Zeros of the Derivative of Random Unitary Matrices' Characteristic Polynomials

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Abstract

Our goal is to describe the distributions of the zeros of $Z'(U, z)$, where $Z(U, z)$ is the characteristic polynomial of a random unitary matrix U.

Contents

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1 Introduction

In this paper we will investigate the distribution of the complex roots of

$$Z'(U,z) = \frac{d}{dz}\det(U - zI) = \frac{d}{dz}\prod_{i=1}^{N}(z - z_i)$$

where U is a random $N \times N$ unitary matrix in the Circular Unitary Ensemble (CUE) and $Z(U,z)$ is its characteristic polynomial. One of the reasons for our study is that the distribution of the roots of $Z'(U,z)$ inside the unit circle is a model of the distribution of the roots of $\zeta'(s)$ on the right of the critical line $Re(z) = 1/2$, where $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ is the Riemann Zeta function.

U is an unitary matrix if and only if $UU^* = I$, where U^* is U's complex conjugate matrix and I is the identity matrix. All the eigenvalues of U lie on the unit circle. The characteristic polynomial $Z(U,z)$ of U is the polynomial whose roots are the eigenvalues of U.

The Unitary Group is the group of all $N \times N$ matrices. A random unitary matrix is a matrix chosen from the Unitary Group with the unique measure that is rotation invariant, called Haar measure. (For a more detailed preview of Random Matrix Theory and the Riemann Zeta function, the reader should refer to, for example, [1]).

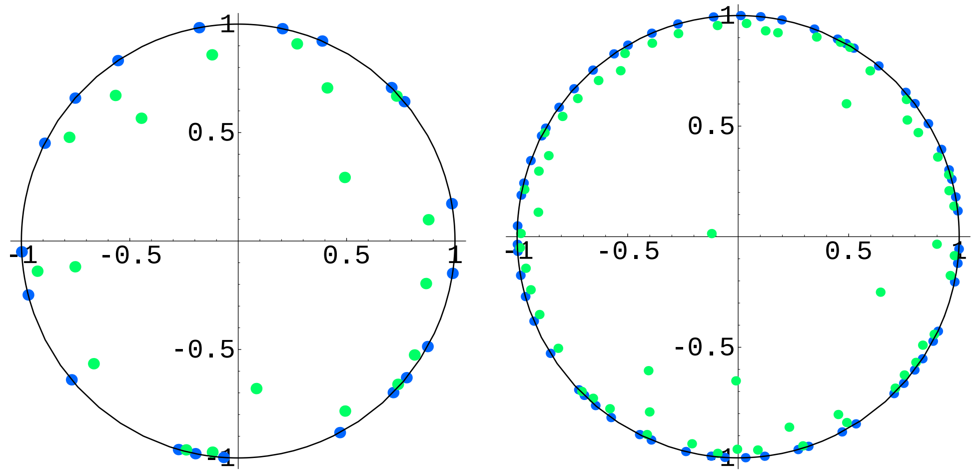

Figure 1: Zeros of $Z(U,z)$ and $Z'(U,z)$ of a 20×20 and a 60×60 random unitary matrix.

Figure 1 is the plot of the roots of the characteristic polynomial $Z(U,z)$ of a unitary matrix U, and the plot of the roots of $Z'(U,z)$. The roots of $Z(U,z)$, or the eigenvalues of U, lie on the unit circle. Figure 1 shows that all the roots of $Z'(U,z)$ lie inside the unit circle. This fact can be easily explained by a classical theorem by Gauss, which says that all the roots of the derivative of a polynomial lie in the convex hull of the roots of the polynomial. Since all the roots $z_1, z_2, \ldots, z_N$ of $Z(U,z)$ lie on the unit circle, all the roots of $Z'(U,z)$ lie inside the circle.

Throughout this paper, unless otherwise defined, $z_1, \ldots, z_N$ will be roots of $Z(U,z)$ (and also the eigenvalues of U); $z'_1, \ldots, z'_{N-1}$ will be the roots of $Z'(U,z)$; z' will be a generic root of $Z'(U,z)$.

In section 2, we are going to study some properties of the coefficients of $Z(U,z)$, which will help us in section 3 to investigate a connection among the distribution of the roots of $Z'(U,z)$, the eigenvalues of U and the coefficients of $Z(U,z)$. A relationship between the distribution of (z')s and the eigenvalues of U will be further investigated in section 4. In section 5, we are going to directly study the distribution of (z')s and see an interesting behavior of it. Finally, in section 6, we are going to look at the big picture of various types of distribution of the eigenvalues of a unitary matrix on the unit circle, other than the distribution in random unitary matrices.

2 Coefficients of $Z(U, z)$

When studying any polynomial, it is important to look at its coefficients. For the characteristic polynomial of an unitary matrix, there is an interesting fact about its coefficients.

Property 2.1. *Let* $Z(U, z) = z^N + a_1 z^{N-1} + \cdots + a_N$*, then*

$$a_{N-k} = (-1)^N \det(U)\ \overline{a_k}. \tag{2.1}$$

(For convenience, we will always choose the first coefficient a_0 *in* $Z(U, z)$ *to be* 1*.)*

Corollary 2.2.

$$|a_{N-k}| = |a_k|, \forall k = 0, \ldots, N \tag{2.2}$$

(Note that this result applies to any polynomial $P(z)$ *all whose roots lie on the unit circle.)*

Proof.

$$\begin{aligned} Z(U, z) &= z^N + a_1 z^{N-1} + \cdots + a_{N-1} z + a_N \\ &= (z - z_1)(z - z_2) \cdots (z - z_N). \end{aligned}$$

Denote $M = \{1, 2, \ldots, N\}$.

Expanding the product and equating the coefficients yields:

$$a_N = (-1)^N \prod_{i=1}^{N} z_i = (-1)^N \det(U)$$

And for all $k = 1, 2, \ldots, N-1$:

$$\begin{aligned}
a_{N-k} &= (-1)^{N-k} \sum_{\substack{B \subset M \\ |B|=N-k}} \prod_{i \in B} z_i \\
&= (-1)^{N-k} (z_1 z_2 \cdots z_N) \sum_{\substack{B \subset M \\ |B|=N-k}} \prod_{i \in B^c} \frac{1}{z_i} \\
&= (-1)^{N-k} \det(U) \sum_{\substack{B \subset M \\ |B|=N-k}} \prod_{i \in B^c} \frac{1}{z_i} \\
&= (-1)^{N-k} \det(U) \sum_{\substack{A \subset M \\ |A|=k}} \prod_{i \in A} \frac{1}{z_i} \\
&= (-1)^{N-k} \det(U) \sum_{\substack{A \subset M \\ |A|=k}} \prod_{i \in A} \frac{\overline{z_i}}{|z_i^2|} \\
&= (-1)^{N-k} \det(U) \sum_{\substack{A \subset M \\ |A|=k}} \prod_{i \in A} \overline{z_i} \\
&= (-1)^{N-k} \det(U) \overline{\sum_{\substack{A \subset M \\ |A|=k}} \prod_{i \in A} z_i} \\
&= (-1)^{N-k} \det(U) (-1)^k \overline{a_k} \\
&= (-1)^N \det(U) \overline{a_k}.
\end{aligned}$$

By applying the absolute value function to all the manipulations above, one will have a proof for $|a_{N-k}| = |a_k|$, where $a_0, a_1, \ldots, a_N$ are the coefficients of any polynomial all whose zeros are on the unit circle. □

This palindromic symmetry among $|a_k|$s will help us prove a property of the roots of $Z'(U, z)$ in the coming section.

