Coefficients of the Moments for the Characteristic Polynomial of Random Unitary Matrices

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Abstract

In order to study moments of the Riemann Zeta Function, this paper looks at lower order terms of a polynomial involved with the calculation of moments for Random Unitary Matrices. Specifically, we will look at the coefficients of these polynomials. First, an algorithm will be developed to work out formulas for these coefficients, then we will find ways to approximate them.

Contents

Chapter 1

Introduction

The moments for the Riemann Zeta Function have been a mystery for years. It is a long standing conjecture that the leading term has the form

$$\frac{1}{T}\int_0^T |\zeta\left(\frac{1}{2}+it\right)|^{2k}dt \sim g_k a(k)\left(\log\frac{T}{2\pi}\right)^{k^2} \tag{1.1}$$

but only the first few were known until recently. Here, g_k is number which we knew little about until recently, and $a(k)$ is a product over primes. Keating and Snaith [KS] suggested that the characteristic polynomial for a Random Unitary Matrix as a model for the Riemann Zeta Function. The characteristic polynomial for a Random Unitary Matrix is defined as

$$\mathbb{E}\{|Z_u|^{2k}\} = g'_k f_k(N)$$

Here, g'_k is known and $f_k(z)$ is a polynomial with degree k^2. It turns out that for $k = 0, 1, 2, 3, 4$, $g_k = g'_k$. It is conjectured that $g_k = g'_k$ for all k for which the moment is defined. If this is true then we can gain insight into the lower terms of the moments of Zeta by studying the lower terms of the expression in (1.2). This paper is primarily interested in the lower terms of the polynomial $f_k(z)$. Our polynomial $f_k(z)$ is defined as

$$f_k(z) := \prod_{i,j\geq 1}^{k}(z+i+j-1). \tag{1.2}$$

Expanding this product, we see

$$\begin{aligned} f_k(z) &= (z+1)(z+2)^2(z+3)^3.....(z+k)^k(z+k+1)^{k-1}....(z+2k-1) \\ &= \sum_{r=0}^{k^2} c_r(k) z^{k^2-r}. \end{aligned}$$

The coefficients of this polynomial, $c_r(k)$ are polynomials that depend on k. This paper will concentrate on finding formulas for $c_r(k)$ and also on the behavior of these coefficients as k gets large. I'd like to acknowledge Michael Rubinstein, David Farmer, Brian Conrey, Steven Miller, and Chris Hughes for all their help on this research project.

Chapter 2

Finding Formulas for $c_r(k)$

2.1 $c_r(k)$ as Symmetric Polynomials

For a set $\alpha = \{a_1, a_2, a_3. \dots, a_n\}$ we can define the elementary symmetric polynomials on α for all $r \geq 1$ as,

$$\begin{aligned} \sigma_1 &= \sum_i a_i \\ \sigma_2 &= \sum_{i \neq j} a_i a_j \\ \sigma_3 &= \sum_{i \neq j \neq k} a_i a_j a_k \\ &\vdots \\ \sigma_n &= \prod_i a_i. \end{aligned}$$

By (1.2), $c_r(k)$ can be represented as the r^{th} elementary symmetric polynomial, σ_r defined on the set $A = \{1, 2, 2, 3, 3, 3, \dots, 2k-2, 2k-2, 2k-1\}$. where each $1 \leq j \leq 2k-1$, occurs $min\{j, 2k-j\}$ times.

So for $x_i \in A$,

$$\begin{aligned} c_0(k) &= 1 && (2.1)\\ c_1(k) &= \sigma_1 = \sum_i x_i \\ c_2(k) &= \sigma_2 = \sum_{i \neq j} x_i x_j \\ &\vdots \\ c_{k^2}(k) &= \prod_{1 \leq i \leq j \leq k} (i+j-1). && (2.2) \end{aligned}$$

Therefore, by (2.1),

$$\begin{aligned} c_1(k) &= (1+2+2+3+3+3+\cdots+2k-1) \\ &= (1^2+2^2+3^2+\cdots+k^2+(k-1)(k+1)+\cdots+2k-1) \\ &= \sum_{n=1}^{k} n^2 + \sum_{n=1}^{k-1} n(2k-n) \\ &= k^3. \end{aligned}$$

Define the generating function $E(t)$ for each $c_r(k)$ as

$$E(t) = \sum_{r \geq 0} c_r(k) t^r = \prod_{i \geq 0} (1 + x_i t). \tag{2.3}$$

This form of the generating function will be useful later on.

