

Reading Classics: Euler ¹

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Abstract

We will discuss many of Euler's gems. *All notes were taken in real-time; all mistakes should be attributed to the typist, not to the lecturer.*

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Chapter 1

Perfect Numbers: Wednesday, October 8, 2003

Lecturer: Scott Arms

1.1 Definitions

Definition 1.1.1 (Proper Divisor). m is a proper divisor of n if $m|n$ and $m < n$.

Definition 1.1.2 (Perfect Number). A natural number n is perfect if and only if it is equal to the sum of all its proper divisors.

Examples are 6, 28, as well as $2^{13466916}(2^{13466917} - 1)$; the last is the largest known to date, more than 8 million digits!

Before Euler, 7 perfect numbers were known. Euler, in 1772, found the eighth perfect number, $2^{30}(2^{31} - 1)$.

1.2 Ancient Results

Theorem 1.2.1 (Euclid). If $2^k - 1$ is prime, then $2^{k-1}(2^k - 1)$ is perfect.

Proof. Let $N = 2^{k-1}(2^k - 1) = 2^{k-1}p$. Then we can write down the sum of the divisors quite easily:

$$\begin{aligned}
\sum_{\substack{d|N \\ N>d>0}} d &= (1 + 2 + \cdots + 2^{k-1}) + p(1 + 2 + \cdots + 2^{k-2}) \\
&= (2^k - 1) + p(2^{k-1} - 1) \\
&= p(2^{k-1} + 1 - 1) \\
&= p \cdot 2^{k-1} = N.
\end{aligned} \tag{1.1}$$

□

Before Euler, only the first seven perfect numbers were known. In addition to finding a new perfect number, Euler provided a characterization for even perfect numbers: now one only needs a characterization of odd perfect numbers to have a complete theory! To date, only partial results concerning odd perfect numbers are known (they must be at least so large, they must have at least so many factors, and so on).

1.3 Euler's Characterization of Even Perfect Numbers

Define a function $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ by

$$\sigma(n) = \sum_{\substack{d|N \\ d>0}} d. \tag{1.2}$$

Lemma 1.3.1. *n is perfect if and only if $\sigma(n) = 2n$.*

Lemma 1.3.2. *p is prime if and only if $\sigma(p) = p + 1$.*

Lemma 1.3.3. *If the greatest common divisor of m and n is 1 (ie, if $(m, n) = 1$), then $\sigma(mn) = \sigma(m)\sigma(n)$.*

Exercise 1.3.4. *Prove $\sigma(2^{k-1}) = 2^k - 1$.*

Theorem 1.3.5 (Euler). *If N is a perfect even number, then $N = 2^{k-1}(2^k - 1)$ for some integer $k \in \mathbb{N}$, and $2^k - 1$ is prime.*

Proof. By unique factorization, we can write $N = 2^{k-1}m$ for some odd $m \in \mathbb{N}$. Thus, $(2^{k-1}, m) = 1$, so

$$\sigma(N) = \sigma(2^{k-1})\sigma(m). \quad (1.3)$$

Further, N is perfect, so

$$\sigma(N) = 2N = 2^k m. \quad (1.4)$$

Therefore,

$$2^k m = (2^k - 1)\sigma(m). \quad (1.5)$$

Since $2^k - 1$ is odd, $2^k - 1$ divides m . Let $m = (2^k - 1)M$. Thus,

$$2^k(2^k - 1)M = (2^k - 1)\sigma(m). \quad (1.6)$$

Therefore,

$$2^k M = \sigma(m). \quad (1.7)$$

Note that $M|m$, implying

$$\begin{aligned} 2^k M &= \sigma(m) \\ &\geq m + M \\ &= (2^k - 1)M + M \\ &= M(2^k - 1 + 1) \\ &= 2^k M. \end{aligned} \quad (1.8)$$

As we have the same at the start and the end, we must have equality everywhere. Thus,

$$\sigma(m) = m + M. \quad (1.9)$$

Thus, m is prime and $M = 1$. □

Remark 1.3.6. *Where do we use that N is even, ie, where do we use that $k > 1$? If $k = 0$, impossible. If $k = 1$, then m and M are not different; if $k \geq 2$, then $M < m$.*

1.4 Odd Perfect Numbers

Regius defined perfect numbers to be even (around 1550). We have a nice characterization of even perfect numbers. What can we say about odd perfect numbers? Do they exist? No one can find any.

Suppose an odd perfect number exists. Can we say anything about the properties it must have?

Frenicle (1657) stated the following, first proved by Euler.

Theorem 1.4.1 (Frenicle-Euler). *If N is an odd perfect number, then $N = p_1^k p_2^{2j_2} \cdots p_r^{2j_r}$ for p distinct primes, $k, j \in \mathbb{N}$, and $p_1 \equiv k \equiv 1 \pmod{4}$.*

Exercise 1.4.2. *Note this implies $N \equiv 1 \pmod{4}$.*

Proof.

$$N = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}. \quad (1.10)$$

N perfect if and only if $\sigma(N) = 2N$. Since the numbers are mutually prime, we obtain

$$\prod_{i=1}^r \sigma(p_i^{e_i}) = 2N. \quad (1.11)$$

So $\sigma(N) \equiv 2 \pmod{4}$, thus at least one σ -term is even. If *two* were even, would have wrong congruence. There is thus a unique i_0 such that $\sigma(p_{i_0}^{e_{i_0}}) \equiv 2 \pmod{4}$. Without loss of generality, let $i_0 = 1$.

Suppose $p_i \equiv 3 \equiv -1 \pmod{4}$. Then

$$\begin{aligned} \sigma(p_i^{e_i}) &= 1 + p_i + p_i^2 + \cdots + p_i^{e_i} \\ &\equiv 1 + (-1) + (-1)^2 + \cdots + (-1)^{e_i} \pmod{4} \\ &\equiv \begin{cases} 0 \pmod{4} & \text{if } e_i \text{ odd} \\ 1 \pmod{4} & \text{if } e_i \text{ even} \end{cases} \end{aligned} \quad (1.12)$$

So p_1 must be $1 \pmod{4}$, and if $p_i \equiv 3 \pmod{4}$, e_i must be even.

If $p_i \equiv 1 \pmod{4}$, then

$$\begin{aligned} \sigma(p_i^{e_i}) &\equiv \sum_{j=0}^{e_i} 1 \\ &\equiv e_i + 1 \pmod{4}. \end{aligned} \quad (1.13)$$

Since $p_1 \equiv 1 \pmod{4}$, we must have $e_1 \equiv 1 \pmod{4}$. For $i > 1$, if $p_i \equiv 1 \pmod{4}$, then e_i is even. This is exactly what we needed, namely, all exponents but the first are even, and the first exponent is $1 \pmod{4}$. □

1.5 Touchard

Theorem 1.5.1 (Touchard 1953). *If N is an odd perfect number, N must be of the form $12m + 1$ or $36m + 9$.*

We will give Holdener's proof from 2002.

Lemma 1.5.2. *N cannot have the form $6m - 1$.*

Proof. Suppose $N = 6m - 1$. Then $N \equiv -1 \pmod{3}$. For any divisor d of N , we have

$$d \cdot \frac{N}{d} = N \equiv -1 \pmod{3}. \quad (1.14)$$

So, either $d \equiv 1 \pmod{3}$ and $\frac{N}{d} \equiv -1 \pmod{3}$ or $d \equiv -1 \pmod{3}$ and $\frac{N}{d} \equiv 1 \pmod{3}$.

Thus,

$$\sigma(N) = \sum_{\substack{d|N \\ 0 < d < \sqrt{N}}} \left(d + \frac{N}{d} \right) \equiv 0 \pmod{3}. \quad (1.15)$$

However,

$$\begin{aligned} \sigma(N) &= 2N \\ &= 2(6m - 1) \\ &= 12m - 2 \\ &\equiv -2 \pmod{3} \\ &\equiv 1 \pmod{3}, \end{aligned} \quad (1.16)$$

and we have a contradiction. □

We now look modulo 6, and prove the theorem.

Proof. By Lemma 1.5.2, N cannot be of the form $6m - 1$. Therefore, $N = 6m + 1$ or $6m + 3$. Hence, N is congruent to either 1 or 3 mod 6. But from Theorem 1.4.1, we know N is congruent to 1 mod 4. Therefore, either N is congruent to 1 mod 4 and mod 6, or N is congruent to 1 mod 4 and 3 mod 6.

Solving these simultaneously yields N has the form $12m + 1$ or $12m + 9$.

We're halfway there, just need to improve the $12m + 9$ case a bit. Assume N is of the form $12m + 9$ and $3 \nmid m$. Then

$$\begin{aligned}\sigma(N) &= \sigma(3(4m + 3)) \\ &= \sigma(3)\sigma(4m + 3) \\ &= 4\sigma(4m + 3) \\ &\equiv 0 \pmod{4}.\end{aligned}\tag{1.17}$$

Therefore, we have a contradiction as $\sigma(N) \equiv 1 \pmod{4}$. (Note: we could skip the above lines by referring to an earlier result).

□

1.6 Bibliography

The reference of Euler/Frenicle came from: Dickson. The proof presented is from: <http://www-maths.swan.ac.uk/pgrads/bb/project/node44.html#tex2html39> This site is also the source of the references of Steuerwald (1937), Kanold (1941), and Hagsis, McDaniel (1972).

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Chapter 2

Euler and Geometry: Wednesday, October 15, 2003

Lecturer: John Christopherson

2.1 Heron's Formula

Theorem 2.1.1 (Heron). *For a triangle Δ with sides of lengths a , b and c , the area of the triangle is*

$$\text{Area}(\Delta) = \sqrt{S(S-a)(S-b)(S-c)}, \quad (2.1)$$

where S is the semi-perimeter

$$S = \frac{a+b+c}{2}. \quad (2.2)$$

This theorem was known in classical times, and Euler provided new proofs.

Exercise 2.1.2. *What about the generalization to a tetrahedron?*

Exercise 2.1.3. *What about a more general polygon in the plane? Open problem if there are sufficiently many sides.*

2.2 Geometry Terms

Definition 2.2.1 (Orthocenter). Consider a triangle with vertices A , B and C . Construct the perpendicular bisectors to each side (the altitudes). The three lines meet in a common point, called the orthocenter.

Remark 2.2.2. Of course, implicit in the above definition is that the three altitudes do meet in a point.

Definition 2.2.3 (Centroid). Intersection of the three medians (lines from vertex to midpoint of opposite side).

Definition 2.2.4 (Circumcenter). The Circumcenter is the center of the circle which passes through the three vertices of the triangle.

Definition 2.2.5 (Incenter). The Incenter is the center of the circle which is tangent to the three sides.

Remark 2.2.6. Take midpoints of three sides, gives us six vectors, two emanating from each vertex. Replace every two vectors at a vertex by their sum (the resultant) going from vertex to opposite side. No rotation, must meet at a point. Assume have a balanced triangle, homogenous material, balancing on a point. Can replace forces with sums. If don't sum to zero, have a net force.

2.3 Euler's Line

Theorem 2.3.1 (Euler's Line). The centroid, orthocenter, and circumcenter meet in a point.

Euler's line has a lot of significance. *Triangle Centers and Central Triangles* indexes a lot of points that are important in triangles, and there are many *special* points that are also on Euler's line.

Remark 2.3.2. The Incenter need not be on Euler's line.

Remark 2.3.3. Euler's Line does not generalize to Hyperbolic Geometry. See Euler's Line in Hyperbolic Geometry, Jeffrey Klaus.

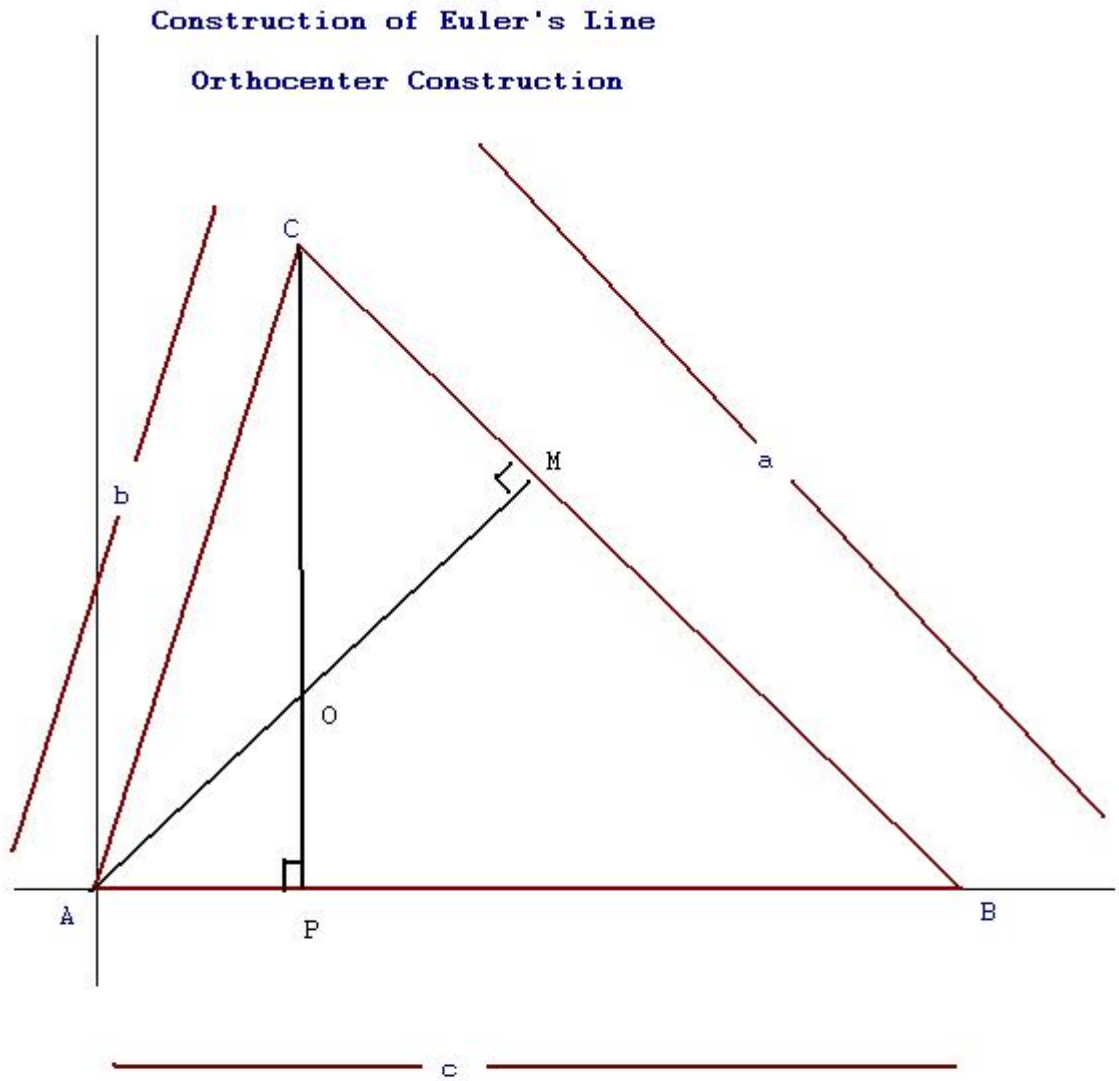
Remark 2.3.4. Altitudes of a Tetrahedron and Traceless Quadratic Forms, in the *American Mathematical Monthly* (October 2003), by Hans Havlicek and Gunter Weiß, talk about generalizations.