3 Connections between a_js, z_j's and z_js

There is certainly a relationship between the distribution of the roots of $Z(U, z)$ and the values of a_is. (When we change the root(s) of a polynomial, its coefficients must change as well.) On the other hand, there is a strong relationship between the distribution of the roots of $Z(U,z)$ and the distribution of the roots of $Z'(U,z)$ (Using Dyson's electrostatic model, one can think of the roots of $Z(U,z)$ as N unit charges on the unit circle, then there is an electric field at any point inside the circle. Then z' is a root of $Z'(U,z)$ if and only if the electric field vanishes at z', i.e.:

$$\mathbf{E}(z') = \sum_{j=1}^{N} \frac{1}{\overline{z'} - e^{-i\theta_j}} = 0.$$

(For more details on Dyson's electrostatic model, see, for example, [3].)) Thus, we expect a bridged connection between the distribution of the roots of $Z'(U,z)$ and the values of a_is. We describe the relationship among a_is, zs and (z')s below.

Proposition 3.1. *The three following statements are equivalent:*
(1) All the roots of $Z'(U,z)$ have absolute value 0.
(2) The roots of $Z(U,z)$ are equally distributed on the unit circle.
(3) $a_1 = a_2 = \cdots = a_{N-1} = 0$, in other words, $Z(U,z) = z^N + a_N$, where $|a_N| = 1$.

Proof. Denote the roots of $Z(U,z)$ by $e^{i\theta_1}, \ldots, e^{i\theta_N}$, where $0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_N \leq 2\pi$. Without loss of generality, assume $\theta_1 = 0$ (if $\theta_1 \neq 0$ we can "rotate" all the roots of $Z(U,z)$ on the unit circle an angle of $-\theta_1$.)

The fact that (1) $\Leftrightarrow$ (3) is easy and can be omitted.

Proof of (2) $\Leftrightarrow$ (3):

Suppose (2) is true, i.e. $\theta_i = \frac{2(i-1)\pi}{N}$, for all $i = 1, \ldots, N$. Consider polynomial $Q(z) = z^N + (-1)^N e^{i(\theta_1 + \cdots + \theta_N)} = z^N + (-1)^N e^{i(N-1)\pi}$. Then it is trivial $Q(z)$ and $Z(U,z)$ have the same zeros $e^{i\theta_1}, \ldots, e^{i\theta_N}$. They also have the same coefficient for z^N. Therefore they are identical. So (3) is true, as desired.

Now suppose (3) is true. Let θ be a real number such that $0 \leq \theta < 2\pi$ and $\theta = \frac{\log((-1)^N a_N)}{iN}$. Then $Z(U,z) = z^N + (-1)^N e^{iN\theta}$. One can check that $e^{ik\theta}$ is a root of Z for any $k = 0, 1, \ldots, N-1$. So $e^0, e^{i\theta}, \ldots, e^{(N-1)\theta}$ are all the N roots of Z, in other words, (2) is true.

□

Proposition 3.2. *(Expansion of proposition 3.1)*
The two following statements are equivalent:

(i) All the roots of $Z'(U,z)$ are bounded by a circle centered at the origin with a small radius $r<1$.
(ii) $|a_1|,|a_2|,\ldots,|a_{N-1}|<\epsilon$*, where* $\epsilon=r^{N/2}\binom{N}{[N/2]}$.

Proof. (*Brief*) Suppose all the roots of $Z'(U,z)$ are inside a circle of radius $r<1$ centered at the origin, i.e.: $|z_1'|,|z_2'|,...,|z_{N-1}'|<r<1$. Then one can prove $|a_i|\leq\epsilon$ for $i=1,\cdots,N-1$, where ϵ is defined as follow:

$$\epsilon=r^{N/2}\binom{N}{[N/2]}. \tag{3.1}$$

Conversely, now suppose $|a_i|\leq\epsilon$ for $i=1,\cdots,N-1$ where ϵ is some positive number, then it can be proven that $|z'|<r$ for all zero z' of P', where r is defined as follow:

$$r=(\frac{(N-1)\epsilon}{2})^{\frac{1}{N-1}} \tag{3.2}$$

(For a detailed proof, see Appendix A1.) □

Let us analyze formulae (3.1) and (3.2).

In (3.1), by using Sterling's formula ($N!\approx N^Ne^{-N}\sqrt{2\pi N}$), ϵ can be approximated as follow:

$$\epsilon=r^{N/2}\binom{N}{[N/2]}\approx\frac{2\sqrt{N}}{\pi}(2\sqrt{r})^N, \tag{3.3}$$

as $N\to\infty$. If r does not exceed $1/4$, $(2\sqrt{r})$ will be less than 1, and therefore ϵ will be small.

In (3.2), when N is large, $(\frac{n-1}{2})^{1/(n-1)}$ is roughly 1, thus r is roughly $\epsilon^{1/(n-1)}$. More rigorously, $\lim_{n\to\infty}r=1$, no matter how small ϵ is. One might think that r is not a good upper bound for $|z'|$, since even when all the coefficients of $Z(U,z)$ have very small absolute values, r is no where near 0. In fact, however, it is no contradiction. $Z_0(z)=(x-e^{i\pi/10})(1+x+x^2+\cdots+x^{49})$ is a numerical example. The absolute values of the coefficients of $Z_0(z)$ are small, and all its roots are equally distributed except for the first root $z_1=e^{i\pi/10}$. (See figure below.)

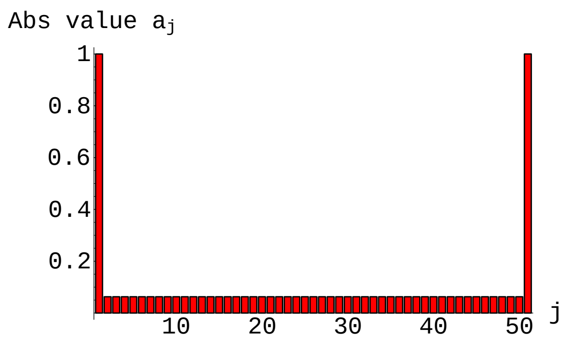

Figure 2a: $|a_j|$s of Z_0

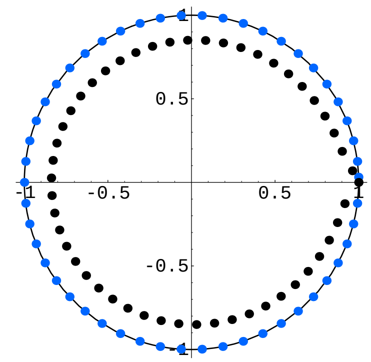

Figure 2b: Roots of $Z_0(z)$ and $Z_0'(z)$

On the other hand, from properties 3.1 and 3.2, it is natural to conjecture that (i) *"All the roots of $Z'(U,z)$ are bounded by a circle centered at the origin with radius $r < 1$"* and (ii) *"$|a_1|, \cdots, |a_{N-1}| < \epsilon$"* in property 3.2 are equivalent to a new statement: (iii) *"the roots of $Z(U,z)$ are on average nearly equally distributed."*

To make (iii) rigorous, we need a function to measure how equally distributed the roots of $Z(U,z)$ are. A candidate for such a function is:

$$\delta(\theta) = \prod_{j=2}^{N} (\theta_j - \theta_1 - \frac{(j-1)2\pi}{n})^2 \tag{3.4}$$

where $0 \le \theta_1 < \theta_2 < \cdots < \theta_N < 2\pi$ are again the eigenphases of U. If we rotate the coordinate axes so that $\theta_1 = 0$, then $\delta(\theta)$ has a simpler form:

$$\delta(\theta) = \prod_{j=2}^{N} (\theta_j - \frac{(j-1)2\pi}{n})^2 \tag{3.5}$$

So the roots of $Z(U,z)$ are on average more equally distributed on the unit circle when $\delta(\theta)$ is smaller. Then we can qualitatively state our conjecture:

Conjecture 3.3. *The following* three *statements are equivalent:*
(i) All the roots of $Z'(U,z)$ are bounded by a circle centered at the origin with a small radius $r < 1$.
(ii) $|a_1|, |a_2|, \ldots, |a_{n-1}| < \epsilon$, where ϵ is a small upper bound.
(iii) $\delta(\theta) < \epsilon'$, where ϵ' is a small upper bound.