2.2 Power Sums

For each $r \geq 1$, the r^{th} power sum, p_r on a set $\beta = \{x_1, x_2, \ldots, x_n\}$ is defined

$$p_r = \sum_{i=1}^{n} x_i^r \tag{2.4}$$

and the generating function for p_r is

$$\begin{aligned} P(t) = \sum_{r\geq 1} p_r t^{r-1} &= \sum_{i\geq 1}\sum_{r\geq 1} x_i^r t^{r-1} \\ &= \sum_{i\geq 1} \frac{x_i}{1-x_i t} \\ &= \sum_{i\geq 1} \frac{d}{dt}\log\frac{1}{1-x_i t}. \end{aligned}$$

We've made this simplification by noting that $\sum_{i\geq 1}\sum_{r\geq 1} x_i^r t^{r-1}$ is a geometric series. Looking back to (2.3) we see that:

$$P(-t) = \frac{d}{dt} logE(t) \; = \; E'(t)/E(t) \tag{2.5}$$

and from (2.5) it follows that:

$$rc_r(k) \; = \; \sum_{n=1}^{r} (-1)^{n-1} p_n c_{r-n}(k) \tag{2.6}$$

Before I describe how to find general formulas for each coefficient I will prove a basic fact about each $c_r(k)$

Theorem 2.2.1. *$c_r(k)$ is a degree 3r polynomial in k.*

Proof. We'll use a proof by induction. We see that $c_0(k) = 1$ and $c_1(k) = k^3$ satisfy this claim. Now assume that for all $r \geq 1, c_{r-1}(k)$ is of degree 3r-3.

For our particular set A, we are able to write the power sum as

$$\begin{aligned} p_n &= 1^n + 2*2^n + 3*3^n + \cdots + k*k^n + (k-1)*(k+1)^n + \\ &\quad \cdots + (2k-1)^n \\ &= \sum_{j=1}^{k} j j^n \; + \; \sum_{j=1}^{k-1} (k-j)(k+j)^n \end{aligned}$$

Here, p_n is a degree $n+2$ polynomial in k. Using the recursion relation from (2.6), we see by the induction hypothesis, $c_{r-n}(k)$ has degree $3r - 3n$ so taking the $n = 1$ term, the highest degree in c_r is $3r$.

□

2.3 Finding $c_r(k)$ through Interpolation

The recursion derived in the previous section is slow computationally. Since the complexity of the power sums grows so fast, finding $c_r(k)$ for $r = 300$ would take an extremely long time. To speed up their calculation, we shall construct the polynomials through interpolation.

Definition 2.3.1. *The Lagrange Interpolating Polynomial*

Given the pairs $(x_1, y_1) \ldots (x_{n+1}, y_{n+1})$, The *Lagrange Interpolating Polynomial* is the n^{th} degree polynomial that passes through these given points. Given these points, one can construct $P(x)$ by

$$P(x) = \sum_{j=1}^{n+1} P_j(x) \tag{2.7}$$

where

$$P_j(x) = \prod_{i \geq 1, i \neq j}^{n+1} \frac{x - x_j}{x_i - x_j} y_j. \tag{2.8}$$

Knowing that each $c_r(k)$ is a polynomial with degree $3r$, we need to collect $3r + 1$ data points to construct each polynomial. These data points are obtained by expanding out our original polynomial in 1.2. As a simple example, lets find $c_1(k)$ using this formula. By the previous theorem, we know that $c_1(k)$ will be of degree 3, which means we will need 4 data points. The points will be of the form $(k, c_1(k))$. We obtain the data points by pulling the coefficient off the x^{k^2-1} term in $f_k(x)$. We see:

$$\begin{aligned}
f_1(x) &= x + 1 \\
f_2(x) &= x^4 + 8x^3 + 23x^2 + 28x + 12 \\
f_3(x) &= x^9 + 27x^8 + 318x^7 + 2142x^6 + \ldots \\
f_4(x) &= x^{16} + 64x^{15} + 1900x^{14} + +34720x^{13} + \ldots
\end{aligned}$$