2.4 Euler's Proof of Euler's Line

Euler proceeds by brute force, calculating the coordinates of the three special points.

Without loss of generality, let A be at $(0, 0)$, let B lie on the x -axis, and let C be in the first quadrant; it is an easy exercise to show that any triangle may be taken in this form.

2.4.1 Orthocenter



Let P be the intersection of the altitude to AB , let M be the intersection of the altitude to BC , and let O be the intersection of the two lines. Let the sides of the original triangle be a , b and c (side of length c is opposite vertex C). Euler uses the Law of Cosines:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos(A) \\ &= b^2 + c^2 - 2bc \frac{\overline{AP}}{b}, \end{aligned} \quad (2.3)$$

which implies that

$$\overline{AP} = \frac{b^2 + c^2 - a^2}{2c}. \quad (2.4)$$

Proceeding similarly, one obtains that

$$\overline{BM} = \frac{a^2 + c^2 - b^2}{2a}. \quad (2.5)$$

Let

$$K = \frac{1}{2} \overline{AM} a, \quad (2.6)$$

thus

$$\overline{AM} = \frac{2K}{a}. \quad (2.7)$$

We have $\triangle AMB \simeq \triangle APO$, which yields

$$\frac{\overline{OP}}{\overline{AP}} = \frac{\overline{AM}}{\overline{BM}}, \quad (2.8)$$

or

$$\overline{OP} = \frac{\overline{AM} \cdot \overline{AP}}{\overline{BM}}. \quad (2.9)$$

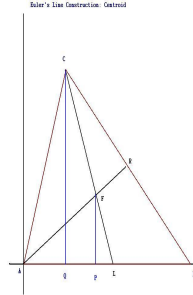
Substituting everything gives

$$\overline{OP} = \frac{(b^2 + c^2 - a^2)(a^2 + c^2 - b^2)}{8cK}, \quad (2.10)$$

which gives the coordinates of O as

$$O = \left(\frac{b^2 + c^2 - a^2}{2c}, \frac{(b^2 + c^2 - a^2)(a^2 + c^2 - b^2)}{8cK} \right). \quad (2.11)$$

2.4.2 Centroid



As these are the medians, they bisect the line. Let the bisector hit \overline{AB} at L , let another hit \overline{BC} at R , and let the two lines meet at F . Now drop a perpendicular from F to \overline{AB} , hitting at P . Drop another perpendicular from C to \overline{AB} , hitting at Q .

Clearly we have $\overline{AL} = \frac{c}{2}$, and $\triangle LFP \simeq \triangle LFP$. This yields

$$\frac{\overline{LE}}{\overline{LC}} = \frac{\overline{LP}}{\overline{LQ}} = \frac{1}{3}. \quad (2.12)$$

Therefore, we find

$$\begin{aligned} \overline{PL} &= \frac{1}{3}\overline{QL} \\ &= \frac{1}{3}(\overline{AL} - \overline{AQ}) \\ &= \frac{1}{3}\left(\frac{c}{2} - \frac{b^2 + c^2 - a^2}{2c}\right). \end{aligned} \quad (2.13)$$

Thus,

$$\begin{aligned} \overline{AP} &= \overline{AL} - \overline{PL} \\ &= \frac{c}{2} - \frac{1}{3}\left(\frac{c}{2} - \frac{b^2 + c^2 - a^2}{2c}\right) \\ &= \frac{3c^2 + b^2 - a^2}{6c}. \end{aligned} \quad (2.14)$$

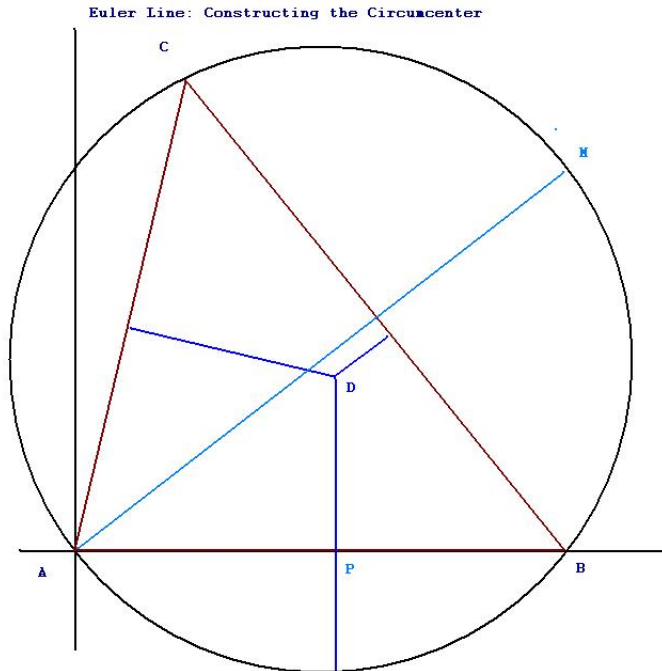
We find

$$\overline{PF} = f \overline{otCA} = \frac{1}{3} \frac{2K}{c} = \frac{2K}{3c}. \quad (2.15)$$

Or, in other words,

$$F = \left(\frac{3c^2 + b^2 - a^2}{bc}, \frac{2K}{3c} \right). \quad (2.16)$$

2.4.3 Circumcenter



Draw the circle with center at the circumcenter C . Draw the altitude from A to \overline{BC} , and extend till it hits the circle at M . Draw the perpendicular lines from the circumcenter C to the three sides of the triangle.

We find

$$\frac{\overline{DP}}{\overline{AP}} = \frac{\overline{CM}}{\overline{AM}}, \quad (2.17)$$

where

$$\overline{CM} = \frac{a^2 + b^2 - c^2}{2a} \quad \text{and} \quad \overline{AM} = \frac{2K}{a}. \quad (2.18)$$

Substituting yields

$$\overline{DP} = \frac{c}{2} \cdot \frac{a^2 + b^2 - c^2}{2a} \cdot \frac{a}{2K} = \frac{c(a^2 + b^2 - c^2)}{8K}. \quad (2.19)$$

We have now found the coordinates of D :

$$D = \left(\frac{c}{2}, \frac{c(a^2 + b^2 - c^2)}{8K} \right). \quad (2.20)$$

2.4.4 Completing the proof

Now that we have the three coordinates of the three special points, we compute and compute and compute.

2.5 A Vector Approach to Euler's Line

From *A Vector Approach to Euler's Line of a Triangle*, by J. Ferrer.

Chapter 3

Euler and Infinite Series:

Wednesday, October 22, 2003

Lecturer: Bill Mance.

We'll mention $\zeta(2) = \frac{\pi^2}{6}$, as well as some generalizations. The handout is from *An Introduction to the Theory of Numbers* (I. Niven, H. Zuckerman, H. Montgomery, fifth edition). Another source is *Pi: A source Book*, by L. Berggren, J. Borwein, P. Borwein.

3.1 Power Series Review

We have the following power series expansions:

$$\begin{aligned}\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots .\end{aligned}\tag{3.1}$$

Theorem 3.1.1 (Viète (1500s)). Let $f(x) = x^N + c_{N-1}x^{N-1} + \cdots + c_0$ with roots $\alpha_1, \dots, \alpha_N$. Then

$$\begin{aligned}
\sum_i \alpha_i &= -c_{N-1} \\
\sum_{i>j} \alpha_i \alpha_j &= c_{N-2} \\
&\vdots \\
\prod_{i=1}^N \alpha_i &= (-1)^N c_0.
\end{aligned} \tag{3.2}$$

Note: set of algebraic complex numbers is a field.

3.2 Evaluating $\zeta(2)$

We will prove

Theorem 3.2.1 (Euler). $\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Proof. Define

$$p(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots \tag{3.3}$$

Note the above equals $\frac{\sin x}{x}$ for all $x \in \mathbb{C} - \{0\}$.

Now, $p(x) = 0$ if and only if $x = k\pi$, $k \in \mathbb{Z}$. Implicit in this assumption is that there are no complex zeros.

Thus, assume we can write $p(x)$ as an infinite product:

$$p(x) = \left(1 - \frac{x}{\pi}\right) \left(1 + \frac{x}{\pi}\right) \left(1 - \frac{x}{2\pi}\right) \left(1 + \frac{x}{2\pi}\right) \dots \tag{3.4}$$

Remark 3.2.2. *It is very important to take the factors in this order, as the above is conditionally convergent, and gives a little better decay. This can be formally justified using Weierstrass products.*

Combining in pairs yields

$$\begin{aligned}
p(x) &= \left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{2^2\pi^2}\right) \left(1 - \frac{x^2}{3^2\pi^2}\right) \dots \\
&= 1 - x^2 \left(\frac{1}{\pi^2} + \frac{1}{2^2\pi^2} + \frac{1}{3^2\pi^2} + \dots\right) + x^4(\dots) + \dots \tag{3.5}
\end{aligned}$$

Equating coefficients, and remembering the expansion of $p(x) = \frac{\sin x}{x}$, we find

$$\frac{1}{3!} = \frac{1}{\pi^2} + \frac{1}{2^2\pi^2} + \frac{1}{3^2\pi^2} + \dots, \quad (3.6)$$

which gives

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \quad (3.7)$$

□

3.3 Generalizations

Consider a power series

$$p(z) = 1 + Az + Bz^2 + Cz^3 + Dz^4 + \dots. \quad (3.8)$$

If the roots of $p(z)$ are $\alpha, \beta, \gamma, \delta, \dots$, then we have

$$\begin{aligned} p(z) &= (1 + \alpha z)(1 + \beta z)(1 + \gamma z)(1 + \delta z) \dots \\ A &= \alpha + \beta + \gamma + \delta + \dots \\ B &= \sum \text{two at a time} \\ C &= \sum \text{three at a time,} \end{aligned} \quad (3.9)$$

and so on. We need certain properties to make all the above convergence (for example, the roots must have certain size properties).

Define

$$\begin{aligned} P &= \alpha + \beta + \gamma + \dots \\ Q &= \alpha^2 + \beta^2 + \gamma^2 + \dots \\ &\vdots \\ V &= \alpha^6 + \beta^6 + \gamma^6 + \dots. \end{aligned} \quad (3.10)$$

Then

$$\begin{aligned}
P &= A \\
Q &= AP - 2B \\
R &= AQ - BP + 3C \\
&\vdots \\
V &= AT - BS + CR - DQ + EP - GF.
\end{aligned} \tag{3.11}$$

In the finite case, these are due to Newton.

We have

$$\alpha = \frac{1}{\pi^2}, \quad \beta = \frac{1}{2^2\pi^2}, \quad \dots \tag{3.12}$$

and

$$Q = \frac{1}{3!} \frac{1}{3!} - 2 \cdot \frac{1}{5!}. \tag{3.13}$$

Therefore,

$$\frac{1}{\pi^4} \left(\frac{1}{1^4} + \frac{1}{2^4} + \dots \right) = \frac{1}{90}, \tag{3.14}$$

which gives

$$\sum_{n \geq 1} \frac{1}{n^4} = \frac{\pi^4}{90}. \tag{3.15}$$

Consider

$$\begin{aligned}
&\cos(u/2) + \tan(g/2) \sin(v/2) \\
&= \left(1 + \frac{v}{\pi - q}\right) \left(1 - \frac{V}{\pi + q}\right) \left(1 + \frac{v}{3\pi - g}\right) \left(1 - \frac{v}{3\pi + g}\right)
\end{aligned} \tag{3.16}$$

Substitute $v = \frac{x}{n}\pi$ and $g = \frac{m}{n}\pi$. Then

$$\begin{aligned}
&\cos(x\pi/2n) + \tan(m\pi/n) \sin(x\pi/2n) \\
&= \left(1 + \frac{x}{n - m}\right) \left(1 - \frac{x}{n - m}\right) \left(1 + \frac{x}{3n - m}\right) \left(1 - \frac{x}{3n - m}\right)
\end{aligned} \tag{3.17}$$

Let $K = \tan(m\pi/n)$. Then, noting there are no K 's in the even terms,

$$1 + \frac{\pi x}{2n}K - \frac{\pi^2 x^2}{2^2 n^2 2!} - \frac{\pi^3 x^3}{2^3 N^3 3!}K^3 + \dots = \text{other side.} \quad (3.18)$$

Collecting gives

$$\frac{1}{n-m} - \frac{1}{n+m} + \frac{1}{3n-m} - \frac{1}{3n+m} + \dots = \frac{\pi}{2n}K. \quad (3.19)$$

Similarly

$$\frac{1}{(n-m)^2} + \frac{1}{(n+m)^2} + \frac{1}{(3n-m)^2} + \frac{1}{(3n+m)^2} + \dots = \frac{K^2+1}{4N^2}\pi^2 \quad (3.20)$$

and

$$\frac{1}{(n-m)^3} - \frac{1}{(n+m)^3} + \frac{1}{(3n-m)^3} - \frac{1}{(3n+m)^3} + \dots = \frac{(K^3+K)}{8n^3}\pi^3. \quad (3.21)$$

In general, we get something of the form $f(k)g(n)\pi^{\text{power}}$, where f and g are nice functions. If $m, n \in \mathbb{Z}$, then K is algebraic:

$$K = K \frac{\sin(m\pi/2n)}{\cos(m\pi/2n)} = \frac{\text{algebraic}}{\text{algebraic}}. \quad (3.22)$$

Thus, all the above sums are transcendental when m and n are integers, as the algebraic numbers are closed under these operations (algebraic numbers are a field).