We already know two formulas relating the magnitudes of r in (i) and ϵ in (ii) by (3.1) and (3.2). However, we have not compared the magnitude of the average value of the roots of $Z'(U,z)$ in (i) and $\delta(\theta)$. This is important in predicting the distribution of the roots of $Z'(U,z)$, but it seems to be a hard task. We will deal with some special cases of the task in the next section.

4 Equality in Distribution of zs

As mentioned in section 3.1, if the roots of $Z(U,z)$ are equally distributed on the unit circle, then all the roots of $Z'(U,z)$ will be at the origin. We are going to investigate the behavior of the roots of $Z'(U,z)$ when the roots of $Z(U,z)$ are a little bit off equal distribution.

State 1: $z_1, z_2, \ldots, z_N$ are equally distributed: $z_j = e^{i\frac{(j-1)2\pi}{N}}$, for all $j = 1, 2, \ldots, N$. Then $z'_1 = \cdots = z'_{N-1} = 0$.

State 2: z_1 is perturbed by an angle of $\Delta(\theta)$: $z_1 = e^{i\Delta(\theta)}$ (before, $z_1 = e^{i0}$). Surprisingly, the roots of $Z'(U,z)$ explode away from the origin and form nearly a circle. (It seems like having all the z'_js at the origin is a very weak "equilibrium")

State 3: z_1 is perturbed a little bit more in the same direction (counter-clockwise). $z'_1, z'_2, \ldots, z'_{N-1}$ slowed down and move sluggishly after the explosion.

State 4: z_1 is even more perturbed until it coincides with z_2. Then $Z(U,z) = (z-z_2)^2(z-z_3)\cdots(z-z_N)$. Therefore z_2 is also a root of $Z'(U,z)$.

(See figure 3 for state 1 to 4 when $N = 20$.)

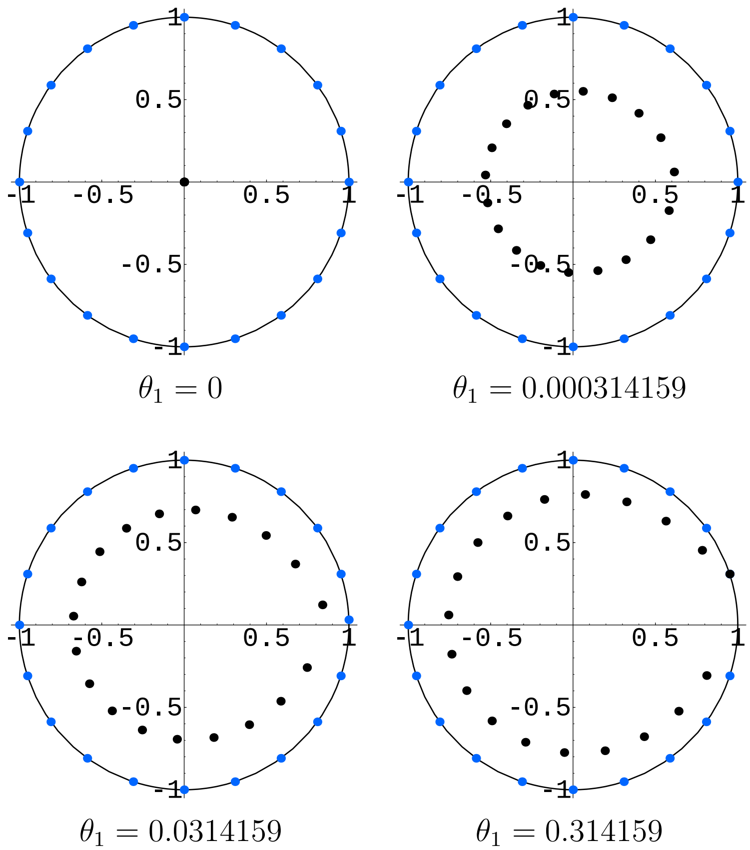

Figure 3: State 1 to 4

If we keep perturbing z_1' counter-clockwise until it comes back to its original location at State 1, z_j's will travel a long the edge of a "(N-1)-petal flower" as in figure 4. By the same reason used in State 4, it is explained that the "petals" touch the circle at the locations of $z_2, z_3, \ldots, z_N$. (See figure below.)

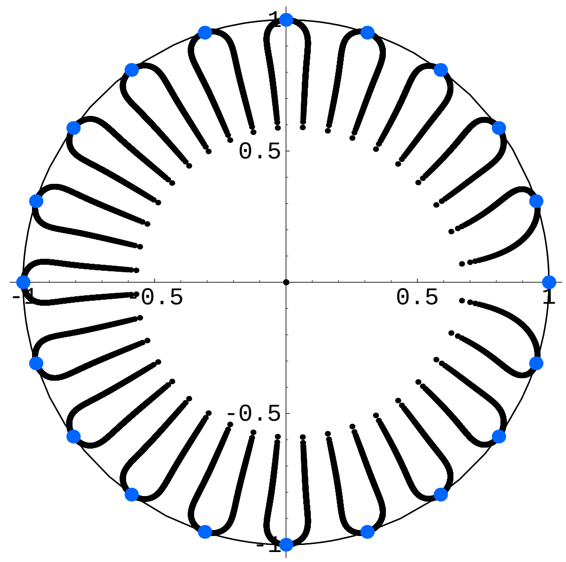

Figure 4: Traces of $z'_0, \ldots, z'_{N-1}$ as z_1 is perturbed continuously by $\Delta(\theta) = 0.5$ rad

In figure 4, the center is empty because of the explosion near the origin. The inability of the snapshots at intervals of 0.5 rad (or even 0.01) implies the z'_js explode away from the origin very fast.

The most interesting fact is that z'_js always seem to form a circle whose center is a little bit away from the origin (see figure 3). After some calculations, it turns out that if z_1 is perturbed by an angle of $\Delta(\theta_1)$, then the radius r of that "circle" is of order $\left(\frac{\Delta(\theta_1)}{N}\right)^{1/(N-1)}$. (In terms of the function $\delta(\theta)$ that measures the equality in distribution of z'_js mentioned in the previous section, r is of order $\left(\frac{\delta(\theta)}{N}\right)^{1/(N-1)}$).So when N is large, $\frac{dr}{d\theta_1}|_{\theta_1=0}$ is very large, which means that r changes very fast when $z_1 = e^{i\theta_1}$ is perturbed from its original location at $z_1 = 1$. This explains the "explosion".

The distributions of the eigenvalues of U in the states we just looked at are very special. When U is a unitary matrix randomly chosen from the Circular Unitary Ensemble, its eigenvalues are very rarely so nearly equally distributed (i.e. $\delta(\theta)$ is usually not very small). However, the "explosion" is an interesting observation and it indicates a general fact that the roots of $Z'(U, z)$ are usually not very close to the origin. As we saw in figure 1, the majority of z'_js are in the proximity of the unit circle when N is big. We are going to investigate the distribution of z_j's further in section 5.

5 Distribution of z' and The Local Maximum Conjecture

As explained in the introduction, all the roots of $Z'(U,z)$ lie inside the convex hull of the eigenvalues of $Z(U,z)$, and therefore lie inside the unit circle. One can see that *when N is large, the majority of roots of $Z'(U,z)$ concentrates in a small region in proximity of the unit circle.* Besides, as N increases, (z')s are more likely to be near the unit circle. (See figure 1.) These two observations can be explained by a fact, which will soon be heuristically proved, that the distance between the majority of (z')s and the unit circle is of size $1/N$. Therefore as N grows, we see more (z')s near the unit circle.