Our data points here are $(1, 1), (2, 8), (3, 27), (4, 64)$. Using the Lagrange Interpolating formula:

$$\begin{aligned}
c_1(k) &= \sum_{j=1}^{4} P_j(k) \\
&= \frac{(k-2)(k-3)(k-4)}{(1-2)(1-3)(1-4)}(1) \\
&\quad + \frac{(k-1)(k-3)(k-4)}{(2-1)(2-3)(2-4)}(8) \\
&\quad + \frac{(k-1)(k-2)(k-4)}{(3-1)(3-2)(3-4)}(27) \\
&\quad + \frac{(k-1)(k-2)(k-3)}{(4-1)(4-2)(4-3)}(64) \\
&= k^3.
\end{aligned}$$

Since the coefficients in $f_k(z)$ grow extremely fast, computing them is difficult. To simplify the interpolation we can take advantage of the fact that when r is even, $c_r(k)$ is an even function and when r is odd, $c_r(k)$ is an odd function. So,

$$c_r(-k) = (-1)^r c_r(k). \tag{2.9}$$

Also, when $k = 0$, $f_0(z) = 1$, so $c_r(0) = 0$ for all $r \geq 0$. These facts allow you to double your collection of data points.

2.4 Truncating $f_k(z)$

Expanding $f_k(z)$ becomes very difficult for larger k due to the number of terms involved. This makes collecting data points for interpolation time consuming. To solve this problem, we need to truncate $f_k(z)$ so it only includes the terms we need to interpolate the polynomial. Namely, if we are seeking $c_R(k)$, then we need only the first R terms of $f_k(z)$. Using the symmetry of $f_k(z)$, we can write $f_k(z)$ as

$$\begin{aligned}
f_{k+1}(z) &= (z+1)(z+2)^2 \ldots (z+k)^k (z+k+1)^{k+1} \ldots & (2.10)\\
&\quad +(z+2k-1)^3(z+2k)^2(z+2k+1) \\
&= f_k(z)(z+k+1)^2(z+k+2)^2 \ldots (z+2k)^2(z+2k+1) & (2.11)
\end{aligned}$$

Now define

$$g_k(z) = (z+k+1)^2(z+k+2)^2 \ldots (z+2k)^2(z+2k+1). \tag{2.12}$$

We now have the recursion:

$$f_{k+1}(z) = f_k(z)g_k(z). \tag{2.13}$$

When collecting data points for $c_R(k)$, one only needs the first R terms of each $f_k(z)$. We can write

$$f_k(z) = \sum_{r=0}^{R} c_r(k) z^{k^2-r} + J(z) \tag{2.14}$$

Where

$$J(z) = \sum_{r=R+1}^{k^2} c_r(k) z^{k^2-r}. \tag{2.15}$$

We see all terms in (2.15) have degree less than or equal to $k^2 - R - 1$, so they are of no use to us in the interpolation process. Looking at the recursion defined in (2.13), we can write

$$\begin{aligned} f_{k+1}(z) &= (\sum_{r=0}^{R} c_r(k) z^{k^2-r} + J(z))g_k(z) \\ &= \sum_{r=0}^{R} c_r(k) z^{k^2-r} g_k(z) + J(z)g_k(z). \end{aligned}$$

Here, $g_k(z)$ has degree $2k+1$ and $J(z)g_k(z)$ has highest degree $(k^2 - R - 1) + (2k+1) = (k+1)^2 - R - 1$, so all terms in $J(z)g_k(z)$ have no useful data in $f_{k+1}(z)$. This leaves us with the truncated polynomial

$$Q_{k+1}(z) = \sum_{r=0}^{R} c_r(k) z^{k^2-r} g_k(z) \tag{2.16}$$

Define $C_n(Q)$ to be the coefficient of z^{k^2-n} in the polynomial Q. Starting with $Q_1(z) = z+1$, we can define a recursion of truncated polynomials as

$$Q_{k+1}(z) = \sum_{r=0}^{R} C_r(Q_k(z)g_k(z)) z^{k^2-r} \tag{2.17}$$

So now we have a truncated version of $f_k(z)$ where all terms with degree lower than $k^2 - R - 1$ are removed. Using the recursion we just developed, 600 terms of $f_k(z)$ were obtained for each k up to 1000. Using these points, the first 300 $c_r(k)$ were interpolated.

