}(x)$), it seems the semicircle is a natural choice in examining this question.

Setting the second moment of the limiting distribution equal to that of the semicircle, we have

$$d\sigma_2 = \frac{1}{4} \Longrightarrow \sigma_2 = \frac{1}{4d}$$

Setting the fourth moments equal yields (since we now know σ_2 has to be $\frac{1}{4d}$) that

$$d\sigma_4 + 2d(d-1)\sigma_2^2 = \frac{1}{8} \Longrightarrow \sigma_4 = \frac{1}{8d^2}$$

Similarly, we can subsequently find that

$$\sigma_6 = \frac{5}{64d^3}$$

Note that the first three even moments of the semicircle are $\frac{1}{4}, \frac{1}{8}, \frac{5}{64}$. The coincidence is rather striking. Consequently, we state the following

Conjecture 4.1. *Suppose $\mathbb{W}(x)$ is a distribution whose moments satisfy*

$$\sigma_k = \begin{cases} 2 \cdot \frac{(k-1)!!}{(k+2)!!} \cdot \frac{1}{d^{k/2}} & \textit{if } k \textit{ is even} \\ 0 & \textit{if } k \textit{ is odd} \end{cases}$$

Then the moments of the limiting distribution of the eigenvalues of large d-regular graphs whose edges are weighted with random weights drawn from $\mathbb{W}(x)$, are those of the semicircle distribution.

In conclusion, there are many interesting directions research in these topics could go. A good starting point would be to prove the unsolved assertions in this study. Alternatively, to gain more intuition about the problem, one could write a computer program to calculate moments using the methods outlined above. From here, it becomes less clear where to go. It does not seem unreasonable that conjecture 4.1, together with the observation of equation (10), could give information about the coefficients c_i. Perhaps there is some connection between the word generation of valid path types, and word generation in some other interesting context. It all remains to be seen.

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References

[M] B. D. McKay, *The Expected Eigenvalue Distribution of Large Regular Graphs,* Lin. Alg. and its App. 40 (1981), $203 - 216$.