There is a stronger but less obvious fact saying that *the majority of roots of $Z'(U,z)$ lie in an annulus in the proximity of the unit circle.*

It is hard to observe this fact by looking at the image of many (z')s plotted together. When N is small, the annulus, if exists, is still not yet fully formed; only the inner circle can be seen. (See figure 6a and 6b.) When N is large, the majority of (z')s has a distance of size $1/N$ from the unit circle, which is very small. It is hard to see the details of their behavior from a picture like 6a or 6b when all the points are so close to the unit circle.

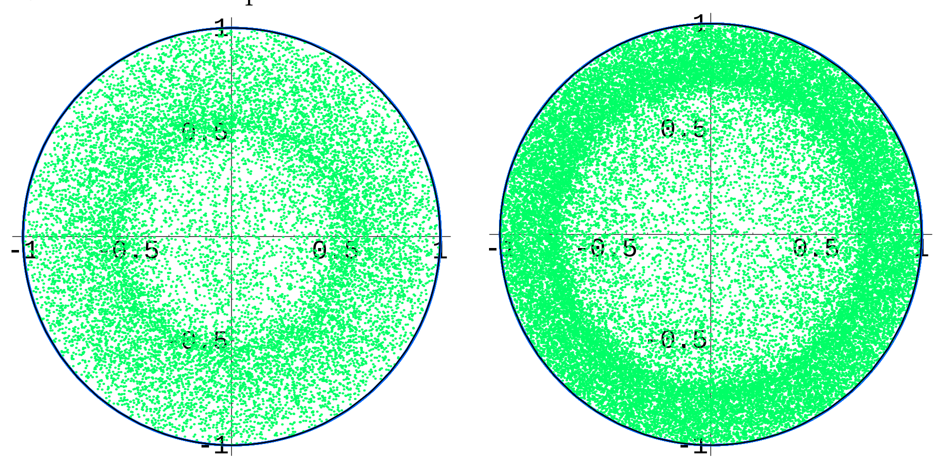

Figure 6a and 6b: Roots of $Z'(U,z)$ of 4000 5×5 and 10×10 matrices.

There is a way to avoid the problem of sizes of N by rescaling the distance from z' to the unit circle. The distance $1-|z'|$ can be rescaled by N, because as said above, $1-|z'|$ is of size $1/N$, so $N(1-|z'|)$ is of size 1. In short, $N(1-|z'|)$ is independent of N. Below are histograms of the rescaled distances $N(1-|z'|)$ in the same 4000 5×5 and 4000 10×10 random unitary matrices used in figure 6a and 6b. Figure 7a and 7b are histograms of $N(1-|z'|)$ in 50×50 and 300×300 random unitary matrices. On the horizontal axes, x stands for the rescaled distance.

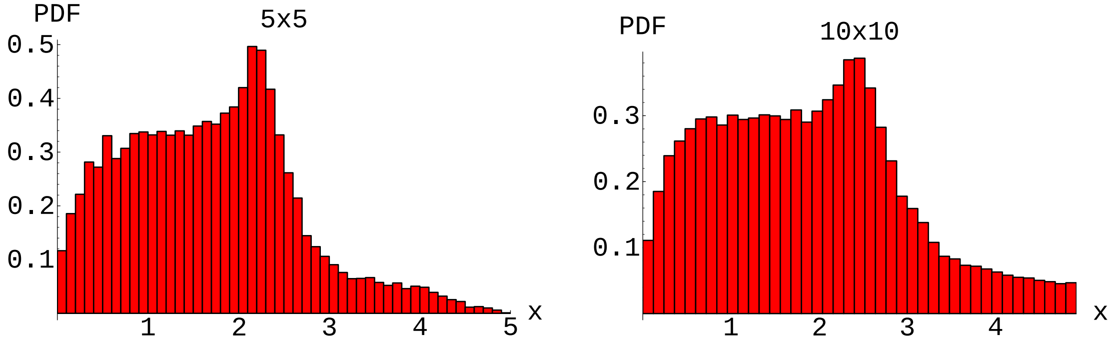

Figure 7a and 7b: Histogram of $N(1-|z'|)$ from 4000 5×5 and 10×10 matrices.

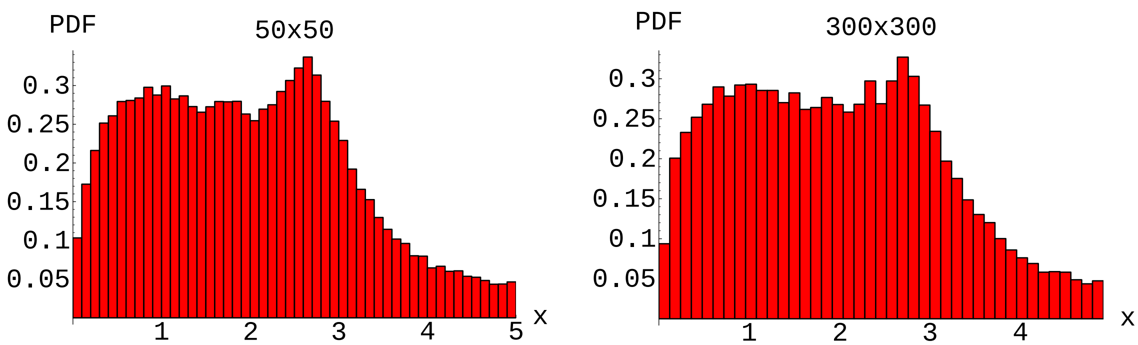

Figure 8: Histogram of $N(1-|z'|)$ from 2000 50×50 matrices and from 100 300×300 matrices.

The above histograms are composed of many thin bins, the area of each represents the fraction of all the (z')s such that the $N(1-|z'|)$ lie in the same interval as the base of the bin. Therefore the area of each bin is like $\text{PDF}(t)dt$, where $\text{PDF}(t)$ is the Probability Density Function of $N(1-|z'|)$. Therefore the histogram is a representation of the probability density function as long as the sample size (the number of generated (z')s) is large.

As seen in figure 7 and 8, when x is near 0, the PDF is small (so there are few (z')s very close to the unit circle) but increases rapidly like $\sqrt{x}$ times some constant. There are two peaks/local maxima in the PDF of the rescaled distance; the first is at roughly 1 and the second is at roughly 2.2 in figure 7a, 2.5 in 7b, 2.8 in 8a and 2.9 in 8b. (For small N, the first peak at $x \approx 1$ is not fully formed yet. For example, in figure 7a only the second peak is visible. It corresponds to the thick region of (z')s near the circle centered at the origin, with radius $1 - 1/5$, in figure 6a.)

The region between the two peaks contains most the area under the PDF curve, and therefore corresponds to an annulus, in which the majority of (z')s lie. The two peaks correspond to the inner and outer circles. The radii of these circle are $1 - x/N$, where x is the location of the corresponding peak in the histogram, which is very small compared to N. Therefore, 1 minus any of the two radii is roughly of order $1/N$. (We do not know for sure if 1 minus one of the radii is *exactly* of order $1/N$. In fact, the location x_2 of the second peak is shifting to the right at a very slow speed with respect to the change in N. x_2 can be a function of N, like $\log(\log(N))$, which grows very slowly, and can be substituted by 1 in our approximation.)

The PDF decreases rapidly as x exceeds the location x_2 of the second peak. The decrease tells us there are few (z')s at distance longer than $1-x_2/N$ from the unit circle.