Chapter 3

Determinant Equation for $c_r(k)$

While the interpolation algorithm is successful for producing polynomials for the coefficients, it starts to slow down when $r > 300$. The purpose in this section is to develop a non-recursive formula for $c_r(k)$ involving a determinant of power sums. We will this new formula will give us the ability of approximating $c_r(k)$ for large k.
Looking back to (2.6) we are able to use properties of symmetric functions [Mc] to write $c_r(k)$ as a determinant.

Lemma 3.0.1.

$$r!c_r(k) = \begin{vmatrix} p_1 & 1 & 0 & 0 & \dots & 0 \\ p_2 & p_1 & 2 & 0 & \dots & 0 \\ p_3 & p_2 & p_1 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ p_{r-1} & p_{r-2} & p_{r-3} & \dots & \ddots & r-1 \\ p_r & p_{r-1} & p_{r-2} & \dots & p_2 & p_1 \end{vmatrix} \tag{3.1}$$

Proof. This can be shown by induction. We see that

$$c_1(k) = k^3 = |p_1|.$$

Let

$$P_r = \begin{pmatrix} p_1 & 1 & 0 & 0 & \dots & 0 \\ p_2 & p_1 & 2 & 0 & \dots & 0 \\ p_3 & p_2 & p_1 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ p_{r-1} & p_{r-2} & p_{r-3} & \dots & \ddots & r-1 \\ p_r & p_{r-1} & p_{r-2} & \dots & p_2 & p_1 \end{pmatrix}.$$

For simplicity, let $P_0 = (1)$. Assume $(r-j)!c_{r-j}(k) = \det(P_{r-j})$ for all $1 \le j \le r-1$. We must show that this implies $(r)!c_r(k) = \det(P_r)$. Consider $\det(P_r)$. Expanding by minors on the rightmost column, we see that

$$\begin{aligned}
\det(P_r) &= p_1 \det(P_{r-1}) - (r-1)(p_2 \det(P_{r-2}) - (r-2)(p_3 \det(P_{r-3}) - \dots \\
&\quad -2(p_{r-2} \det(P_2) - (p_{r-1} \det(P_1) - p_r)) \dots) \\
&= p_1 \det(P_{r-1}) - (r-1)p_2 \det(P_{r-2}) + (r-1)(r-2)p_3 \det(P_{r-3}) \\
&\quad \dots + (-1)^{r-1}(r-1)!p_r \\
&= \sum_{j=1}^{r} (-1)^{j-1} p_j \frac{(r-1)!}{(r-j)!} \det(P_{r-j}) \qquad (3.2)
\end{aligned}$$

By induction, $\det(r-j) = (r-j)!c_{r-j}(k)$. So,

$$\sum_{j=1}^{r} (-1)^{j-1} p_j \frac{(r-1)!}{(r-j)!} \det(r-j) = (r-1)! \sum_{j=1}^{r} (-1)^{j-1} p_j c_{r-j}(k). \qquad (3.3)$$

Finally, by (2.6),

$$(r-1)! \sum_{j=1}^{r} (-1)^{j-1} p_j c_{r-j}(k) = (r-1)! \cdot r c_r(k) = r! c_r(k). \qquad (3.4)$$

□

Now we have an expression for $c_r(k)$ involving only power sums, which are easy to express in terms of Bernoulli Polynomials. This form will enable us to approximate $c_r(k)$.

3.0.1 Eigenvalues for P_r

For large r, computing the determinant in (3.1) is very difficult. Since the determinant of a matrix is also the product of its eigenvalues, studying the behavior of the eigenvalues of P_r would give us insight into $c_r(k)$.