3.4 More Rational Multiples of π

Consider

$$\cos(v/2) + \cot(g/2) \sin(v/2). \quad (3.23)$$

Making the same substitutions as before,

$$\begin{aligned}
& 1 + \frac{\pi x}{2nK} - \frac{\pi^2 x^2}{2^2 N^2 2!} - \frac{\pi^3 x^3}{2^3 N^3 3! K} + \dots \\
= & \left(1 + \frac{x}{m}\right) \left(1 - \frac{x}{2n-m}\right) \left(1 + \frac{x}{2n+m}\right) \left(1 - \frac{x}{4n-m}\right) \left(1 + \frac{x}{4n+m}\right) \dots
\end{aligned} \tag{3.24}$$

Therefore, as before we get

$$\begin{aligned}
\frac{1}{m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \dots &= \frac{\pi}{2nK} \\
\frac{1}{m^2} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \dots &= \frac{(K^2+1)\pi^2}{4n^2 K^2} \\
\frac{1}{m^3} - \frac{1}{(2n-m)^3} + \frac{1}{(2n+m)^3} + \dots &= \frac{(K^2+1)\pi^4}{8n^3 K^3},
\end{aligned} \tag{3.25}$$

and so on.

Taking $m = 1$, $n = 2$ gives $K = 1$, and we get

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \dots = \frac{\pi}{4} \tag{3.26}$$

and

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \dots = \frac{\pi^3}{32}. \tag{3.27}$$

Catalan's constant is

$$\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \dots, \tag{3.28}$$

and we don't even know if it is irrational, let alone transcendental!

One can show

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}. \tag{3.29}$$

To see this, let $S = \sum_{n \geq 1} \frac{1}{n^2}$. Then $\frac{1}{2}S = \sum_{n \geq 1} \frac{1}{(2n)^2}$. Then $S - 2 \cdot \frac{1}{4}S$, and noting $S = \frac{\pi^2}{6}$, solves the above.

Note many of these are a rational number times π .

Euler conjectured that $\zeta(3) = \alpha(\log 2)^3 + \beta \log 2$, with α, β probably rational.

This is no longer believed to be true.

3.5 Irrational Multiples of π

Continuing as before, let $m = 1$, $n = 3$, and use the last relations with these values. This implies that $K = \tan(\pi/6) = \frac{1}{\sqrt{3}}$. Therefore, the first relation will yield

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{10} + \dots = \frac{\pi}{6\sqrt{3}}. \quad (3.30)$$

Thus, here we have an *irrational* multiple of π , which is a lot harder to detect.

3.6 Adding Series from Both

Adding series from both expansions gives us (the most recent one and the cotangent one) gives

$$\frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \dots = \frac{K\pi}{2n} + \frac{\pi}{2n\pi}. \quad (3.31)$$

Unfortunately, the above are not absolutely convergent! Now, if you truncate each series, then one has finitely many terms, and the above can be justified (a bit).

Chapter 4

Euler and Sums of Four Squares: Wednesday, October 29, 2003

Lecturer: Brinkmeier.

4.1 History

Starts with Bachet in 1621. We will show that a prime number is the sum of two squares if and only if $p = 4k + 1$ (Bachet first claimed this).

In 1685, Fermat claims to have a proof of the above, but again, no proof is given.

In the 1740s, Euler becomes interested in this problem, which he proves in 1747. Then in 1770 Lagrange, using the ideas of Euler, finally shows that any number can be written as the sum of four squares.

4.2 Sums of Two Squares

Theorem 4.2.1. *A prime p is the sum of two squares if and only if $p = 4k + 1$.*

Lemma 4.2.2 (Leonardo of Pisa, 1202). *If x and y are sums of two squares, then so is xy .*

Proof. Say

$$x = a^2 + b^2 \quad \text{and} \quad y = c^2 + d^2. \tag{4.1}$$

Then

$$(a^2 + b^2)(c^2 + d^2) = (ad + bc)^2 + (ac - bd)^2. \quad (4.2)$$

Can interpret the above in terms of complex numbers. □

We will use the method of infinite descent: if you have one solution, we show that there is a strictly smaller solution (in some sense); this process cannot be continued indefinitely with integers.

Lemma 4.2.3. *For any prime $p = 4k + 1$, there exist $m, z \in \mathbb{Z}$ such that $mp = z^2 + 1$.*

Proof. From Wilson's Theorem, we know $(p - 1)! \equiv -1 \pmod{p}$. Thus, in our case, $(4k)! \equiv -1 \pmod{p}$. Then

$$\begin{aligned} 2k + 1 &\equiv -2k \\ 2k + 2 &\equiv -2k - 1 \\ &\vdots \\ 4k - 1 &\equiv -2 \\ 4k &\equiv -1. \end{aligned} \quad (4.3)$$

Therefore, we have $[(2k)!]^2 \equiv -1 \pmod{p}$. In the above calculations, there will be an even number of minus signs coming out. □

Now that we know $mp = z^2 + 1$, we can find (straightforward calculation) that $-\frac{1}{2}p < z < \frac{1}{2}p$ (with possibly a different m , but z in this range). This implies

$$m = \frac{z^2 + 1}{p} < \frac{\frac{1}{4}p^2 + 1}{p} < p. \quad (4.4)$$

For $mp = x^2 + y^2$, $p = 4k + 1$ we can find u and v such that $u \equiv x \pmod{m}$ and $v \equiv y \pmod{m}$. We may choose u and v so that $-\frac{1}{2}m \leq u, v \leq \frac{1}{2}m$.

This implies $u^2 + v^2 \equiv 0 \pmod{m}$, which yields that there exists an r such that $mr = u^2 + v^2$. If $r \neq 0$, then $u = v = 0$.

Hence, if $m > 1$, there exists an $r < m$ such that $rp = x^2 + y^2$, so $p = \tilde{x}^2 + \tilde{y}^2$. Note we had

$$(mr) \cdot (mp) = (u^2 + v^2) \cdot (x^2 + y^2). \quad (4.5)$$

By Fibonacci's identity, this is also a sum of two squares, say $(xu + yv)^2 + (xv - yu)^2$. Thus, we get $rp = \tilde{x}^2 + \tilde{y}^2$.

4.3 The Representation is (basically) Unique

Say $p = a^2 + b^2 = x^2 + y^2$, and we have that the congruence $z^2 + 1 \equiv 0 \pmod{p}$ has two solutions, say $\pm h$.

Since p is prime and a, b non-zero, a^{-1} and b^{-1} exist (the inverses are multiplicative inverses mod p). Then, mod p ,

$$\begin{aligned} 0 &\equiv a^2 + b^2 \\ &\equiv a^2(b^{-1})^2 + b^2(b^{-1})^2 \\ &\equiv (ab^{-1})^2 + 1 \\ &\equiv 0. \end{aligned} \quad (4.6)$$

Thus, $ab^{-1} \equiv \pm h \pmod{p}$; relabel h if necessary so that $a \equiv hb \pmod{p}$.
Now

$$\begin{aligned} p^2 &\equiv (a^2 + b^2) \cdot (x^2 + y^2) \\ &\equiv (ax + by)^2 + (ay - bx)^2. \end{aligned} \quad (4.7)$$

Using our result that $a \equiv hb \pmod{p}$, we can find a similar statement concerning x and y , and we find that one of the two factors above is congruent to zero mod p . Let's assume that $ay - bx \equiv 0 \pmod{p}$, or $p \mid (ax + by)$.

Dividing by p^2 above (be very careful doing such divisions mod p), we find

$$1 \equiv \left[\frac{ax + by}{p} + \frac{ay - bx}{p} \right]^2. \quad (4.8)$$

Suppose $ax + by = 0$. As a and b are relatively prime (as their squares sum to the prime p), we find a divides by , so a divides y . Similarly, one can find that b divides x , and we can interchange the rolls of (a, b) and (x, y) . We find that the only solutions are of the form $(\pm x, \pm y)$ or $(\pm y, \pm x)$.

Exercise 4.3.1. Show that we are correct above when we state that $ay - bx \equiv 0 \pmod p$.

4.4 Sums of Four Squares

We have seen that primes of the form $4k + 3$ cannot be the sum of two squares (look at what squares are congruent to mod 4).

Exercise 4.4.1. Show that 7 and 15 cannot be written as the sum of three squares.

So, three squares is not enough to get all numbers. In 1750, Euler discovered the following identity

$$\begin{aligned} & (a_1^2 + a_2^2 + a_3^2 + a_4^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2) \\ = & (a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4)^2 + (a_1b_2 + a_2b_1 + a_3b_4 - a_4b_2)^2 \\ & + (a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2)^2 + (a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1)^2. \end{aligned} \quad (4.9)$$

Thus, if x and y are the sum of four squares, so is their product!

Remark 4.4.2. Modern day proofs of this use

$$\begin{pmatrix} z & w \\ \bar{w} & \bar{z} \end{pmatrix}. \quad (4.10)$$

The determinant above is the sum of four squares....

Noting that $2 = 1^2 + 1^2 + 0^2 + 0^2$ and primes p of the form $4k + 1$ can be written $a^2 + b^2 + 0^2 + 0$, we see it is sufficient to write odd primes of the form $4k + 3$ as the sum of four squares.

So, we must show that there exists an m such that $0 < m < p$ and $mp = a^2 + b^2 + c^2 + d^2$. We will do this by descent. To show such an m exists, it is enough to show that $x^2 + y^2 + 1 \equiv 0 \pmod p$ is solvable.

We rewrite as $x^2 + 1 \equiv -y^2 \pmod p$. Clearly, mod p , y^2 is a perfect square. As $p \equiv -1 \pmod 4$, -1 is *not* a perfect square (this follows from $(-1)^{\frac{(4k+3)-1}{2}} \equiv -1$).

We introduce the Legendre symbol $\left(\frac{a}{p}\right)$. If $a \equiv 0 \pmod p$, $\left(\frac{a}{p}\right) = 0$. Otherwise, we have

$$\left(\frac{a}{p}\right) = \begin{cases} -1 & \text{if } a \text{ is not a square mod } p \\ +1 & \text{if } a \text{ is a non-zero square mod } p \end{cases} \quad (4.11)$$

We find $\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right)$.

Thus, rewriting $x^2 + y^2 + 1 \equiv 0 \pmod{p}$ gives $x^2 + 1 \equiv (-1)(y^2) \pmod{p}$, implying that $x^2 + 1$ is *not* a square mod p .

So, $x^2 + 1$ cannot be a square. so, we want to find a perfect square which, when we add 1, is not a square. Look at the list of numbers 1, 2, 3, and so on. At least one such number will work. There are only $\frac{p-1}{2}$ squares, the same number of non-squares, and 0.

We find

$$mp = a^2 + b^2 + c^2 + d^2, \quad m < p. \quad (4.12)$$

Let $A \equiv a \pmod{m}$, $B \equiv b \pmod{m}$, $C \equiv c \pmod{m}$, and $D \equiv d \pmod{m}$. We may take $-\frac{1}{2}m \leq A, B, C, D \leq \frac{1}{2}m$. So there exists an r such that

$$mr = A^2 + B^2 + C^2 + D^2. \quad (4.13)$$

If $r = 0$, each is 0, so p is divisible by m , which contradicts the primality of p . Thus, $r > 0$. Therefore

$$\begin{aligned} r &= \frac{A^2 + B^2 + C^2 + D^2}{m} \\ &\leq \frac{\frac{1}{4}m^2 + \frac{1}{4}m^2 + \frac{1}{4}m^2 + \frac{1}{4}m^2}{m} = m. \end{aligned} \quad (4.14)$$

If $r = m$, then all these terms have to achieve a maximum of $\frac{1}{2}m$, which implies that $a = b = c = d = \frac{1}{2}m \pmod{m}$. Thus, $mp = 0 \pmod{m^2}$, so $r < m$.

We have

$$\begin{aligned} mr &= A^2 + B^2 + C^2 + D^2 \\ mp &= a^2 + b^2 + c^2 + d^2. \end{aligned} \quad (4.15)$$

Multiplying out (using a slightly different version of Euler's identity for sums of four squares) gives

$$m^2pr = w^2 + x^2 + y^2 + z^2 \quad (4.16)$$

where

$$w = aA + bB + cC + dD \equiv a^2 + b^2 + c^2 + d^2 \equiv 0 \pmod{m}. \quad (4.17)$$

Likewise, we find $x, y, z \equiv 0 \pmod{m}$. Thus,

$$pr = \tilde{w}^2 + \tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2. \quad (4.18)$$

The descent is following in this form: we are probably assuming $m > 1$, and if $m > 1$, then we can find a smaller r .

4.5 Later Years

Waring (1770) conjectured that every n is the sum of 9 cubes. Wieferich and Kempner (1912) proved that Waring's conjecture is true. Hilbert (1909) states that for any N , there is a natural number $g(N)$ such that every n is the sum of at most $g(N)$ N^{th} powers: $n = \sum_{i=1}^{g(N)} x_i^N$. Chen (1986) showed that $g(5) = 37$, and others in 1986 showed that $g(4) = 19$.

For 9 cubes, 257 is the largest number that needs 9 cubes. A natural problem is from some finite point on, how many terms does one need for each N ? For example, $g(4)$ is 16 (give or take).

Exercise 4.5.1. *Is every positive integer the sum of a finite number of squares?*

Exercise 4.5.2. *Same as above, but can one show that bounded number of squares work?*

Chapter 5

Euler and Graph Theory: Friday, November 5, 2003

Lecturer: Corey

5.1 Graph Theory Review

A graph $G = (V, E)$ is a set of vertices V (the vertex set) and a set E of pairings of vertices.

If $v \in V$, the degree of v is the number of edges leaving v .

If $v \in V$, then e is incident to v . If v is in two edges e_1 and e_2 , then e_1 is adjacent to e_2 .

A path P in G is a sequence of edges $\{e_i\}_{i \leq n}$ such that e_i is adjacent to e_{i+1} .

The vertex sequence of P is the sequence $\{v_i\}$ such that e_i is incident to v_i .

If $i \neq j$ implies $e_i \neq e_j$, then the path P is simple.

If $v_n = v_1$ (the last vertex is the same as the first), then P is closed.

If P is simple and closed, then P is a circuit.