Property 5.1. *For large N, the majority of the roots of $Z'(U,z)$ lies in an annulus, whose inner and outer circles are of distance of order approximately $1/N$ away from the unit circle.*

The existence of such an annulus says that there are few (z')s with very small or large rescaled distance from (z')s to the unit circle. One explanation for this behavior of (z')s stems from the following observation: *If the rescaled spacing*[1] *between two consecutive roots z_j and z_{j+1} of $Z(U,z)$ (or any polynomial with roots on the unit circle) is small, then there is usually a root of $Z'(U,z)$ near the midpoint $(z_j + z_{j+1})/2$.* In figure 5, it appears that *the distance from the midpoint has a monotonic relationship with the neighbor spacing.* The distance is longer when the spacing is larger, and the distance is smaller when the spacing is smaller.

[1] The rescale constant for the (nearest) neighbor spacing is $\frac{N}{2\pi}$. After being rescaled, the average spacing is 1, independent of N

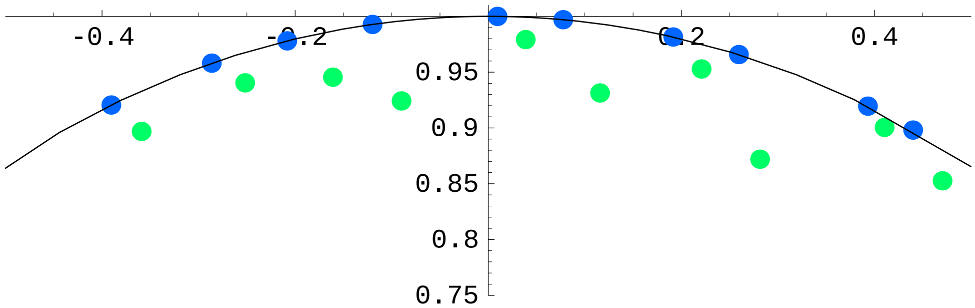

Figure 5: Roots of $Z(U, z)$ and $Z'(U, z)$ of a 60×60 unitary matrix.

The neighbor spacings of eigenphases in random unitary matrices are usually neither very small nor very big. (The distribution of the neighbor spacings between the eigenphases/eigenvalues will be studied in section 6. See Figure 9 for an illustration of the probability density function in this distribution.) Assuming the monotonic relationship between a neighbor spacing and the corresponding distance from the midpoint, then the distances from a z' to the midpoint nearest to it is neither too small nor too big.

Let d be the distance from a z' to the unit circle, t be the distance from z' to its nearest midpoint $(e^{i\theta_{j+1}} + e^{i\theta_j})/2)$, and ϵ be the distance from the midpoint to the unit circle. Then $d \approx t + \epsilon$. By simple geometry, one can see ϵ is roughly $(\theta_{j+1} - \theta_j)^2/8$, which is of size $1/N^2$. On the other hand, for (z')s near the unit circle and for large N, t is of size $1/N$ (see Appendix A2 for a heuristic proof). Thus, $d \approx t$. This also explains why d is of size $1/N$, as stated in the beginning of this section.

Since t is neither too small nor too large and $d \approx t$, d is also neither too small nor too large. In other words, there are very few (z')s extremely close to the unit circle, and also very few too far away from the unit circle. Thus, (z')s must be concentrated in an annulus near the unit circle.

The interesting thing is that the maximum concentration is at the inner and the outer circles, not at somewhere inside the annulus. The existence of the two peaks (two local maxima) instead of a single peak in the PDF of the rescaled distance is surprising.

As N increases continuously, the second peak (the one farther to the right) gradually gets shorter and sluggishly shifts to the right. These changes can be seen in figure 7 and 8. We guess this peak will vanish eventually as $N \to \infty$, leaving the graph of the PDF a nice curve with only one peak.

However, up until the time this paper was written, no one has known the truth about the behaviors of the distribution of the rescaled distance. Even though there have been good approximations of the PDF of the rescaled distance x when $x \to 0$ and $x \to \infty$,[2] but there is no such approximation when x is neither too small nor too big. The only thing we know for sure is the existence of at least one local maximum, since the PDF increases as x is small and then decreases as $x \to \infty$ and the PDF is continuous all along the real line. We conjecture that there is *only* one peak in the asymptotic case and at least two peaks in the finite case.

Conjecture 5.2. *(The Local Maximum Conjecture)*
For a finite $N > 10$, there are two major local maxima of the PDF of the rescaled distance $N(1 - |z'|)$.
The leftmost maximum is at $x \approx 1$. The rightmost maximum gets shorter and shifts to the right as N increases; it will vanish when $N \to \infty$. Accordingly, there exists only one local maximum at $x \approx 1$ when $N \to \infty$.

6 Various Distributions of the Neighbor Spacings

The distribution of (z')s is strictly connected to the distribution of the eigenvalues on the unit circle, since given N roots of $Z(U, z)$, the locations of $N - 1$ roots of $Z'(U, z)$ are defined[3]. In the previous section, we already used the fact that - any pair of consecutive eigenvalues of U do not like to be too close or too far from each other - to heuristically explain why there are very few (z')s too close or too far from the unit circle. In this section, we are going to further investigate the connection between the distribution of eigenvalues/eigenphases and the distribution of z'.

The PDF in the distribution of the rescaled neighbor spacings between the eigenphases of unitary matrices (which is $\frac{N}{2\pi}|\theta_{j+1} - \theta j|$) behaves like $t^2 e^{-t^2}$. In our random unitary matrix case, the PDF $\boldsymbol{f}(t)$ is well approximated by:

$$\rho(t) = \frac{32t^2}{\pi^2} e^{-\frac{4t^2}{\pi}}. \tag{6.1}$$

[2]An approximation of the PDF of the rescaled distance for small x is $\frac{24}{18\pi}x^{1/2} - \frac{320}{450\pi}x^{3/2} + \frac{896}{4410\pi}x^{5/2} + O(x^3)$. On the other hand, the PDF when $x \to \infty$ is $1 - 1/x$ (see [3]).

[3]Dyson electrostatic model gives a good sense of these locations as where the electric field inside the unit circle vanishes.

In [2], Mehta has introduced a power series of $\boldsymbol{f}(t)$:

$$p_2(0,t) = \frac{\pi^2 t^2}{3} - \frac{2\pi^4 t^4}{45} + \frac{\pi^6 t^6}{315} - \frac{\pi^6 t^7}{4050} - \frac{2\pi^8 t^8}{14175} + \dots .^4 \tag{6.2}$$

The following figures illustrate $\boldsymbol{f}$ and the two approximations:

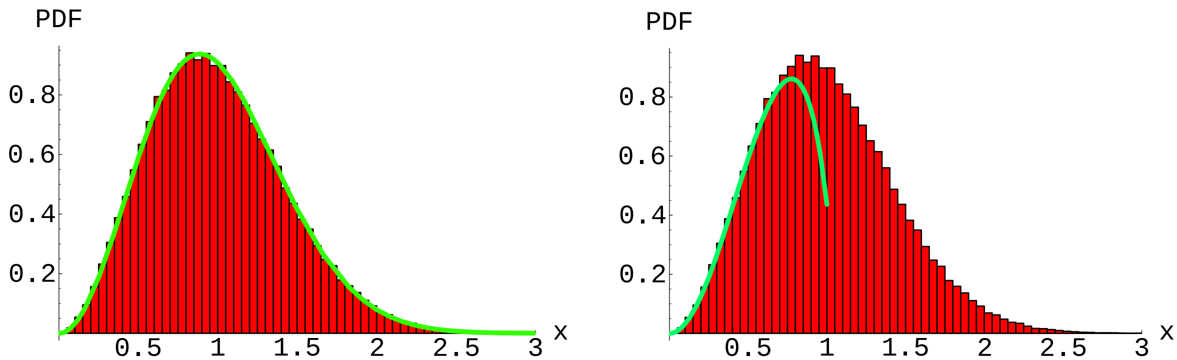

Figure 9: Histogram of rescaled gaps in 200 100 × 100 random unitary matrices and graph of (a) $\rho(t)$, (b) $p_2(0,t)$

The shape of the PDF of the neighbor spacing like a Bell curve tells us that consecutive eigenvalues of a random unitary matrix do not like to be too close or too far from each other.