Lemma 3.0.2. *The characteristic equation for P_r can be written as:*

$$\det(P_r - \lambda I) = \sum_{m=0}^{r} (-1)^{r-m} \binom{r}{m} \lambda^{r-m} \det(P_m). \tag{3.5}$$

Proof. Again, we shall prove this by induction. We see that

$$\det(P_1 - \lambda I) = |p_1 - \lambda| = -\lambda + p_1 \tag{3.6}$$

satisfies this claim. Now assume for all $1 \leq j \leq r-1$, (3.5) holds. We'll show this implies that it holds for $j = r$. Consider $\det(P_r - \lambda I)$:

$$\begin{vmatrix} p_1-\lambda & 1 & 0 & 0 & \dots & 0 \\ p_2 & p_1-\lambda & 2 & 0 & \dots & 0 \\ p_3 & p_2 & p_1-\lambda & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ p_{r-1} & p_{r-2} & p_{r-3} & \dots & \ddots & r-1 \\ p_r & p_{r-1} & p_{r-2} & \dots & p_2 & p_1-\lambda \end{vmatrix}. \tag{3.7}$$

Expanding by minors on the right most column, we get an expression similar to what we've seen for $\det(P_r)$. So,

$$\begin{aligned} \det(P_r - \lambda I) &= (p_1-\lambda)\det(P_{r-1}-\lambda I) - (r-1)(p_2 \det(P_{r-2}-\lambda I) \\ &\quad -(r-2)(p_3 \det(P_{r-3}-\lambda I) \cdots - 2(p_{r-2}\det(P_2 - \lambda I \\ &\quad -(p_{r-1}\det(P_1-\lambda I) - p_r)) \dots) \\ &= p_1 \det(P_{r-1}-\lambda I) - (r-1)p_2 \det(P_{r-2}-\lambda I) \\ &\quad +(r-1)(r-2)p_3 \det(P_{r-3}-\lambda I) \cdots + (-1)^{r-1}(r-1)! p_r \\ &= (-\lambda)\det(P_{r-1}-\lambda I) + \sum_{j=1}^{r} (-1)^{j-1} p_j \frac{(r-1)!}{(r-j)!} \det(P_{r-j}-\lambda I). \end{aligned}$$

By the induction hypothesis, $\det(P_{r-j} - \lambda I) = \sum_{i=0}^{r-j}(-1)^{r-i}\binom{r}{i}\lambda^{r-i}\det(P_i)$. So we now have,

$$\begin{aligned}
\det(P_r - \lambda I) &= \sum_{i=0}^{r-1}(-1)^{r-i+1}\binom{r-1}{i}\lambda^{r-i}\det(P-i) \\
&\quad + \sum_{j=1}^{r}\sum_{i=0}^{r-j}(-1)^{r+i-1}p_j\frac{(r-1)!}{(r-j)!}\binom{r-j}{i}\lambda^{r-j-i}\det(P_i) \\
&= \sum_{m=0}^{r}\lambda^{r-m}((-1)^{r-m}\binom{r-1}{m}\det(P_m) \\
&\quad + \sum_{j=1}^{m}(-1)^{r-m-j-1}p_j\frac{(r-1)!}{(r-j)!}\binom{r-j}{m-j}\det(P_{m-j})).
\end{aligned}$$

Note that

$$\frac{(r-1)!}{(r-j)!}\binom{r-j}{m-j} = \frac{(r-1)!}{(m-j)!(r-m)!} = \binom{r-1}{m-1}\frac{(m-1)!}{(m-j)!}.$$

Now, $\det(P_r - \lambda I)$ becomes

$$\begin{aligned}
&\sum_{m=0}^{r}\lambda^{r-m}((-1)^{r-m}\binom{r-1}{m}\det(P_m) \\
&+(-1)^{r-m}\binom{r-1}{m-1}\sum_{j=1}^{m}(-1)^{j-1}p_j\frac{(m-1)!}{(m-j)!}\det(P_{m-j}).
\end{aligned}$$