5.2 Bridges of Königsberg

Question: can you walk around town, crossing each bridge exactly once, ending up where you started?

Two islands in a river, [] — — — — [], river flows around the two islands, two bridges from each side of the first island to the opposite shores; on the second island, one bridge from each side to the other banks.

Gives rise to a graph: four vertices, say 1, 2, 3, 4.

An **Euler Circuit** is a closed path in G (the graph) that uses each edge exactly once.

Theorem 5.2.1. *If an Euler Circuit exists in G , then all vertices have even degree.*

Proof. Start at a vertex in the Euler Circuit: every time you come to another vertex, you contribute two (once coming in, once leaving). In the end, when you have the last edge, since it is a closed path, you end up at the original vertex, which now gives everything having an even degree. \square

Theorem 5.2.2 (Euler). *If G has all even degree vertices, then there exists an Eulerian Circuit.*

Proof. We proceed by induction on the number of vertices. Assume you have at least two vertices (otherwise trivial). The case of two vertices is trivial.

We proceed by strong induction. Take *any* path such that you end back where you started. Such a path exists as all vertices have even degree. Start at a v_1 and walk. Every time you hit a new vertex, you leave it; thus, you decrease their degrees by an even number each time. Eventually, as there are only finitely many vertices, you must return to where you started. Why? Each vertex has even degree, so when you come in and leave, you decrease its degree by 2. If you haven't returned yet, then this vertex is no longer available if you've used up all its edges. As we keep decreasing the number of edges, eventually it will work.

Then, by strong induction, we can find Euler Circuits for each connected component of G minus the path we've just constructed. Then we just piece those pieces to the original path. \square

Remark 5.2.3. *We count a self-loop as two edges; a self-loop is an edge from v to v .*

5.3 Fleury's Algorithm

We describe Fleury's Algorithm to construct an Eulerian Circuit. The input is a finite connected graph G with all vertices of even degree.

Step One: Start at any vertex v . Let $VS = \{v\}$, and ES the empty sequence; VS stands for the Vertex Set, ES stands for the Edge Set.

Step Two: While there are edges incident with v :

If there is no edge incident with v , stop.

If there is exactly one edge, say $e = \{v, w\}$. Add the edge e to ES , add w to VS , delete v from the graph, and now move on and consider w .

If there is more than one, choose one edge such that its removal does not disconnect what is left; we claim that there is always such an edge.

5.4 Euler Characteristic

Definition 5.4.1. *A polyhedron is a three-dimensional figure whose faces are polygons, fitting together well.*

Definition 5.4.2. *A polygon is a simple closed curve with straight sides, non-intersecting, divides the plane into two sets.*

We often want to deal with convex figures. For example, we'll deal with a polyhedron as the convex hull of a set of points in the plane.

Let F be the number of faces of the polyhedron, let E be the number of edges, and let V be the number of vertices.

Theorem 5.4.3 (Euler). *For any convex polyhedron,*

$$F - E + V = 2. \quad (5.1)$$

This was known to Descartes (1639); Euler rediscovered this (1751). We give Cauchy's proof (1811).

Proof. Take a face-off. We haven't removed any edges or vertices – we've just removed the interior of a polygon. There is now a whole, and we have something topologically equivalent to a cell. We've decreased F by 1, and everything else unchanged.

We now flatten everything out, and triangulate. Now that we have something flat, we just need to show $F - E + V = 1$. Then start removing triangles and boundaries. See what happens in each case. \square

Consider polyhedra with regular faces, and the same number of edges meeting at each vertex. Let a be the number of edges on each face, let b be the number of edges meeting at each vertex. One finds $aF = 2E$, and $bV = 2E$. As $F - E + V = 2$, a little algebra yields

$$\frac{1}{a} - \frac{1}{2} + \frac{1}{b} = \frac{1}{E}. \quad (5.2)$$

There are only so many answers: the five answers are the Platonic solids.

Chapter 6

Wednesday, November 12, 2003

Lecturer: Dan File

6.1 History of the Fundamental Theorem of Algebra

First was d'Alembert (1746) – he was interested in integrating rational function. A consequence of the FToA is that any polynomial can be separated into linear and quadratic terms, so to integrate $\frac{f(x)}{g(x)}$, we can succeed using partial fractions.

Euler became interested in this problem: Euler worked on the quartic and quintic. For the quartic, Euler showed that there was an x-intercept. He was relying on the fact that if you have roots β_i ($i \in \{1, 2, 3, 4\}$), then $-(\beta_1 \cdots \beta_4)^2$ is negative. This is fine if the β s are real or in complex conjugate pairs, but had some trouble with the quintic.

Nicolas Bernoulli claimed a certain quartic was irreducible over \mathbb{R} , but Euler found a factorization:

$$x^4 - 4x^3 + 2x^2 + 4x + 4 = (x^2 - \sqrt{2 \pm \sqrt{4 + 2\sqrt{7}}})x + (1 \pm \sqrt{4 + 2\sqrt{7} + \sqrt{7}}), \quad (6.1)$$

where above the two factors come from taking the + sign each time, or the – sign each time. Note factoring a quartic into two real quadratics is different than trying to find four complex roots.

A function f is *analytic* on an open subset $R \subset \mathbb{C}$ if f is complex differentiable everywhere on R ; f is *entire* if it is analytic on all of \mathbb{C} .

6.2 Proof of the Fundamental Theorem via Liouville

Theorem 6.2.1 (Liouville). *If $f(z)$ is analytic and bounded in the complex plane, then $f(z)$ is constant.*

We now prove

Theorem 6.2.2 (Fundamental Theorem of Algebra). *Let $p(z)$ be a polynomial with complex coefficients of degree n . Then $p(z)$ has n roots.*

Proof. It is sufficient to show any $p(z)$ has one root, for by division we can then write $p(z) = (z - z_0)g(z)$, with g of lower degree.

Note that if

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad (6.2)$$

then as $|z| \rightarrow \infty$, $|p(z)| \rightarrow \infty$. This follows as

$$p(z) = z^n \cdot \left| a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right|. \quad (6.3)$$

Assume $p(z)$ is non-zero everywhere. Then look at $\frac{1}{p(z)}$, with $|z| = R$. Since $P(z) \neq 0$ for all z , we find $\frac{1}{p(z)}$ is bounded (look at $|z|$ small and large separately). Thus, $\frac{1}{p(z)}$ is a bounded, entire function, which must be constant. Thus, $p(z)$ is constant, a contradiction which implies $p(z)$ must have a zero (our assumption). \square

6.3 Proof of the Fundamental Theorem via Rouché

Theorem 6.3.1 (Rouché). *If f and h are each analytic functions inside and on a domain C with bounding curve ∂C , and $|h(z)| < |f(z)|$ on ∂C , then f and $f + h$ have the same number of zeros in C .*

We now prove the Fundamental Theorem of Algebra:

Proof. Let

$$\begin{aligned}
p(z) &= a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \\
f(z) &= a_n z^n \\
h(z) &= a_{n-1} z^{n-1} + \cdots + a_0.
\end{aligned} \tag{6.4}$$

Take

$$R > \frac{|a_{n-1}| + \cdots + |a_0|}{|a_n|}. \tag{6.5}$$

Then $|h(z)| < |f(z)|$ on the boundary of the circle centered at the origin of radius R . As clearly f has n zeros, we are done. \square

6.4 Proof of the Fundamental Theorem via Picard's Theorem

This proof is due to Boas (1935).

Theorem 6.4.1. *If there are two points missed in the image of an entire function $p(z)$ (ie, $\exists z_1 \neq z_2$ such that for all $z \in \mathbb{C}$, $p(z) \neq z_1$ or z_2), then $p(z)$ is constant.*

We now prove the Fundamental Theorem of Algebra:

Let $p(z)$ be a non-constant polynomial missing two points. Without loss of generality, we may assume $p(z)$ is never 0.

Claim 6.4.2. *If $p(z)$ is as above, $p(z)$ does not take on one of the values $\frac{1}{k}$ for $k \in \mathbb{N}$.*

Proof. Assume not; thus, $\exists z_k \in \mathbb{C}$ such that $p(z_k) = \frac{1}{k}$. If we take a circle centered at the origin with sufficiently large radius, then $|P(z)| > 1$ for all z outside some circle D . Thus, each $z_i \in D$. By Bolzano-Weierstrass, as all the points $z_k \in D$, we have a convergent subsequence. Thus, we have $z_{n_i} \rightarrow z'$. But

$$p(z') = \lim_{n_i \rightarrow \infty} p(z_{n_i}) = 0. \tag{6.6}$$

Thus, there must be some k such that $p(z) \neq \frac{1}{k}$. As $p(z)$ misses two values, by Picard it is now constant. This contradicts our assumption that $p(z)$ is non-constant. Therefore, $p(z_0) = 0$ for some z_0 . \square

Remark 6.4.3. *One can use a finite or countable version of Picard. Rather than missing just two points, we can modify the above to work if Picard instead stated that if we miss finitely many (or even countably many) points, we are constant. Just look at the method above, gives $\frac{1}{k_1}$. We can then find another larger one, say $\frac{1}{k_2}$. And so on. We can even get uncountably many such points by looking at numbers such as $\frac{\pi}{k}$ (using now the transcendence of \mathbb{C} is 1).*

6.5 Proof of the Fundamental Theorem via Cauchy's Integral Theorem

This proof is due to Boas (1964).

Theorem 6.5.1 (Cauchy Integral Theorem). *Let $f(z)$ be analytic inside on on the boundary of some region C . Then*

$$\int_{\partial C} f(z) dz = 0. \quad (6.7)$$

We now prove the Fundamental Theorem of Algebra:

Proof. Let $p(z)$ be a non-constant polynomial and assume $p(z) = 0$. For $z \in \mathbb{R}$, assume $p(z) \in \mathbb{R}$; in other words, we are assuming $p(z)$ has real coefficients.

Without loss of generality, $p(z)$ doesn't change signs for $z \in \mathbb{R}$, or by the Intermediate Value Theorem it would have a zero.

$$\int_0^{2\pi} \frac{d\theta}{p(2 \cos \theta)} = \neq 0. \quad (6.8)$$

This follows from our assumption that $p(z)$ is of constant sign for real arguments, bounded above from 0. We also have

$$\frac{1}{i} \int_{|z|=1} \frac{dz}{z p(z + z^{-1})} = \frac{1}{i} \int_{|z|=1} \frac{z^{n-1}}{Q(z)}, \quad (6.9)$$

where

$$Q(z) = z^n P(z + z^{-1}). \quad (6.10)$$

If $z \neq 0$, $Q(z) \neq 0$.

If $z = 0$, then

$$\begin{aligned}
p(z + z^{-1}) &= a_n(z + z^{-1})^n + \dots \\
z^n p(z + z^{-1}) &= z^n (\dots a_n z^{-n}) + \dots \\
&= a_n + z(\dots).
\end{aligned} \tag{6.11}$$

Thus, $Q(z) = a_n$, which is non-zero. □

Remark 6.5.2. *If $p(z)$ doesn't have real coefficients, then consider $g(z) = p(z)\overline{p(z)}$. By differentiating, one can pick off the coefficients.*

6.6 Proof of the Fundamental Theorem via Maximum Modulus Principle

This proof is due to C. Fefferman (1967).

Theorem 6.6.1 (Maximum (Minimum) Modulus Principle). *No entire function attains its maximum in the interior.*

We now prove the Fundamental Theorem of Algebra:

Proof. Assume $p(z)$ is non-constant and never zero. $\exists M$ such that $|p(z)| \geq |a_0| \neq 0$ if $|z| > M$. Let z_0 be the value in the circle of radius M where $p(z)$ takes its minimum value. All we are using is a continuous function on a closed, bounded domain attains its maximum (minimum).

But $|p(z_0)| \leq |p(0)| = |a_0|$. Therefore, $|p(z_0)| \leq |p(z)|$ for all $z \in \mathbb{C}$.

Translate the polynomial. Let $p(z) = p((z - z_0) + z_0)$; let $p(z) = Q(z - z_0)$. Note the minimum of Q occurs at $z = 0$: $|Q(0)| \leq |Q(z)|$ for all $z \in \mathbb{C}$.

$$Q(z) = c_0 + c_j z^j + \dots + c_n z^n, \tag{6.12}$$

where j is such that c_j is the first coefficient (after c_0) that is non-zero. Note if $c_0 = 0$, we are done.

We may rewrite such that

$$Q(z) = c_0 + c_j z^j + z^{j+1} R(z). \tag{6.13}$$

We will extract roots. Let

$$re^{i\theta} = -\frac{c_0}{c_j}. \quad (6.14)$$

Further, let

$$z_1 = r^{\frac{1}{j}} e^{\frac{i\theta}{j}}. \quad (6.15)$$

Let $\epsilon > 0$ be a small real number. Then

$$\begin{aligned} Q(\epsilon z_1) &= c_0 + c_j \epsilon^j z_1^j + \epsilon^{j+1} z_1^{j+1} R(\epsilon z_1) \\ |Q(\epsilon z_1)| &\leq |c_0 + c_j \epsilon^j z_1^j| + \epsilon^{j+1} |z_1|^{j+1} |R(\epsilon z_1)| \\ &< |c_0| - \epsilon^j |c_0| + \epsilon^{j+1} |z_1|^{j+1} N, \end{aligned} \quad (6.16)$$

where N is chosen such that $N > |R(\epsilon z_1)|$. Thus,

$$|Q(\epsilon z_1)| < |c_0|, \quad (6.17)$$

but this was supposed to be our minimum. Thus, a contradiction! \square

6.7 Proof of the Fundamental Theorem via Radius of Convergence

The proof below is from Velleman (1997).

We now prove the Fundamental Theorem of Algebra: As always, $p(z)$ is a non-constant polynomial. Consider

$$f(z) = \frac{1}{p(z)} = b_0 + b_1 z + \cdots, \quad (6.18)$$

and

$$p(z) = a_n z^n + \cdots + a_0, \quad a_0 \neq 0. \quad (6.19)$$

Lemma 6.7.1. $\exists c, r \in \mathbb{C}$ such that $|b_k| > cr^k$ for infinitely many k .