In contrast, if we choose N random points on the unit circle with uniform distribution, the PDF of the rescaled neighbor spacings between the points is like e^{-x}. The distribution of the neighbor spacing in this case is called the Poisson distribution.

In this case, the neighbor spacings are likely to be small. Therefore, if the eigenvalues of U were distributed like in N random points on the unit circle, (z')s would very much be likely to be near the unit circle.

[4] $p_2(k,t)$ refers to the PDF of the rescaled difference between an eigenphase and the $(k+1)$-st nearest eigenvalue of those that are larger than it. The subscript 2 refers to the GUE.

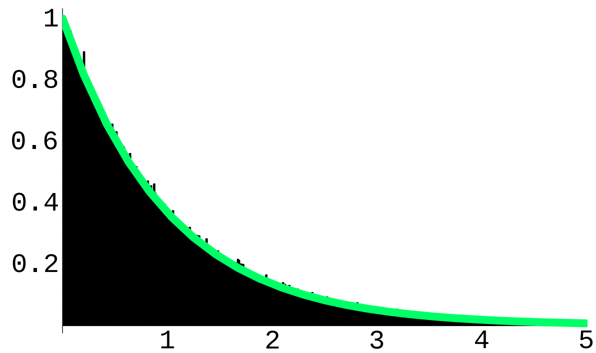

Figure 10a: Histogram of rescaled neighbor spacing in Poisson distribution ($N = 100$, 200 polynomials) is well approximated by e^{-x}

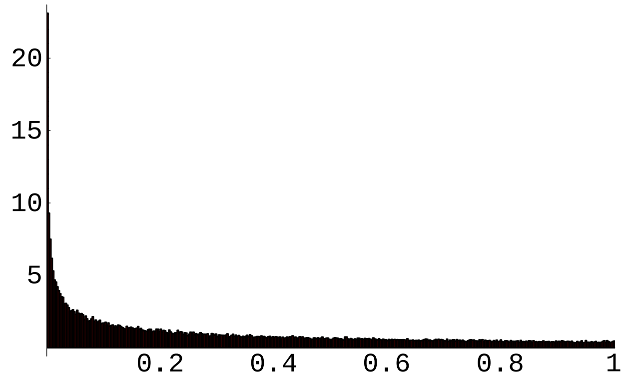

Figure 10b: Histogram of the rescaled distances from (z')s to the unit circle in Poisson distribution ($N = 70$).

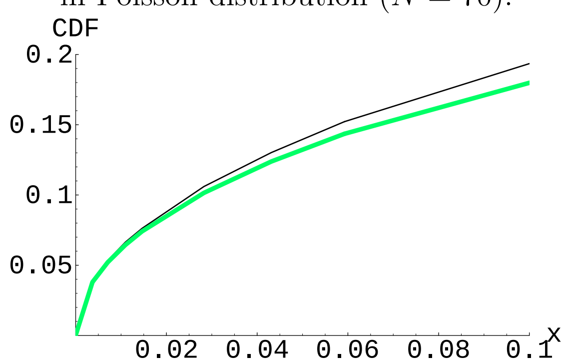

Figure 10c: Cumulative Distribution Function of the rescaled distances from (z')s to the unit circle ($N = 70$, 2000 polynomials) and approximation of $\int_0^x \boldsymbol{g}(t)dt$ using (6.3).

So the peak-behavior does not happen here. We would like to find a different distribution of the eigenvalues so that the distribution of the rescaled distances from (z')s to the unit circle is similar to that in the random unitary matrix case. If we can find such a distribution, then we can conclude that the peak-behavior is not peculiar only to random unitary matrices.

Elaborately, we want some distribution of the neighbor spacings between N points on the unit circle, such that if we construct a polynomial taking those N points as its roots, then the PDF of the rescaled distance from the roots (z')s of the derivative of that polynomial to the unit circle satisfies has one or two local maxima like those in the random unitary matrix case.

Since the PDF of the rescaled distance in random unitary matrices starts as $\sqrt{x}$, we want $\boldsymbol{g}(x)$, the PDF of the rescaled distance in the new distribution, to do so.

Let $\boldsymbol{f}(x)$ be the PDF of the rescaled neighbor spacing in the new distribution. There is a connection between $\boldsymbol{f}(x)$ and $\boldsymbol{g}(x)$ when x is small:

$$\boldsymbol{g}(x) \approx \frac{\mathbf{f}\left(\frac{2\sqrt{t}}{\pi}\right)}{\frac{4\sqrt{x}}{\pi}}. \tag{6.3}$$

(For details, see Appendix A3)

Therefore we want $\boldsymbol{f}(x) \approx cx^2$ for small x, where c is some constant. After various calculations and experiments, the following distribution was found:

Split the circle into N equal arcs. In each arc, choose a point by randomly choosing its eigenphase in the corresponding interval with 'semicircle' distribution. One way to do this is to choose $\theta_j = \frac{(2j-1)\pi}{N} + \frac{\pi}{N}w$, where w is randomly chosen from the interval $(-1, 1)$ with the probability density function of $\frac{2}{\pi}\sqrt{1-x^2}$. [5] Then the PDF of the neighbor spacings between these N points on the unit circle behaves like x^2 times some constant for small x. To generate the following histogram, $N = 100$ points were randomly chosen in the arcs for 300 times.

[5] For a computer program that generates random numbers from an interval with a given probability distribution, the reader can refer to the link provided in Appendix A4.

$N = 100$.

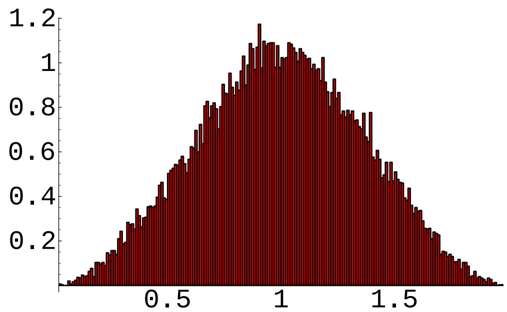

Figure 11a:
Histogram of rescaled neighbor spacings from 300 runs in 'semicircle' distribution. (i.e. $30,000$ spacings)

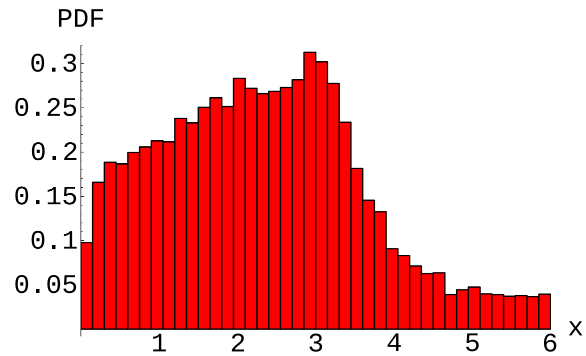

Figure 11b:
Histogram of $N(1-|z'|)$ from 225 runs in 'semicircle' distribution. (i.e. roughly $22,275$ (z')s)

As expected, the PDF of the rescaled distance from z' to the unit circle behaves like $\sqrt{x}$ as x is small. Interestingly enough, the rescaled distance distribution in this case is pretty similar to the distribution in the random unitary matrix case. The histogram in figure 11b illustrates the distribution of the rescaled distance $N(1-|z'|)$ from 225 runs with $N = 100$.