Finally, by (3.2), this becomes

$$\begin{aligned}
&\sum_{m=0}^{r}\lambda^{r-m}(-1)^{r-m}\det(P_m)(\binom{r-1}{m} + \binom{r-1}{m-1}) \\
&= \sum_{m=0}^{r}(-1)^{r-m}\binom{r}{m}\lambda^{r-m}\det(P_m).
\end{aligned}$$

□

Note that by using (3.1), we can write the characteristic polynomial in terms of $c_r(k)$:

$$\det(P_r - \lambda I) = r!\sum_{m=0}^{r}(-1)^{r-m}\frac{\lambda^{r-m}}{(r-m)!}c_m(k). \tag{3.8}$$

With the characteristic polynomial for P_r, we can now solve for the eigenvalues of the matrix in terms of k. Solving for the case when $r = 2$, we get the 2 eigenvalues to be:

$$\lambda = k^3 \pm \frac{k\sqrt{7k^2 - 1}}{\sqrt{6}}$$

3.0.2 Approximating Eigenvalues of P_r using Polynomials

When $r > 5$, we can no longer solve explicitly for the roots of 3.5. Because of this we are going to have to develop approximations for the eigenvalues for P_r.

Lemma 3.0.3. *Each eigenvalue of P_r approach k^3 as $k \to \infty$.*

Proof. By dividing P_r through by k^3, the upper half of the matrix approaches zero as k grows. This tells us that for large k,

$$\begin{aligned} \det(P_r - \lambda I) \to (k^3 - \lambda)^r &= 0 \\ \Rightarrow \lambda &= k^3 \end{aligned}$$

and since $c_r(k)$ is the product of the eigenvalues, we see

$$c_r(k) \approx \frac{k^{3r}}{r!}. \tag{3.9}$$

□

It would be useful to come up with a better approximation for $c_r(k)$ than just its leading term. Consider the case when $r = 9$. Since we know that each eigenvalues approaches k^3, we should consider $\lambda_i - k^3$ to study how this difference behaves as a function of k. This graph shows the 9 eigenvalues of P_9 with k^3 subtracted off:

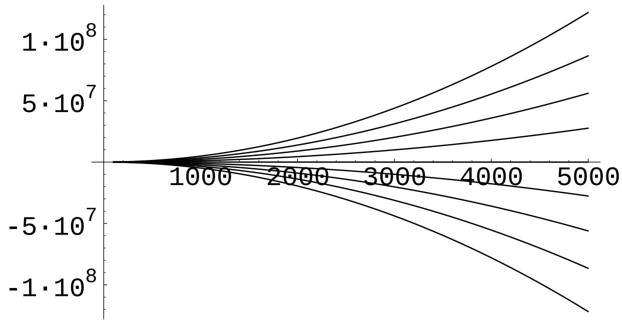

Figure 3.1: $\lambda_i - k^3$ for P_9. We see that 8 of these eigenvalues are symmetric about k^3.

There are eight eigenvalues symmetric about k^3. If we close in on the k axis then we see there is also a linear term:

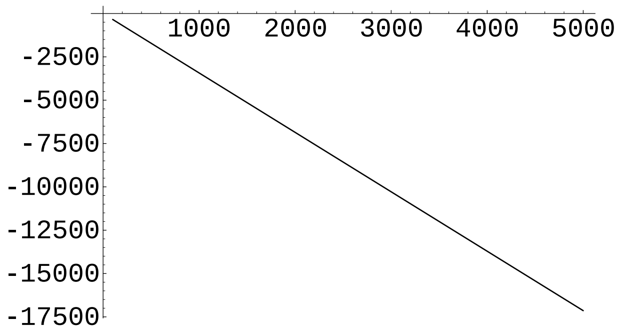

Figure 3.2: One $\lambda_i - k^3$ is a linear function of k. It turns out that if r is odd, one eigenvalue will always be k^3 plus a linear term.

If r is odd, then we see this one linear term when we subtract off k^3. The others are symmetric about k^3. If you let r be even, then each eigenvalue is symmetric about k^3.