Now, $1 = p(z)f(z)$. Thus, $a_0 b_0 = 1$. This is our basis step. Assume we have some coefficient such that $|b_k| > cr^k$. We claim we can always find another. Suppose there are no more. Then the coefficient of z^{k+n} in $p(z)f(z)$ is

$$a_0 b_{k+n} + a_1 b_{k+n-1} + \cdots + a_n b_k = 0. \quad (6.20)$$

Thus, as we have $|b_j| > cr^j$ in this range, we have the coefficient satisfies

$$|a_0| r^n + |a_1| r^{n-1} + \cdots + |a_{n-1}| r \leq |a_n| \quad (6.21)$$

if

$$r \leq \min\left\{1, \frac{|a_n|}{|a_0| + \cdots + |a_{n-1}|}\right\}. \quad (6.22)$$

This will give that

$$\begin{aligned} |b_k| &= \frac{|a_0 b_{k+n} + \cdots + a_{n-1} b_{k+1}|}{|a_n|} \\ &\leq \frac{|a_0 b_{k+n}| + \cdots + |a_{n-1} b_{k+1}|}{|a_n|} \leq cr^k \end{aligned} \quad (6.23)$$

for sufficiently small.

Let $z = \frac{1}{r}$. Then

$$|b_k z^k| = \frac{|b_k|}{r^k} > c. \quad (6.24)$$

Chapter 7

Wednesday, November 19, 2003

Lecturer: Rafal Pikula

7.1 An Interesting Sum

Theorem 7.1.1.

$$\sum_{m,n \geq 2} \frac{1}{m^n - 1} = 1. \quad (7.1)$$

Proof. Let

$$x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \quad (7.2)$$

By the Geometric Series Formula,

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \quad (7.3)$$

Therefore, subtracting yields

$$x - 1 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{9} + \dots \quad (7.4)$$

Similarly, we know

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots \quad (7.5)$$

Subtracting again gives

$$x - 1 - \frac{1}{2} = 1 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \dots \quad (7.6)$$

Again by the Geometric Series,

$$\frac{1}{4} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \dots \quad (7.7)$$

Subtracting again gives

$$x - 1 - \frac{1}{2} - \frac{1}{2} = 1 + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \dots \quad (7.8)$$

Continuing in this manner gives

$$x - 1 - \frac{1}{2} - \frac{1}{5} - \frac{1}{6} - \frac{1}{9} - \dots = 1. \quad (7.9)$$

This implies

$$x - 1 = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{5} + \frac{1}{6} + \frac{1}{9} + \dots \quad (7.10)$$

Note the RHS's denominators are all numbers with denominators *not* of the form $m^n - 1$ with $m, n \geq 2$. Subtracting $x - 1$ (Equation 7.10) from the expansion of x gives

$$1 = \frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \dots \quad (7.11)$$

Of course, we are subtracting divergent series.... □

Remark 7.1.2. *You need to be careful in trying to add convergent factors, to make the series convergent.*

7.2 Summation Methods

Consider

$$\sum_{n=0}^{\infty} a_n. \quad (7.12)$$

We can consider the Power Series Method: Assume

$$\sum_{n=0}^{\infty} a_n x^n \quad (7.13)$$

converges for small $|x|$; assume for such x , the above converges to $f(x)$. If the function is regular in the region (open, connected) that contains the origin and the point $x = 1$, then we define

$$\sum_{n=0}^{\infty} a_n = f(1). \quad (7.14)$$

We call convergence of sums of this type \mathcal{E} -convergence.

Another type of summation is to again consider the power series $\sum a_n x^n$. Let $x = \frac{y}{1-y}$, so $y = \frac{x}{x+1}$. Note $y = \frac{1}{2}$ corresponds to $x = 1$.

Assume $\sum a_n x^n$ converges for small $|x|$. Then

$$\begin{aligned} x f(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= \sum_{n=0}^{\infty} a_n \frac{y^{n+1}}{(1-y)^{n+1}} \\ &= \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} \binom{n+m}{m} y^{n+m+1} \\ &= \sum_{n=0}^{\infty} a_n \sum_{k=n}^{\infty} \binom{k}{k-n} y^{k+1} \\ &= \sum_{k=0}^{\infty} y^{k+1} \sum_{n=0}^k \binom{k}{k-n} a_n \\ &= \sum_{k=0}^{\infty} b_k y^{k+1}, \quad b_k = \sum_{n=0}^k \binom{k}{n} a_n. \end{aligned} \quad (7.15)$$

If the above converges for $y = \frac{1}{2}$, say to $h(y)$, then we define

$$\sum_{n=0}^{\infty} a_n = h\left(\frac{1}{2}\right). \quad (7.16)$$

We call this $(E, 1)$ -summation. To evaluate at $y = \frac{1}{2}$, we must study

$$\sum_{n=0}^{\infty} \frac{b_n}{2^{n+1}}. \quad (7.17)$$

7.3 Examples

$$1 - 1 + 1 - 1 + 1 - 1 + \dots \quad (7.18)$$

Using \mathcal{E} -Summation:

$$\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}. \quad (7.19)$$

Thus, as $f(1) = \frac{1}{2}$, we obtain

$$\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{2}. \quad (7.20)$$

Now let us use $(E, 1)$ -Summation. We need to determine b_n . We find $b_0 = a_0 = 1$, and

$$b_n = \sum_{k=0}^n \binom{n}{k} (-1)^k = 0 \quad \text{if } n \geq 1. \quad (7.21)$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{b_n}{2^{n+1}} = \frac{1}{2}. \quad (7.22)$$

Let's consider

$$1 - 2 + 4 - 8 + \dots \quad (7.23)$$

By \mathcal{E} -summation,

$$\sum_{n=0}^{\infty} (-1)^n (2x)^n = \frac{1}{1+2x}. \quad (7.24)$$

As $f(1) = \frac{1}{3}$, we get this sum is $\frac{1}{3}$.

Using $(E, 1)$ -summation, we get $b_0 = a_0 = 1$, $b_1 = -1$, and in general

$$b_n = \sum_{k=0}^n \binom{n}{k} (-2)^k = (1 - 2)^n = (-1)^n. \quad (7.25)$$

Thus,

$$1 - 2 + 4 - 8 + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} = \frac{1}{3}, \quad (7.26)$$

as we have a geometric series.

7.4 Another Example

Consider

$$1 + 2 + 4 + 8 + \dots \quad (7.27)$$

By \mathcal{E} -Summation, we get

$$\sum_{n=0}^{\infty} 2^n x^n = \frac{1}{1 - 2}. \quad (7.28)$$

Thus, as $f(1) = -1$, our initial sum is -1 .

Using $(E, 1)$ -Summation, we get

$$b_n = \sum_{k=0}^n \binom{n}{k} 2^k = 3^n. \quad (7.29)$$

Thus,

$$\sum_{n=0}^{\infty} \frac{b_n}{2^{n+1}} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n = \infty. \quad (7.30)$$

Remark 7.4.1. In the above, if we use \mathcal{E} -Summation to handle the $\left(\frac{3}{2}\right)^n$, we get -2 , which regains the -1 from before.

Let's try using $(E, 1)$ -Summation on $\left(\frac{3}{2}\right)^n$. We now get

$$c_n = \sum_{k=0}^n \binom{n}{k} \frac{3^k}{2^{k+1}} = \frac{1}{2} \left(\frac{5}{2}\right)^n. \quad (7.31)$$

We then substitute, and get $\sum \frac{c_n}{2^{n+1}}$, and see we have things like $\left(\frac{5}{4}\right)^n$. If we apply the Geometric Series now, we get -1 again.

7.5 Related Sums and Values

Consider

$$a_0 + a_1 + a_2 + a_3 + \cdots + \quad (7.32)$$

and

$$a_0 + 0 + 0 + a_2 + 0 + 0 + 0 + a_3 + \cdots \quad (7.33)$$

While this will not change the value of the sum if it converges, if we are using the new convergence methods on divergent series, we will get new results.

Specifically, consider

$$1 - 1 + 1 - 1 + \cdots \quad (7.34)$$

We showed the above is $\frac{1}{2}$. Now look at

$$1 - 1 + 0 + 0 + 1 - 1 + 0 + 0 + 1 - 1 + 0 + 0 + \cdots \quad (7.35)$$

Now we have something like

$$1 - x + x^4 - x^5 + x^8 - x^9 + \cdots \quad (7.36)$$

which gives

$$(1 - x)(1 + x^4 + x^8 + \cdots) = \frac{1 - x}{1 - x^4} = \frac{1}{(1 + x)(1 + x^2)} = f_1(x). \quad (7.37)$$

As $f_1(1) = \frac{1}{4}$, we've obtained a new value!

7.6 Another Example

$$1 - 1! + 2! - 3! + \dots \quad (7.38)$$

Let

$$f(x) = 1 - 1!x + 2!x^2 - 3!x^3 + \dots \quad (7.39)$$

Then $f(1)$ is the value of our original sum. Let $g(x) = xf(x)$. We have

$$\begin{aligned} g'(x) &= f(x) - xf'(x) \\ &= 1 - 2!x + 3!x^2 - 4!x^3 + \dots \end{aligned} \quad (7.40)$$

Consider the combination

$$\begin{aligned} x^2g'(x) + g(x) &= x^2(1 - 2!x + 3!x^2 - \dots) + x - x^2(1! - 2!x + 3!x^2 - \dots) \\ &= x. \end{aligned} \quad (7.41)$$

This gives the differential equation

$$x^2g'(x) + g(x) = x. \quad (7.42)$$

We have

$$(g(x)e^{-1/x})' = \frac{1}{x} \cdot e^{-1/x}, \quad (7.43)$$

which yields

$$g(x) = e^{1/x} \int_0^x \frac{e^{-1/t}}{t} dt. \quad (7.44)$$

The integral is well-defined for x positive. We have

$$f(x) = \frac{g(x)}{x} = \frac{e^{1/x}}{x} \int_0^x \frac{e^{-1/t}}{t} dt. \quad (7.45)$$

Substitute by $t = \frac{x}{1+xw}$. This yields

$$f(x) = \int_0^{\infty} \frac{e^{-w}}{1+xw} dw. \quad (7.46)$$

Let $u = \frac{1}{t}$, so $t = \frac{1}{u}$. We now have

$$\begin{aligned} \int_{1/x}^{\infty} \frac{e^{-u}}{u} du &= \int_1^{\infty} \frac{e^{-u}}{u} du - \int_0^1 \frac{(1-e^{-u})}{u} du - \int_1^{1/x} \frac{du}{u} + \int_0^{1/x} \frac{(1-e^{-u})}{u} du. \\ &= -\gamma - \log \frac{1}{x} + \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{x}\right)^n \cdot \frac{1}{n \cdot n!}. \end{aligned} \quad (7.47)$$

Therefore,

$$f(1) = e \cdot \left(-\gamma + \sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot n!} \right) \approx .596347. \quad (7.48)$$

7.7 Cesaro Summation

Lecturer: Steven Miller

Let us consider the partial sums of a series:

$$s_n = \sum_{k=1}^n a_k. \quad (7.49)$$

Then we define Cesaro Summation by

$$\lim_{n \rightarrow \infty} \frac{s_1 + \cdots + s_n}{n}. \quad (7.50)$$

Note for

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots \quad (7.51)$$

we have $s_{2k} = 0$ and $s_{2k+1} = 1$. Thus, the average of the partial sums is $\frac{1}{2}$ if $n = 2k$, and $\frac{1}{2} + \frac{1}{n}$ if $n = 2k + 1$. Therefore, we see again that this series sums to $\frac{1}{2}$.

7.8 A Nice Integral From Euler

$$\int_0^1 x^{-x} dx = \sum_{n=1}^{\infty} n^{-n}. \quad (7.52)$$

Chapter 8

Wednesday, December 3, 2003

Lecturer: Ari

8.1 Euler and Mechanics

Euler is often attributed as the first to solve a differential equation. He used dots for time derivative (a la Newton's Fluxion Notation). Uses very modern looking notation. He proceeds through a treatise of Mechanics (vibrations of strings, for example).

See Euler, *Mechanica sive Matus Scientia Analytica*

8.2 Preliminaries

A body B is a collection of points $X = (x_1, x_2, x_3)$; we will assume the body is smooth (a smooth manifold) which can be embedded in Euclidean 3-Space.

Newton did mechanics through points: his notation and framework didn't generalize well. Euler's work put mechanics on a solid foundation, and increased the complexity of systems that could be mathematically handled.

Let E_1, E_2, E_3 be a basis. We will have a reference configuration at time $t = 0$: call this B_0 .

A motion of the body is a map $\phi_t : B_0 \rightarrow B_t$ such that $x(t) = \phi_t(X) = \phi(X, t)$. If we had two points $X_1, X_2 \in B_0$, we would have $x_1(t), x_2(t)$. We assume these motions are invertible; thus, $X = \phi^{-1}(x, t)$.

8.3 Lagrangian Formulation

Lagrange invented the Calculus of Variations – Euler was independently working on the subject, but held off on publishing so that this young man from Turan could get his name out. Lagrange also wrote a beautiful appendix to Euler’s Algebra.

Lagrange’s description of mechanics is as follows: start with a reference coordinate system. We’ll have points X evolving to points $\phi(X, t)$. We get velocities $v(X, t) = \frac{\partial \phi(X, t)}{\partial t}$.

For the Eulerian formulation, we have $v(X, t) = \widehat{v}(\phi(X, t), t) = \widetilde{v}(x, t)$. Thus, all of these equal $\frac{\partial \phi(x, t)}{\partial t}$.

A present configuration basis $\{e_i\}$, $i = 1$ to 3 . We are suppressing the subscript t . We adopt Einstein’s summation convention, which states any repeated subscript is summed over. Thus, $x = \sum_i x_i e_i = x_i e_i$.

8.4 Euler’s Theorem on Rotations about a Point

From Whittaker (1927, possibly Analytical Mechanics)

Theorem 8.4.1. *Any rotation about a point is equivalently a rotation about a line through that point (in three dimensions).*

Consider a rigid body in an initial configuration B_0 . There is a point in this body, X , such that, when $B_0 \rightarrow B_t$, x is fixed by this motion. Or, for all t , $x = \phi_t(X) = X$. For a rigid body, the distance between any two points in B_0 is equal to the distance between what they are mapped to in B_t .

Thus, a rotation about a point holds a point fixed; a rotation about a line holds a line fixed. We look at two snapshots: look at two times, say t_0 and t_1 . Then if we look at two snapshots such that it is a rotation about a point, one can also show that it is a rotation about a line.

Proof. Let X be a point in the body fixed by the rotation, let X_1 and X_2 be two fixed points in the body. This is all at time 0 .