There do not appear to be two peaks in figure 11b, but it appears that when $N \to \infty$, there will be one peak that is similar to the peak in the PDF of the rescaled distance when $N \to \infty$ in the random unitary matrix case.

Stimulated by the previous distribution, a better distribution was found:

Like before, split the circle into N equal arcs. In each arc, choose a point by randomly choosing its eigenphase in the corresponding interval with 'two semicircle' distribution. One way to do this is to choose $\theta_j = \frac{(2j-1)\pi}{N} + \frac{\pi}{N} w$, where w is randomly chosen from the interval $(-1, 1)$ with the probability density function $\phi(t)$:

$$\phi(t) = \begin{cases} \frac{2}{\pi}\sqrt{1-(2x+1)^2} & \text{if } t \leq 0 \\ \frac{2}{\pi}\sqrt{1-(2x-1)^2} & \text{if } t > 0 \end{cases} \tag{6.4}$$

Again the PDF of the neighbor spacings between these N points on the unit circle behaves like x^2 times some constant for small x, but the constant is larger this time than the constant in the 'semicircle' case. Following are some histograms for $N = 100$

$$N = 100.$$

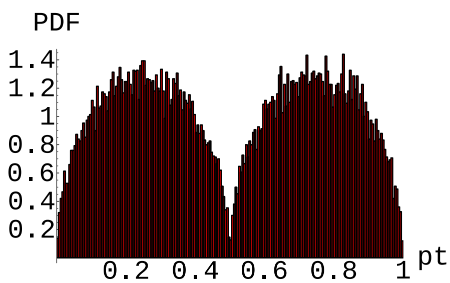

Figure 12a:
Histogram of $60,000$ random points in $(-1, 1)$ with 'two semicirce' distribution

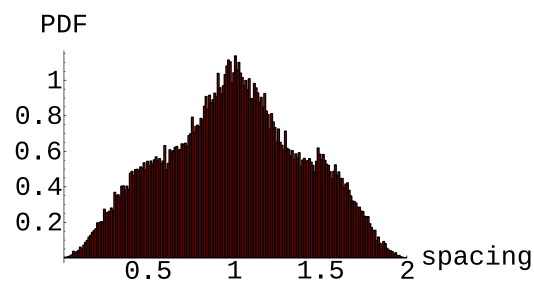

Figure 12b:
Histogram of rescaled neighbor spacing from 600 runs in 'two semicircle' distribution (i.e. $60,000$ spacings).

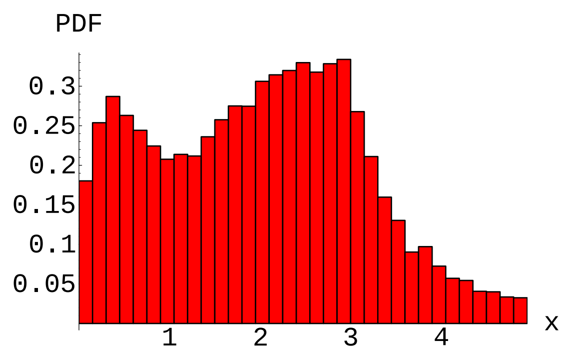

Figure 12c: Histogram of $N(1 - |z'|)$ from 300 runs in 'two semicircle' distribution (i.e. roughly $29,700$ (z')s).

As seen in figure 12c, there appear to be two local maxima in the PDF of the rescaled distance in our 'two semicircle' distribution, although the local maxima are at different locations from those in the random unitary matrix case. So the 'two peak behavior' in the finite case and the 'one peak behavior' in the asymptotic case of the probability density function of the rescaled distance $N(1 - |z'|)$ is likely to be not peculiar only to random unitary matrices.

We conclude this section by two open questions:

Question 6.1. *Is the Local Maximum Conjecture true? If it is, why is it true?*

Question 6.2. *What are the types of distribution of the roots of a polynomial, so that the roots of the derivative of the polynomial satisfy the Local Maximum Conjecture?*

7 Conclusion

In section 3 and 4, we have seen the connection among the distribution of the roots of $Z'(U, z)$, function $\delta(\theta)$ that measure how equally distributed the eigenvalues of U are, and the magnitude of the coefficients of $Z(U, z)$. In section 5, we have seen the roots of $Z'(U, z)$ concentrated in an annulus inside the unit circle. We have also conjectured the local maximum behavior of the probability density function of the rescaled distance from z' to the unit circle, for finite as well as infinite N. In section 6, we have guessed the local maximum behavior does not only appear in random unitary matrices with a Gaussian distribution of the neighbor spacings between the matrices' eigenvalues.

As discussed in the introduction, the distribution of z' – the roots of $Z'(U, z)$ – inside the unit circle is a model of the distribution of the roots of $\zeta'(s)$ on the right of the critical line $Re(s) = 1/2$. Therefore, we also expect that the roots of $\zeta'(s)$ concentrate in an annulus on the right of the critical line. Besides, the PDF of the rescaled distance from the roots of $\zeta'(s)$ to the critical line should also have two local maxima if we go up to a finite height above the x-axes, and one local maximum for infinite height. If the Local Maximum Conjecture for random unitary matrices is proven, its equivalent version for $\zeta'(s)$ will also be proven, and vice versa.

8 Appendix

A1. Proof of Proposition 3.2

Proof. Suppose all the roots of $Z'(U, z)$ are inside a circle of radius $r < 1$ centered at the origin, i.e.: $|z'_1|, |z'_2|, ..., |z'_{N-1}| < r < 1$. We have:

$$\begin{aligned} P(z) &= z^N + a_1 z^{N-1} + \cdots + a_{N-1} z + a_N \\ \Rightarrow P'(z) &= N z^{N-1} + (N-1) a_1 z^{N-2} + \cdots + a_{N-1} \end{aligned}$$

Expanding and equating coefficients yields:

$$\begin{aligned}
\sum_{|A|=k}\prod_{i\in A} z_i' &= (-1)^k \frac{N-k}{N} a_k \forall k = 1,2,\ldots,n \\
\Rightarrow \left|\sum_{|A|=k}\prod_{i\in A} z_i'\right| &= \frac{N-k}{N}|a_k|. \qquad (8.1)
\end{aligned}$$

(8.2)

The sum on the left hand side of (1.2) contains $\binom{N-1}{k}$ terms; the modulus value of each term is less than r^k. Therefore:

$$\begin{aligned}
|a_k| &\leq \binom{N-1}{k} r^k \frac{N}{N-k} \\
&= \binom{N}{k} r^k \qquad (8.3)
\end{aligned}$$

For $k \geq N/2$, both $\binom{N}{k}$ and r^k are decreasing, so the maximum occur at $k = [N/2]$, where $[N/2]$ is the largest integer not exceeding $N/2$. In other words:

$$|a_k| \leq \binom{N}{[N/2]} r^{N/2}$$

for $k = N-1, N-2, \cdots, [N/2]$.

By the palindromic symmetry of $|a_{N-1}|, \cdots, |a_1|$, we conclude that

$$|a_k| \leq \binom{N}{[N/2]} r^{N/2} \qquad (8.4)$$

for $k = 1, 2, \cdots, n-1$.

In conclusion, if $|z_i'| \leq r$ for $i = 1, \cdots, N-1$ then $|a_i| \leq \epsilon$ for $i = 1, \cdots, N-1$, where:

$$\epsilon = r^{N/2}\binom{N}{[N/2]}. \qquad (8.5)$$

Conversely, now suppose $|a_i| \leq \epsilon$ for $i = 1, \cdots, N-1$ and ϵ is some positive number. We need to prove that $|z'| \leq r$ for all (z')s, where r is some positive number smaller than 1.