A conjecture is that each eigenvalue of P_r can be approximated by k^3+ak^2+bk+c, for some constants a, b, c. This would mean that we can write each $c_r(k)$ as the product of r eigenvalues, each of which are cubic functions. The values of these constants are different for each eigenvalue and also for each value of r. For even r, each eigenvalue is symmetric about k^3, meaning that if $\lambda_i = k^3 + (a_i(r)k^2 + b_i(r)k + c_i(r))$ is an eigenvalue then $\lambda_j = k^3 - (a_i(r)k^2 + b_i(r)k + c_i(r))$ is also an eigenvalue. For odd r, we see similar behavior with the exception that one of the eigenvalues is k^3 plus a linear term, like we saw in 3.0.2. Here we have a plot of the eigenvalue evaluated at various k up to 5000.

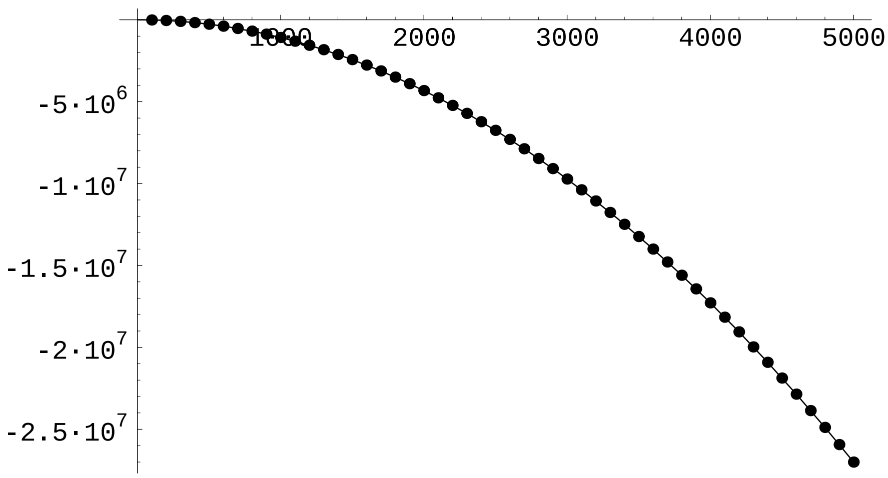

Figure 3.3: Fitting a degree 2 polynomial to $\lambda_i - k^3$ gives a very accurate approximation.

Numerically, it seems feasible to fit a cubic polynomial to each eigenvalue. As a start, consider the following:

$$\begin{aligned} c_r(k) &= \prod_{i=1}^{r} \lambda_i \\ &= \prod_{i=1}^{r} k^3 + a_i(r)k^2 + b_i(r)k + c_i(r) \\ &= k^{3r} \prod_{i=1}^{r} \left(1 + \frac{a_i(r)}{k} + \frac{b_i(r)}{k^2} + \frac{c_i(r)}{k^3}\right) \end{aligned} \tag{3.10}$$

As k gets large, $\frac{b_i(r)}{k^2} + \frac{c_i(r)}{k^3}$ approaches zero quickly, so we can write $c_r(k)$ in the approximate form

$$k^{3r} \prod_{i=0}^{r} (1 + \frac{a_i(r)}{k}). \tag{3.11}$$

This means finding values of $a_i(r)$ would be a good start to approximating $c_r(k)$ for larger values of k. The hope is that using this formula will not only model the behavior of $c_r(k)$ for fixed r, but also if we let r be a function of k. For example, we would be able to study the behavior of $c_{k^2-\mu k}(k)$, where μ is a number between 0 and 1.