At time t , we now have points X', X'_1, X'_2 . As X is the fixed point, $X' = X$.

Consider the plane that is perpendicular to the plane containing XX_1 and XX'_1 and bisects the angle X'_1XX_1 . Similarly for X, X_2, X_3 .

There should be a unique line of intersection of the two planes. Let C be on that line. So the angle CXX_1 equals CXX'_1 , and the angle CXX_2 equals CXX'_2 . This follows from the definition of C , as it lies in both planes which bisect.

Look at the four points C, X_1, X_2, X . Rotate these points about X . We know X_1 and X_2 go into X'_1 and X'_2 respectively. Then the line XC must be mapped into itself, and this gives us an entire fixed line.

If the two planes are co-incident, it is trivially modified. □

If we let t_0 approach zero, this line becomes the infinitesimal axis of rotation. This is useful in many applications.

Remark 8.4.2. *We can prove this using a more modern formulation. Since rigid, all distances remain the same. Thus, ϕ_t can be extended to an isometry of three dimensional space. Thus, the whole space rotates with this motion. Thus, it should be an element of $SO(3)$, the group of three dimensional rotations. Thus, $\phi_{t_0} \in SO(3)$. We just need the simple result that such a matrix has 1 as an eigenvalue, which gives us a line of symmetry. This follows from the eigenvalues occur in complex conjugate pairs, and are of modulus one. Thus, either all eigenvalues are real (and since the determinant is 1, at least one eigenvalue is 1) or two eigenvalues are modulus one complex conjugates, and the third is 1.*

8.5 Co-Rotating Coordinates

Euler introduces three angles ϕ, θ, ψ . We will have rotations of each of these angles, and consider

$$e_i = Q(\phi)Q(\theta)Q(\psi)E_i = QE_i. \quad (8.1)$$

These are called the Eulerian angles, and are given explicitly as follows. We represent these rotations with respect to the fixed E_i basis. Let

$$Q(\psi) = \begin{pmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (8.2)$$

This is a rotation about E_3 by an angle ψ . This gives

$$\begin{aligned} e'_1 &= \cos \psi E_1 + \sin \psi E_2 \\ e'_2 &= \cos \psi E_2 - \sin \psi E_1 \\ e'_3 &= E_3. \end{aligned} \quad (8.3)$$

Now we look at $Q(\theta)$, and we represent it with respect to the $\{e'_i\}$ basis. In this basis, it is a rotation about e'_2 , and we find

$$Q(\theta) = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}. \quad (8.4)$$

This yields

$$\begin{aligned} e''_1 &= \cos \theta e'_1 - \sin \theta e'_3 \\ e''_2 &= e'_2 \\ e''_3 &= \cos \theta e'_3 + \sin \theta e'_1. \end{aligned} \quad (8.5)$$

Now we consider $Q(\phi)$, written in the $\{e''\}$ basis. We now rotate about e''_1 . We have

$$Q(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix}. \quad (8.6)$$

Expanding we obtain

$$\begin{aligned} e_1 &= e''_1 \\ e_2 &= \cos \phi e''_2 - \sin \phi e''_3 \\ e_3 &= \sin \phi e''_2 + \cos \phi e''_3. \end{aligned} \quad (8.7)$$

Composing all the rotations, we find that e_1, e_2, e_3 are related to E_1, E_2, E_3 by

$$Q(\theta, \phi, \psi) = \begin{pmatrix} \cos \theta \cos \psi & \cos \theta \sin \psi & -\sin \theta \\ \sin \phi \sin \theta \cos \psi - \cos \phi \sin \psi & \sin \phi \sin \theta \sin \psi + \cos \phi \cos \psi & \sin \phi \cos \theta \\ \cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi & \cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi & \cos \phi \cos \theta \end{pmatrix}. \quad (8.8)$$

Specifically,

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = Q(\theta, \phi, \psi) \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}. \quad (8.9)$$

Angular velocity is $w = \dot{\phi}e_1'' + \dot{\theta}e_2' + \dot{\psi}E_3$. This leads to

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} -\sin\theta & 0 & 1 \\ \cos\theta \sin\phi & \cos\phi & 0 \\ \cos\theta \cos\phi & -\sin\phi & 0 \end{pmatrix} \begin{pmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix}. \quad (8.10)$$

Chapter 9

Euler and Fountains: Thursday, January 15

Lecturer: Seth Hulett

Historical and social note of Euler. Books says he's the master of us all (master of physics), Euler has been called in many circles a second rate physicist. Will go from an article *Euler and Fountains of Sanssouci* by Michael Eckert.

Some of the examples used against Euler: didn't take into account friction in dealing with some fountains. Euler was part of the project for a few years; during the entire reign of King Frederick, the fountains were never successful. It wasn't till later (steam engines and metal pipes) that it worked. Steam engines were originally proposed, but the king didn't want to spend the money on steam engines. Also, they used wooden pipes rather than iron pipes.

Before Euler, pipes burst at bottom. Wasn't till one trial run before Euler was involved: hollowed out trees and coated with metal on the outside. Then didn't burst on the bottom.

Still led to many physicists making claims such as "Euler didn't know conservation of energy" or "Euler's theories on fluids didn't lead to practical answers."

This paper explores whether or not Euler's knowledge was sufficient to build a system that would work. Eckert looked into the history. In WWII, much of the history of the building was lost, but he was able to reconstruct some of the history. Euler wasn't involved until after the fountains were somewhat successful, after a great flood of rain.

Have a river that flows, have a castle with fountains. Castle is on a higher ground. Had to raise the water to the level of the castle. Wanted the water to drop

100 feet at the castle. The river is not next to the castle – built a windmill to pump the water up. This failed (only one was somewhat successful right before Euler).

One semi-successful day: lots of rain, helped fill the reservoir up at the castle! Worked for half an hour or an hour. Pipes brought water up the mountain to the castle, about 150 feet.

King wanted grandiose fountains (to be better than Versailles) but wasn't willing to spend the right amount of money. Euler never brought up the cost factor. Euler might have assumed the pipes are built out of metal (lead), so they wouldn't burst.

Other great fountains of the era used metal pipes and not wooden pipes. Why are these historians of science saying Euler was a second rate physicist?

During Euler's communication, he described what he thought the pipes should be. In the article, on page 458, is a depiction of the system.

Euler assumed constant pressure to fill the reservoir. Many terms in the equation don't seem to match with the diagram; certain variables were extrapolated from the diagram. Bottom has 7 times more pressure than top (Euler wrote this down, though others thought that it was the other way around). Euler knew this was hydro-dynamical not hydro-static. Pressure at the bottom is much higher than one would think.

One of the concerns of historians is that Euler ignored certain things / his theory wasn't practical. Euler, in his letters to the king, asked that if they change anything (about the lead pipes), please let him know. Euler was extrapolating from some of the Versailles fountains. Euler wanted to work, was doing some experiments. Euler was put in charge of many administrative tasks. While he might play with math for fun (manipulating infinities), he was far more careful on the practical, applied calculations.

Another part, not dealt with this: Euler's paper on ballistics and gunnery was useless to the practical person. The way the first computer was funded was to come up with firing tables / ballistic tables. Every variable from wind, temperature, gunpowder temperature, et cetera: needed a different table for each combination. Euler's paper only worked for those cannons he studied: even the hardness of the ground influences greatly the tables. A historian not knowing the science well enough can look at Euler's book and say it's useless; however, it probably would not have been published or used back then if it didn't work.

The equation is

$$p = (k - y)g\rho + (b - r)g\rho \left(1 - \frac{w_p}{g} \frac{dw_p}{dr}\right) - a^2 \rho w_p \frac{dw_p}{dr} \int \frac{ds}{z^2} + \frac{w_p^2}{2} \rho \left(1 - \frac{a^4}{z^4}\right). \quad (9.1)$$

Chapter 10

Thursday, January 29th, 2004: Euler and Continued Fractions

Lecturer: Dan File

10.1 Series Expansions

This talk is based on a translation by Dan File of a paper of Euler on continued fractions. A generic continued fraction is of the form

$$a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \frac{1}{\ddots}}}}, \quad (10.1)$$

where $a, b, c, d, \dots \in \mathbb{N}$. We have convergents (truncating the continued fraction after a finite number of digits). We have

$$\frac{p_0}{p_1} = a, \quad \frac{p_1}{q_1} = a + \frac{1}{b}, \quad \dots \quad (10.2)$$

We have relations between the numerators and denominators:

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2} \\ q_n &= a_n q_{n-1} + q_{n-2}. \end{aligned} \quad (10.3)$$

This implies that

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}; \quad (10.4)$$

this implies, in particular, that $(p_n, q_n) = 1$.

Direct computation gives

$$\frac{p_{n+1}}{q_{n+1}} = \frac{p_{n-1} + a_{n+1}p_n}{q_{n-1} + a_{n+1}q_n}. \quad (10.5)$$

We have the following

$$\begin{aligned} p_0 &= a \\ p_1 &= p_0b + 1 \\ p_2 &= p_1c + p_0 \\ p_3 &= p_2d + p_1 \end{aligned} \quad (10.6)$$

and

$$\begin{aligned} q_0 &= 1 \\ q_1 &= b \\ q_2 &= q_1c + q_0 \\ q_3 &= q_2d + q_1. \end{aligned} \quad (10.7)$$

Looking at successive differences gives

$$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{(-1)^{n-1}}{q_n q_{n-1}}. \quad (10.8)$$

Another way of writing this is

$$\begin{aligned} \frac{p_0}{q_0} &= a \\ \frac{p_1}{q_1} &= a + \frac{1}{q_0 q_1} \\ \frac{p_2}{q_2} &= a + \frac{1}{q_0 q_1} - \frac{1}{q_1 q_2}, \end{aligned} \quad (10.9)$$

and so on. Thus, we are getting series expansions, and this series converge because these denominators grow exponentially (at least as fast as the Fibonacci numbers).

10.2 Another Perspective

What if we have an infinite series

$$s = \frac{1}{\alpha} - \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} - \cdots, \quad \alpha, \beta, \gamma, \delta \in \mathbb{Z}. \quad (10.10)$$

From our earlier results, matching gives

$$\begin{aligned} q_0 q_1 &= \alpha \\ q_1 q_2 &= \beta \\ q_2 q_3 &= \gamma. \end{aligned} \quad (10.11)$$

We set $q_0 = 1$, and then find that $q_1 = \alpha$, $q_2 = \frac{\beta}{\alpha}$, $q_3 = \frac{\alpha\gamma}{\beta}$. Alternatively, we have

$$\begin{aligned} q_1 &= \alpha \\ q_2 &= \frac{\beta}{\alpha} \\ q_3 &= \frac{\alpha\gamma}{\beta} \\ q_4 &= \frac{\beta\delta}{\alpha\gamma} \\ q_5 &= \frac{\alpha\gamma\epsilon}{\beta\delta} \\ q_6 &= \frac{\beta\delta\zeta}{\alpha\gamma\epsilon}. \end{aligned} \quad (10.12)$$

Using our earlier charts for conversion, we find that

$$\begin{aligned} b &= q_1 \\ c &= \frac{q_2 - q_0}{q_1} \\ d &= \frac{q_3 - q_1}{q_2}. \end{aligned} \quad (10.13)$$

Since $q_1 = \alpha$, we have $b = \alpha$. Then $q_2 - q_0 = cq_1 = \frac{\beta - q_0\alpha}{q_1}$. As $q_1 = \alpha$, we have that $c = \frac{q_2 - q_0}{q_1} = \frac{\beta - \alpha}{\alpha^2}$. As $d = \frac{q_3 - q_1}{q_2}$, we need to find $q_3 - q_1$, which is just $\frac{\gamma - \beta}{q_2}$. Thus, $d = \frac{\alpha^2(\gamma - \beta)}{\beta^2}$.

Collecting our results gives

$$\begin{aligned}
 b &= \alpha \\
 c &= \frac{\beta - \alpha}{\alpha^2} \\
 d &= \frac{\alpha^2(\gamma - \beta)}{\beta^2} \\
 e &= \frac{\beta^2(\delta - \gamma)}{\alpha^2} \gamma^2 \\
 f &= \frac{\alpha^2 \gamma^2 (\epsilon - \delta)}{\beta^2 \delta^2}.
 \end{aligned} \tag{10.14}$$

The pattern is clearer if we look at every other:

$$\begin{aligned}
 b &= \alpha \\
 d &= \frac{\alpha^2(\gamma - \beta)}{\beta^2} \\
 f &= \frac{\alpha^2 \gamma^2 (\epsilon - \delta)}{\beta^2 \delta^2}
 \end{aligned} \tag{10.15}$$

and

$$\begin{aligned}
 c &= \frac{\beta - \alpha}{\alpha^2} \\
 e &= \frac{\beta^2(\delta - \gamma)}{\alpha^2} \gamma^2.
 \end{aligned} \tag{10.16}$$

Re-writing gives

$$\begin{aligned}
 b &= \alpha \\
 \beta^2 d &= \alpha^2(\gamma - \beta) \\
 \beta^2 \delta^2 f &= \alpha^2 \gamma^2 (\epsilon - \delta) \\
 \beta^2 \delta^2 \zeta^2 h &= \alpha^2 \gamma^2 \epsilon^2 (\eta - \zeta)
 \end{aligned} \tag{10.17}$$

and

$$\begin{aligned}
 \alpha^2 c &= \beta - \alpha \\
 \alpha^2 \gamma^2 e &= \beta^2(\delta - \gamma) \\
 \alpha^2 \gamma^2 \epsilon^2 g &= \beta^2 \delta^2 (\zeta - \epsilon) \\
 \alpha^2 \gamma^2 \epsilon^2 \eta^2 i &= \beta^2 \delta^2 \zeta^2 (\theta - \eta).
 \end{aligned} \tag{10.18}$$

Remember we had

$$s = 0 + \frac{1}{b + \frac{1}{c + \dots}}, \quad (10.19)$$

as we are taking $a = 0$. Thus, we have

$$\frac{1}{b + \frac{\alpha^2}{\alpha^2 c + \frac{\alpha^2 \beta^2}{\beta^2 d + \frac{\alpha^2 \beta^2 \gamma^2}{\alpha^2 \gamma^2 e + \frac{\alpha^2 \beta^2 \gamma^2 \delta^2}{\beta^2 \delta^2 f + \dots}}}} \quad (10.20)$$

Now we do our big substitution:

$$\frac{1}{\alpha + \frac{\alpha^2}{(\beta - \alpha) + \frac{\alpha^2 \beta^2}{\alpha^2 (\gamma - \beta) + \frac{\alpha^2 \beta^2 \gamma^2}{\beta^2 (\delta - \gamma) + \frac{\alpha^2 \beta^2 \gamma^2 \delta^2}{\alpha^2 \gamma^2 (\epsilon - \delta) + \dots}}}} \quad (10.21)$$

Cancelling gives

$$\alpha + \frac{1}{(\beta - \alpha) + \frac{\alpha^2}{(\gamma - \beta) + \frac{\beta^2}{(\delta - \gamma) + \frac{\gamma^2}{(\epsilon - \delta) + \dots}}}} \quad (10.22)$$

We now have great continued fraction expansions, but these continued fractions are no longer simple.