Let z' be any zero of P'. Then:

$$\begin{aligned} P'(z') &= Nz'^{(N-1)} + (N-1)a_{N-1}z'^{(N-2)} + \cdots + 2a_2z' + a_1 = 0 \\ \Rightarrow |Nz'^{(N-1)}| &= |(N-1)a_{N-1}z^{N-1} + \cdots + a_1|. \end{aligned}$$

Since all the zeros of P' lie in inside the unit circle, $|z'| < 1$. So the right hand side of the previous equality is less than $(N-1)|a_{N-1}| + (N-2)|a_{N-2}| + \cdots + 1|a_1|$. In other words:

$$\begin{aligned} |Nz'^{(N-1)}| &< (N-1)|a_{N-1}| + (N-2)|a_{N-2}| + \cdots + |a_1| \\ &\leq \frac{N(N-1)}{2}\epsilon \\ \Rightarrow |z'| &< (\frac{(N-1)\epsilon}{2})^{\frac{1}{N-1}}. \end{aligned}$$

So if we choose

$$r = (\frac{(N-1)\epsilon}{2})^{\frac{1}{N-1}} \tag{8.6}$$

then $|z'| < r$ for all zero z' of P'.

□

A2. Approximating t

In section 5, we denoted t the distance from a z' to its nearest midpoint $(e^{i\theta_{j+1}} + e^{i\theta_j})/2$. We are going to sketch a proof for the fact that for z' near the unit circle and for large N, t is of size $1/N$.

Proof. Without lost of generality, we can assume the nearest midpoint to z' is $(e^{i\theta_2} + e^{i\theta_1})/2$. Also, we can assume $e^{i\theta_1}$ and $e^{i\theta_2}$ are symmetric through the real axes (since we can rotate the coordinate system.) Then $(e^{i\theta_2} + e^{i\theta_1})/2 = a$, where a is a real number.

Let $w = z' - a$, then $|w| = t$.

Let $\phi(z) = \frac{Z'(U,z)}{Z(U,z)} = \sum_{j=1}^{N} \frac{1}{z - e^{\theta_j}}$, where $z_j = e^{\theta_j}$s are the roots of $Z(U,z)$.

$\phi(z)$ has the following Taylor series:

$$\begin{aligned} \phi(z) &= \phi(a) + \phi'(a)(z-a) + \phi''(a)(z-a)^2 + \phi^{(3)}(a)(z-a)^3 + \cdots \\ &= \phi(a) + \phi'(a)w + \phi''(a)w^2 + \phi^{(3)}(a)w^3 + \cdots . \end{aligned}$$

So if z' is a root of $Z'(U,z)$ then $\phi(z') = 0$

$$\Rightarrow w = -\frac{\phi(a)}{\phi'(a)} + \frac{\phi''(a)w^2 + \phi^{(3)}(a)w^3 + \cdots}{\phi'(a)} \tag{8.7}$$

On the other hand, for very large N:

$$\begin{aligned}\phi(a) &= \sum_{j=1}^{N} \frac{1}{a - e^{\theta_j}} \\ &= \frac{1}{a - e^{i\theta_1}} + \frac{1}{a - e^{i\theta_2}} + \sum_{j=3}^{N} \frac{1}{a - e^{i\theta_j}} \\ &= \sum_{j=3}^{N} \frac{1}{a - e^{i\theta_j}} \\ &\approx N \int_{i\theta_2}^{i\theta_1 + 2\pi} \frac{1}{a - e^{i\theta}} d\theta \\ &= \frac{N}{2\pi} \frac{\pi - 2\theta_1}{a}\end{aligned}$$

Therefore:

$$\phi(a) \approx \frac{N}{2\pi} \frac{\pi - 2\theta_1}{a} \approx \frac{N}{2}$$

(The approximation signs ('$\approx$') can be replaced by the equal signs ('$=$') as $N \to \infty$.)

So $\phi(a)$ is of order $1/N$.

Without much more difficulty, one can similarly show that $\phi^{(k)}(a)$ is of order $1/N^k$.

For a given w, suppose w is of order N^{-r}, then the right hand-side of equation (8.7) is of order N^{1-2r}. Thus, $-r = 1 - 2r$. Therefore, $r = -1$ for any given w. In conclusion, $t = |w|$ is of order $1/N$. □

A3. $f(t)$ and $g(t)$

We are going to prove equation (6.3) by assuming the following conjecture:

Conjecture 8.1. *(Mezzadri) Let x be the rescaled distance from a root of $Z'(U,z)$ to the unit circle. For small x, as $N \to \infty$, the distance of the roots of $Z'(U,z)$ from the unit circle is distributed like the square of the spacing between phases of consecutive eigenvalues of unitary matrices in the Circular Unitary Ensemble (appropriately rescaled).*

The conjecture can be rephrased as: After being appropriately rescaled, *the PDF of $1 - |z'|$ is similar to the PDF of G^2*, where G is a variable representing a gap. A concrete form of this statement is:

$$PDF \;\; of \;\; \frac{N}{\pi^2/4}(1-|z'|) \approx PDF \;\; of \;\; (\frac{N}{2\pi}G)^2. \tag{8.8}$$

(In [3], Mezzadri used $N-1$ to rescale $1-|z'|$ instead of N, but the difference between the two is infinitesimal as $N \to \infty$.)

Question:

Let X be any variable and let its PDF be $f(t)$. Suppose $g(t)$ is the PDF of X^2. What is g in terms of f?

Answer:

According to the definition of PDF:

$$\mathbf{P}[a < X^2 < b] = \int_a^b g(t)dt.$$

Therefore:

$$\begin{aligned}
\int_a^b g(t)dt &= \mathbf{P}\left[\sqrt{a} < X < \sqrt{b}\right] \\
&= \int_{\sqrt{a}}^{\sqrt{b}} f(t)dt \\
&= \int_a^b f\left(\sqrt{w}\right) d\sqrt{w} \\
&= \int_a^b \frac{f\left(\sqrt{t}\right)}{2\sqrt{t}}dt.
\end{aligned}$$

Accordingly:

$$g(t) = \frac{f\left(\sqrt{t}\right)}{2\sqrt{t}}. \tag{8.9}$$

As a remind, $\boldsymbol{f}(t)$ represents the PDF of $\frac{N}{2\pi}G$ and $\boldsymbol{g}(t)$ represents the PDF of $N(1-|z'|)$. From (8.8) and (8.9) it suffices that:

$$PDF \;\; of \;\; \frac{N}{\pi^2/4}(1-|z'|) \approx \frac{\boldsymbol{f}\left(\sqrt{t}\right)}{2\sqrt{t}}. \tag{8.10}$$

Therefore:

$$\begin{aligned}
\mathbf{P}\left[1-|z'|<\frac{x}{N}\right] &= \mathbf{P}\left[\frac{N}{\pi^2/4}(1-|z'|)<\frac{x}{\pi^2/4}\right] \\
&= \int_0^{4x/\pi^2} PDF \;\; of \;\; \left(\frac{N}{\pi^2/4}(1-|z'|)\right) \; dt \\
&\approx \int_0^{4x/\pi^2} \frac{\boldsymbol{f}(\sqrt{t})}{2\sqrt{t}} dt \\
&= \int_0^{x} \frac{\boldsymbol{f}\left(\frac{2\sqrt{w}}{\pi}\right)}{\frac{4\sqrt{w}}{\pi}} dw. \\
\Rightarrow \boldsymbol{g}(t) &\approx \frac{\mathbf{f}\left(\frac{2\sqrt{t}}{\pi}\right)}{\frac{4\sqrt{t}}{\pi}}. \;\; (Q.E.D.)
\end{aligned}$$

A4. Mathematica Programs

Major Mathematica programs used for this paper are available at
`http://www.eg.bucknell.edu/~tphan/AIM/Mathematica/`

An electric copy of this paper is also available at
`http://www.eg.bucknell.edu/~tphan/AIM/ZeroDistr.pdf`

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