Chapter 4

Lower Terms of $c_r(k)$

Using the eigenvalues of P_r we shown that the leading term of $C_r(k)$ is

$$\frac{k^{3r}}{r!}. \tag{4.1}$$

In this section we will use our determinant formula to produce more terms of $c_r(k)$. It seems that we can express (3.1) as

$$\det(P_r) = p_1^r + \sum_{j=2}^{r-1}(j-1)\cdot\binom{r}{j}p_1^{r-i})R_i \tag{4.2}$$

where

$$R_r = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ p_2 & 0 & 2 & 0 & \dots & 0 \\ p_3 & p_2 & 0 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ p_{r-1} & p_{r-2} & p_{r-3} & \dots & \ddots & r-1 \\ p_r & p_{r-1} & p_{r-2} & \dots & p_2 & 0 \end{pmatrix}.$$

Using (2.6), we can deduce that if r is even, $c_r(k)$ will have the form

$$c_r(k) = \sum_{i=0}^{\frac{3r}{2}-1}\gamma_i k^{3r-2i}. \tag{4.3}$$

and if r is odd,

$$c_r(k) = \sum_{i=0}^{\frac{3r-1}{2}} \gamma_i k^{3r-2i}. \tag{4.4}$$

Here, each γ_i is a constant that depends on r. We see that the second highest degree of $c_r(k)$ will be k^{3r-2}. We would like to write

$$c_r(k) = \frac{k^{3r}}{r!} + \gamma_1 k^{3r-2} + O(k^{3r-4}). \tag{4.5}$$

In order to do this we need to find γ_1. To start, consider a truncated version of P_r:

$$T_{i,r} \quad = \quad \frac{1}{r!}\begin{pmatrix} p_1 & 1 & 0 & 0 & \dots & 0 \\ p_2 & p_1 & 2 & 0 & \dots & 0 \\ p_3 & p_2 & p_1 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ p_i & p_{i-1} & p_{i-2} & \dots & \ddots & i \\ 0 & p_i & p_{i-1} & \dots & \ddots & i-1 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ 0 & \dots & p_i & \dots & p_2 & p_1 \end{pmatrix}. \tag{4.6}$$

So for each $i < j \leq r$, p_j is replaced with 0 in P_r. It turns out that the determinant of this matrix will match the first i terms of $c_r(k)$ exactly. So, if $n < \frac{3r}{2}$,

$$\det(T_{n,r}) = \sum_{i=0}^{n} \gamma_i k^{3r-2i} + O(k^{3r-2(n+1)}). \tag{4.7}$$

This implies that $\det(T_i)$ is a good approximation for $c_r(k)$ for larger k.

To find γ_1, we can use (4.2) to expand out $T_{2,r}$ and then collect terms of k^{3r-2}. Note that R_i in this case will be T_i with a zero replacing each diagonal entry. In $\det(T_{2,n})$, we are only interested in terms with degree k^{3r-2}. So using (4.2),

$$\begin{aligned} \det(T_{2,r}) &= p_1^r - \binom{r}{2} p_1^{r-2} p_2 + O(p_1^{r-4} p_2^2) \\ &= k^{3r} - \binom{r}{2} \frac{7}{6} k^{3r-2} + O(k^{3r-4}) \end{aligned}$$

So,

$$\gamma_1 = \frac{7}{12(r-2)!} \tag{4.8}$$

and

$$\begin{aligned} c_r(k) &\approx \frac{1}{r!}\det(T_{2,r}) \\ &= \frac{k^{3r}}{r!} - \frac{7}{12(r-2)!}k^{3r-2} + O(k^{3r-4}). \end{aligned}$$

Using this same method with $T_{3,r}$, we deduce that

$$\gamma_3 = \left(\frac{1}{12(r-2)!} + \frac{1}{2(r-3)!} + \frac{49}{288(r-4)!}\right) \tag{4.9}$$

So, we get this better approximation for $c_r(k)$:

$$\begin{aligned} c_r(k) &\approx \frac{k^{3r}}{r!} + \frac{7}{12(r-2)!}k^{3r-2} & (4.10) \\ &\quad + \left(\frac{1}{12(r-2)!} + \frac{1}{2(r-3)!} + \frac{49}{288(r-4)!}\right) k^{3r-4} & (4.11) \\ &\quad + O(k^{3r-6}) & (4.12) \end{aligned}$$

Until now, only γ_1 and γ_2 are known, but more can be found by examining (4.2). Perhaps a conjecture can be made once more of these terms are worked out. It would be interesting to consider γ_j for higher j since they will provide asymptotics of $c_r(k)$. This is another open problem that is worth pursuing.

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