10.3 Example: $\log 2$

Consider

$$s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots = \log 2. \quad (10.23)$$

In this example,

$$\alpha = 1, \beta = 2, \gamma = 3, \delta = 4, \dots \quad (10.24)$$

This gives

$$\frac{1}{\log 2} = 1 + \frac{1^2}{1 + \frac{2^2}{1 + \frac{3^2}{1 + \frac{4^2}{1 + \dots}}}} \quad (10.25)$$

For this, we trivially went from $\log 2$ to $\frac{1}{\log 2}$.

10.4 Example: $\frac{\pi}{4}$

We have

$$s = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}. \quad (10.26)$$

We obtain

$$\frac{4}{\pi} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \cdots}}} \quad (10.27)$$

10.5 Example: Another Method

Consider

$$s = \frac{1}{ab} - \frac{1}{bc} + \frac{1}{cd} - \cdots \quad (10.28)$$

In this case, note that we have a common factor between adjacent terms. We have

$$s = \frac{1}{ab + \frac{a^2b^2}{(bc-ab) + \frac{b^2c^2}{(cd-bc) + \cdots}}} \quad (10.29)$$

We can do some cancellation and factoring, and obtain

$$\frac{1}{as} = b + \frac{ab}{(c-a) + \frac{bc}{(d-b) + \cdots}} \quad (10.30)$$

For example, consider

$$\begin{aligned}
 \log 2 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \\
 \log 2 - 1 &= -\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \\
 2\log 2 - 1 &= \frac{1}{2} - \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} - \frac{1}{4 \cdot 5} + \dots
 \end{aligned} \tag{10.31}$$

Now substituting into the previous gives

$$\frac{1}{2\log 2 - 1} = 2 + \frac{1 \cdot 2}{2 + \frac{2 \cdot 3}{2 + \frac{3 \cdot 4}{2 + \dots}}} \tag{10.32}$$

In the taxonomy of continued fractions, this is similar to $\frac{\pi}{4}$. Similarly, we have

$$\begin{aligned}
 \frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \dots \\
 \frac{\pi}{4} - 1 &= -\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \\
 \frac{\pi}{2} - 1 &= \frac{2}{3} - \frac{2}{3 \cdot 5} + \frac{2}{5 \cdot 7} \\
 \frac{\pi}{4} - \frac{1}{2} &= \frac{1}{3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \dots
 \end{aligned} \tag{10.33}$$

Similarly as before, we find

$$\frac{\pi}{4} - \frac{1}{2} = \frac{1}{3 + \frac{1 \cdot 3}{2 + \frac{3 \cdot 5}{2 + \frac{5 \cdot 7}{2 + \dots}}}} \tag{10.34}$$

10.6 Example: Yet Another Method

Consider

$$s = \frac{a}{\alpha} - \frac{b}{\beta} + \frac{c}{\gamma} - \frac{d}{\delta} + \dots \quad (10.35)$$

We find

$$as = \alpha + \frac{\alpha^2 b}{a\beta - b\alpha + \frac{ac\beta^2}{b\gamma - c\beta + \frac{bd\gamma^2}{c\delta - d\gamma + \dots}}} \quad (10.36)$$

For example, consider

$$s = \frac{1}{1} - \frac{2}{2} + \frac{3}{3} - \dots \quad (10.37)$$

In some sense, we can interpret this as $\frac{1}{2}$. Thus, inverting $\frac{1}{2}$ to 2 we get

$$2 = 1 + \frac{2}{0 + \frac{3 \cdot 4}{0 + \frac{8 \cdot 9}{0 + \frac{15 \cdot 16}{0 + \dots}}}} \quad (10.38)$$

This collapses to the fraction

$$2 = 1 + \frac{2 \cdot 1^2 \cdot 2 \cdot 4 \cdot 3^2 \cdot 4 \cdot 6 \cdot 5^2 \cdot 6 \dots}{1 \cdot 3 \cdot 2^2 \cdot 3 \cdot 5 \cdot 4^2 \cdot 5 \cdot 7 \cdot 6^2 \dots} \quad (10.39)$$

10.7 Another Variety of Examples

Let

$$s = \frac{1}{\alpha} - \frac{1}{\alpha\beta} + \frac{1}{\alpha\beta\gamma} - \frac{1}{\alpha\beta\gamma\delta} + \dots \quad (10.40)$$

One can show

$$\frac{1}{s} = \alpha + \frac{\alpha}{\beta - 1 + \frac{\beta}{\gamma - 1 + \frac{\gamma}{\delta - 1 + \dots}}} \quad (10.41)$$

We have

$$\begin{aligned} \frac{1}{e} &= 1 - \frac{1}{1} + \frac{1}{1 \cdot 2} - \frac{1}{1 \cdot 2 \cdot 3} + \dots \\ 1 - \frac{1}{e} = \frac{e-1}{e} &= \frac{1}{1} - \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} - \dots \end{aligned} \quad (10.42)$$

We find that

$$\frac{e}{e-1} = 1 + \frac{1}{1 + \frac{2}{2 + \frac{3}{3 + \frac{4}{4 + \dots}}}} \quad (10.43)$$

Chapter 11

Thursday, February 5th, 2004: Euler and $\zeta(s)$: some formulas

11.1 Definition of $\zeta(s)$

Lecturer: Warren Sinnott

A good survey article is by Ayoub, *Euler and the zeta function*, from around 1975. Another is by Weil, *Number Theory* (or something like that); other good sources are Davenport's *Multiplicative Number Theory*.

The *zeta function* is defined by

$$\begin{aligned}\zeta(s) &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s}\end{aligned}$$

which converges for real $s > 1$ (and complex s with $\Re(s) > 1$); the *alternating zeta function* $\zeta_{\pm}(s)$ is defined by

$$\begin{aligned}\zeta_{\pm}(s) &= 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}.\end{aligned}$$

which converges for real $s > 0$ (and complex s with $\Re(s) > 0$).

11.2 $\zeta(2), \zeta(4)$

Around 1700, a classical problem was to evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Similar sums had been evaluated, for example,

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \quad (11.1)$$

If we want to approximate the sum of $\frac{1}{n^2}$, one needs many terms (summing $n \leq x$ gives an error of size $\frac{1}{x}$); for example, Stirling did nine digits, the first eight being correct.

Euler was born in 1707; Euler tried to find methods to improve ways to calculate this sum. He invented many ways to speed up the convergence of this series. One such method is the Euler-MacLauren method: let $f(x)$ be analytic, and say we want to evaluate $\sum_{n=1}^{\infty} f(n)$. In the end, Euler in the 1730s calculates $\sum \frac{1}{n^2}$ to 20 digits; doing this naively would take an enormous amount of time!

In 1734, Euler made a breakthrough and calculated exact values not just for this series, but for series of the form $\sum \frac{1}{n^{2k}}$. We have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{\pi^2}{6} \\ \sum_{n=1}^{\infty} \frac{1}{n^4} &= \frac{\pi^4}{90}. \end{aligned} \quad (11.2)$$

Euler started with

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad (11.3)$$

It has zeros at $x = n\pi$, $n \in \mathbb{Z}$. He guessed that maybe we could factor and get

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x}{n\pi}\right) \cdot \left(1 + \frac{x}{n\pi}\right) = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right). \quad (11.4)$$

Comparing with the expansion for $\sin x$, we find that

$$\sin x = x - \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} \cdot x^3 + \sum_{\substack{n,m \\ 1 \leq n < m}} \frac{1}{n^2m^2\pi^4} \cdot x^5 - \dots \quad (11.5)$$

Thus, comparing coefficients gives

$$-\sum_{n=1}^{\infty} \frac{1}{n^2 \pi^2} = -\frac{1}{6}, \quad (11.6)$$

which gives our sum $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$. For the x^5 term, we have

$$\sum_{\substack{n,m \\ 1 \leq n < m}} \frac{1}{n^2 m^2 \pi^4} = \frac{1}{120}. \quad (11.7)$$

We can rewrite and obtain

$$\left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right) \cdot \left(\sum_{m=1}^{\infty} \frac{1}{m^2} \right) = \sum_{k=1}^{\infty} \frac{1}{k^4} + 2 \sum_{\substack{n,m \\ 1 \leq n < m}} \frac{1}{n^2 m^2}. \quad (11.8)$$

Using our result for $\sum \frac{1}{n^2}$, simple arithmetic gives $\sum \frac{1}{k^4} = \frac{\pi^4}{90}$. It gets harder to go for the x^7 term and higher; Euler did up to $k = 34$ (just the even k s) by hand!

11.3 Another Approach

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \\ &= x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2 \pi^2} \right); \end{aligned} \quad (11.9)$$

taking the logarithmic derivative of the product formula leads to the series expansion of the cotangent:

$$x \cot x = 1 - 2 \sum_{k=1}^{\infty} \frac{\zeta(2k)}{\pi^{2k}} x^{2k}$$

To see this, note that we would have

$$\begin{aligned}
 \cot x &= \frac{1}{x} + \sum_{n=1}^{\infty} \frac{-2x}{1 - \frac{x^2}{n^2\pi^2}} \cdot \frac{1}{n^2\pi^2} \\
 &= \frac{1}{x} - 2x \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{x^{2k}}{(n\pi)^{2k+2}} \\
 x \cot x &= 1 - 2 \sum_{k=0}^{\infty} \frac{x^{2k+2}}{\pi^{2k+2}} \zeta(2k+2). \tag{11.10}
 \end{aligned}$$

Thus, if we can find a nice Taylor expansion for $x \cot x$, we would have formulas for $\zeta(2k+2)$. Shifting variables, it is enough to study

$$x \cot x = 1 - 2 \sum_{k=1}^{\infty} \frac{x^{2k}}{\pi^{2k}} \zeta(2k). \tag{11.11}$$

Unfortunately, it isn't pleasant to take high derivatives of tan or cot; however, one can clearly see that the coefficients in the Taylor expansion are rational.

The *Bernoulli numbers* B_k , $k = 0, 1, 2, \dots$ are defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = 1 - \frac{1}{2}t + \frac{1}{6} \frac{t^2}{2!} - \frac{1}{30} \frac{t^4}{4!} + \dots \tag{11.12}$$

which implies that the Bernoulli numbers are all rational; we find

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = B_5 = B_7 = \dots = 0 \tag{11.13}$$

They also arise in studying $\sum_{n=1}^N n^k$. We also have (from the definition of B_k , bringing $e^t - 1$ to the right hand side) that

$$1 = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} \cdot \sum_{k'=0}^{\infty} \frac{t^{k'}}{(k+1)!}. \tag{11.14}$$

We additionally have

$$B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}, B_{12} = -\frac{691}{2730}, \dots \tag{11.15}$$

Note that

$$\frac{t}{e^t - 1} + \frac{t}{2} = \frac{t e^t + 1}{2 e^t - 1} = \frac{t e^{t/2} + e^{-t/2}}{2 e^{t/2} - e^{-t/2}} \quad (11.16)$$

is an *even* function: this tells us that $B_k = 0$ if k is odd and > 1 , and

$$\frac{t}{e^t - 1} + \frac{t}{2} = \frac{t e^{t/2} + e^{-t/2}}{2 e^{t/2} - e^{-t/2}} = \sum_{k=0}^{\infty} B_{2k} \frac{t^{2k}}{2k!}. \quad (11.17)$$

If we replace t by $2ix$, we get again a series expansion for the cotangent; since

$$\begin{aligned} \cos x &= \frac{e^{ix} + e^{-ix}}{2} \\ \sin x &= \frac{e^{ix} - e^{-ix}}{2i}. \end{aligned} \quad (11.18)$$

We find

$$x \cot x = \sum_{k=0}^{\infty} B_{2k} \frac{(2ix)^{2k}}{2k!} = \sum_{k=0}^{\infty} (-1)^k B_{2k} 2^{2k} \frac{x^{2k}}{2k!}. \quad (11.19)$$

Comparing the two expansions for the cotangent we find:

$$\zeta(2k) = (-1)^{k-1} 2^{2k-1} \pi^{2k} \frac{B_{2k}}{(2k)!}, \text{ for } k = 1, 2, 3, \dots \quad (11.20)$$

11.4 $\zeta(2k + 1)$

What about the values of $\zeta(2k + 1)$? It is now known that $\zeta(3)$ is odd, though it is not known if it is transcendental. Other results include that at least so many of certain sets of odd values must be irrational. The following is from a paper of Euler from 1749. We are interested (say) in

$$\begin{aligned} \odot &= 1^m - 2^m + 3^m - 4^m + \dots \\ \oslash &= 1^{-n} - 2^{-n} + 3^{-n} - 4^{-n} + \dots \end{aligned} \quad (11.21)$$

Let m be a non-negative integer. We can evaluate ζ for even values of n .

$$\begin{aligned}
\zeta_{\pm}(-m) &= 1 - 2^m + 3^m - 4^m + \dots \\
&= x - 2^m x^2 + 3^m x^3 - 4^m x^4 + \dots \Big|_{x=1} \\
&= \left(x \frac{d}{dx} \right)^m \left(\frac{x}{1+x} \right) \Big|_{x=1} \\
&= \left(\frac{d}{dt} \right)^m \left(\frac{e^t}{1+e^t} \right) \Big|_{t=0}. \tag{11.22}
\end{aligned}$$

So we need to find the Taylor expansion of $\frac{e^t}{1+e^t}$. Note that (!):

$$\frac{e^t}{1+e^t} = 1 + \frac{2}{e^{2t}-1} - \frac{1}{e^t-1}, \tag{11.23}$$

so that

$$\begin{aligned}
\frac{te^t}{1+e^t} &= t + \frac{2t}{e^{2t}-1} - \frac{t}{e^t-1} \\
&= t + \sum_{k=0}^{\infty} B_k (2^k - 1) \frac{t^k}{k!}. \tag{11.24}
\end{aligned}$$

and so

$$\frac{e^t}{1+e^t} = 1 + \sum_{k=0}^{\infty} B_{k+1} (2^{k+1} - 1) \frac{t^k}{(k+1)!}. \tag{11.25}$$

Note that the constant term is $1 + B_1 = \frac{1}{2}$. Thus

$$\begin{aligned}
\zeta_{\pm}(0) &= \frac{1}{2} \\
\zeta_{\pm}(m) &= \begin{cases} (2^{m+1} - 1) \frac{B_{m+1}}{m+1} & \text{for } m = 1, 2, 3, \dots \\ 0 & \text{if } m = 2, 4, 6, \dots \end{cases} \tag{11.26}
\end{aligned}$$

We are using beautiful formulas, such as at $x = 1$, we have

$$1 - 2 + 3 - 4 + \dots = \frac{1}{4}. \tag{11.27}$$

The values of ζ_{\pm} at positive integers: Recall that $\zeta_{\pm}(s) = (1 - 2^{1-s})\zeta(s)$, so we have

$$\begin{aligned}\zeta_{\pm}(1) &= \log 2 \\ \zeta_{\pm}(2k) &= (1 - 2^{1-2k})(-1)^{k-1}2^{2k-1}\pi^{2k}\frac{B_{2k}}{(2k)!}, \quad k = 1, 2, 3, \dots\end{aligned}\quad (11.28)$$

What does $\zeta_{\pm}(2k + 1)$ equal?

$$\zeta_{\pm}(0) = \frac{1}{2}, \quad \zeta_{\pm}(-1) = \frac{1}{4}, \quad \zeta_{\pm}(-2) = 0, \quad \zeta_{\pm}(-3) = \frac{1}{8}, \quad \zeta_{\pm}(-4) = 0, \dots \quad (11.29)$$

If he had gone more (he stopped at 9), he would've seen the 691 from Bernoulli numbers resurface. Thus for $k = 1, 2, 3, \dots$

$$\frac{\zeta_{\pm}(1 - 2k)}{\zeta_{\pm}(2k)} = \frac{2^{2k} - 1}{2^{2k-1} - 1}(-1)^{k-1}\frac{(2k - 1)!}{\pi^{2k}} \quad (11.30)$$

and so

$$\frac{\zeta_{\pm}(1 - m)}{\zeta_{\pm}(m)} = \begin{cases} \frac{2^m - 1}{2^{m-1} - 1}(-1)^{\frac{m}{2} - 1}\frac{(m-1)!}{\pi^m} & \text{if } m \text{ is even, } \geq 2 \\ 0 & \text{if } m \text{ is odd, } \geq 3 \\ \frac{1}{2\log 2} & \text{if } m = 1 \end{cases} \quad (11.31)$$

Euler observes that this can be written

$$\frac{\zeta_{\pm}(1 - m)}{\zeta_{\pm}(m)} = \begin{cases} -\frac{2^m - 1}{2^{m-1} - 1} \cos(\pi m/2) \frac{\Gamma(m)}{\pi^m} & \text{if } m \geq 2 \\ \frac{1}{2\log 2} & \text{if } m = 1 \end{cases} \quad (11.32)$$

and conjectures

$$\frac{\zeta_{\pm}(1 - s)}{\zeta_{\pm}(s)} = -\frac{2^s - 1}{2^{s-1} - 1} \cos(\pi s/2) \frac{\Gamma(s)}{\pi^s} \quad \text{for all } s. \quad (11.33)$$

This does give value $\frac{1}{2\log 2}$ at $s = 1$ and also reduces to

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad (11.34)$$

when $s = \frac{1}{2}$. He continues and tries $s = \frac{2k+1}{2}$, and does some numerics. Using $\zeta_{\pm}(s) = (1 - 2^{1-s})\zeta(s)$, we can rewrite this conjecture in the form

$$\zeta(1 - s) = 2^{1-s} \cos(\pi s/2) \Gamma(s) \pi^{-s} \zeta(s), \quad (11.35)$$

which is Riemann's functional equation.

11.5 Appendix on $\zeta(s)$

Handout from Steve Miller

Let $[x]$ denote the greatest integer less than or equal to x , and let $\{x\} = x - [x]$. Following Davenport [Da], we have the following:

$$\begin{aligned}
 \zeta(s) &= \sum_{n=1}^{\infty} n \cdot \left[\frac{1}{n^s} - \frac{1}{(n+1)^s} \right] \\
 &= s \sum_{n=1}^{\infty} n \int_n^{n+1} t^{-s-1} dt \\
 &= s \sum_{n=1}^{\infty} \int_n^{n+1} [t] t^{-s-1} dt \\
 &= s \int_1^{\infty} [t] t^{-s-1} dt \\
 &= s \int_1^{\infty} t^{-s} dt - s \int_1^{\infty} \{t\} t^{-s-1} dt \\
 &= \frac{s}{s-1} + O(s).
 \end{aligned} \tag{11.36}$$

Therefore, we have shown

Lemma 11.5.1. $\zeta(s) = \frac{s}{s-1} + O(s)$.

In fact, in the above the $O(s)$ term is at most $|s|$. Let

$$\zeta_x(s) = \sum_{n \leq x} \frac{1}{n^s}. \tag{11.37}$$

We want to compare $\zeta(s)$ with $\zeta_x(1)$ and $\zeta_x(s)$. We have

$$\zeta_x(1) = \log x \text{ plus lower order terms.} \tag{11.38}$$

A similar argument as before gives

$$\begin{aligned}
 \sum_{n=x+1}^{\infty} \frac{1}{n^s} &= \sum_{n=x}^{\infty} n \left[\frac{1}{n^s} - \frac{1}{(n+1)^s} \right] - \frac{1}{x^{s-1}} \\
 &= \frac{s}{s-1} x^{1-s} + O(x^{1-s}).
 \end{aligned} \tag{11.39}$$

Let us choose

$$s = s(x) = 1 + \frac{1}{\log x}. \quad (11.40)$$

Then

$$\begin{aligned} \sum_{n=x+1}^{\infty} \frac{1}{n^s} &= \log x \cdot \left(1 + \frac{1}{\log x}\right) \cdot x^{-\frac{1}{\log x}} + O(1) \\ &= (\log x + 1) \cdot e^{-\frac{\log x}{\log x}} + O(1) \\ &= \frac{\log x}{e} + O(1). \end{aligned} \quad (11.41)$$

Therefore,

$$\begin{aligned} \zeta_x(s) &= \zeta(s) - \sum_{n=x+1}^{\infty} \frac{1}{n^s} \\ &= \frac{s}{s-1} + O(s) - \frac{\log x}{e} + O(1) \\ &= (\log x + 1) - \frac{\log x}{e} + O(1) \\ &= \frac{e-1}{e} \log x + O(1). \end{aligned} \quad (11.42)$$

Therefore, for $s = 1 + \frac{1}{\log x}$, $\zeta(s)$, $\zeta_x(s)$ and $\zeta_x(1)$ are all a constant times $\log x$. Up to lower order terms, $\zeta(s)$ and $\zeta_x(1)$ both equal $\log x$; $\zeta_x(s)$ is slightly smaller, approximately $\frac{e-1}{e} \log x \approx .632 \log x$.

Thus, the most efficient of these (to determine $\zeta(s)$ for s close to 1) is $\zeta(s) = \frac{s}{s-1} + O(s)$, which involves one division. We can make the $O(s)$ error explicit, as $s \cdot O(1)$; in fact, for $s \geq 1$, the $O(1)$ error is at most $\frac{\pi^2}{6} + 1$.

Chapter 12

Thursday, February 12th, 2004:

Lecturer: Vitaly Bergelson

12.1 Euler and Continued Fractions II

Bostwick Wyman and his mother transferred a paper by Euler on continued fractions. Two years later, Euler wrote another paper on the subject (50+ pages, many expressions of continued fractions). Turns out that Russians (historians) are very good at checking what Euler did in his notebooks – people have careers describing what he did in his notebooks. There are at least 50 notes concerning continued fractions. He was using continued fractions to calculate definite integrals. Some of the crazy divergent series we've seen earlier came from continued fractions.

Four things Euler did with continued fractions:

1. Pell Equation (Euler didn't care too much on proofs, cared about results and speed of approximation; seems he knew the algorithm though Lagrange was the one who proved it);
2. Euler showed that e and $e^r, r \in \mathbb{Q}$, are irrational; he had explicit, infinite continued fraction expressions for these, which imply they are irrationals, although he never stressed this point (Lambert proved irrationality of π by using continued fraction expansions of functions);
3. Euler was able to explain, by playing with sequences versus products, Brouncker's formula (for π); in 1776 Euler observes that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \quad (12.1)$$

and

$$\frac{\pi}{4} = \frac{1}{1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \dots}}}} \quad (12.2)$$

are identical; relates to the Wallis product

$$\frac{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \dots}{3 \cdot 3 \cdot 5 \cdot 5 \dots} \quad (12.3)$$

4. calculating integrals and solving differential equations (it was his knowledge of some differential equations that led to his formulas for e).

12.2 \sqrt{n}

$$\sqrt{2} = 1.41421356, \quad (12.4)$$

which leads to a continued fraction expansion

$$1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}} \quad (12.5)$$

Does the same with $\sqrt{3}$,

$$\sqrt{3} = [1; 1, 2, 1, 2, \dots], \quad (12.6)$$

where the number before the semicolon is the greatest integer less than or equal to our number.

What is Euler's proof? Start with fractions like

$$a + \frac{1}{b + \frac{1}{b + \frac{1}{b + \dots}}} = x. \quad (12.7)$$

. Thus, we find

$$\begin{aligned} x - a &= \frac{1}{b + x - a} \\ x &= a - \frac{b}{2} + \sqrt{1 + \frac{b^2}{4}}. \end{aligned} \quad (12.8)$$

Letting $b = 2$, $a = 1$, we get the expansion for $x = \sqrt{2}$. Thus, we have the expansion for $\sqrt{a^2 + 1}$.

Take now numbers of the form

$$a + \frac{1}{b + \frac{1}{c + \frac{1}{b + \frac{1}{c + \dots}}}} \quad (12.9)$$

. He sees / believes from this that any nice periodic continued fraction will lead to a quadratic irrational.

Theorem 12.2.1 (Lagrange). *A simple (all ones along the numerators) continued fraction of x is eventually periodic if and only if x is a quadratic irrational (called a quadratic surd in this field).*

Galois gave a necessary and sufficient condition for a continued fraction to be purely periodic. Euler opened the gate for these two theorems.

12.3 Denominators in Arithmetic Progression

Consider a continued fraction $x = [0; a_1, a_2, a_3, \dots]$. We call the a_i the denominators (sometimes also call them digits). What about numbers whose denominators are in arithmetic progression?

Euler was the first one to use e , though sometimes they would use c or a or b for 2.71828182845904... He divides by 10 to a large power, and arrives at

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots]. \quad (12.10)$$

Then he studies

$$\sqrt{2} = 1.6487212707 = [1; 1, 1, 1, 5, 1, 1, 9, 1, 1, 13, \dots]. \quad (12.11)$$

He finds

$$\frac{\sqrt[3]{e} - 1}{2} = [0; 5, 18, 30, 42, 54, \dots], \quad (12.12)$$

and then he finds

$$\frac{e^2 - 1}{2} = [3; 5, 7, 9, 11, 13, \dots], \quad (12.13)$$

finally giving an uninterrupted arithmetic progression.

Again, Euler takes a special case, where we have an arithmetic progression interrupted by two terms:

$$\begin{aligned} x &= [a; m, n, b, m, n, c, m, n, d, \dots] \\ &= \frac{1}{mn + 1} [(mn + 1)a + n; (mn + 1)b + m + n, (mn + 1)c + m + n, \dots] \end{aligned} \quad (12.14)$$

Proof? Take partial quotients, and see they are identical. Consider rational functions of e – will these be of this form? If something is of this form, is it a rational function of e ?

Why didn't Euler ask Lagrange's Theorem? Why not attempt to try to characterize all continued fractions that are periodic or eventually periodic? What is the general characterization of such expressions?

Theorem 12.3.1 (Euler). Consider

$$[a; m, n, b, m, n, c, m, n, \dots] - [a; n, m, b, n, m, c, n, m, d, \dots]. \quad (12.15)$$

This equals $\frac{n-m}{1+mn}$.

Remark 12.3.2 (Sinnott). This looks a lot like the formula $\tan(x + y)$.

Consider now

$$\begin{aligned} e &= [2; 1, 2, 1, 1, 4, 1, 1, 6, \dots] \\ \frac{1}{e - 2} &= [1; 2, 1, 1, 4, 1, 1, 6, \dots], \end{aligned} \quad (12.16)$$

and then ends with (after using a modification of the arguments we gave with $[a; m, n, b, m, n, c, m, n, d, \dots]$; he shifted so that it would start in this form)

$$e = 2 + \frac{1}{1 + \frac{2}{5 + \frac{1}{10 + \frac{1}{14 + \frac{1}{18 + \dots}}}}}, \quad (12.17)$$

which is an arithmetic progression, and almost simple (just one non-one).

12.4 Riccati Equation

Consider the equation

$$ady^2 + ydx = x. \quad (12.18)$$

After some substitutions, it is equivalent to

$$adq + q^2 dp = dp. \quad (12.19)$$

As an exercise, show this is the same as

$$q = \left[\frac{a}{p}; \frac{3a}{p}, \frac{5a}{p}, \frac{7a}{p}, \dots \right]. \quad (12.20)$$

We have

$$\frac{adq}{1 - q^2} = dp. \quad (12.21)$$

From this, we can obtain

$$\frac{a}{2} \log \frac{1+q}{1-q} = p + c. \quad (12.22)$$

He finds

$$e^{\frac{1}{s}} = [1; s-1, 1, 1, 3s-1, 1, 1, 5s-1, \dots]. \quad (12.23)$$

Nowadays, we know there is a connection between $\mathrm{SL}_2(\mathbb{R})$ and continued fractions. This group is the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant one; the action on $z \in \mathbb{C}$ is defined by $z \mapsto \frac{az+b}{cz+d}$.

12.5 Digits and Normality

Take a typical $x \in [0, 1]$. What can one say about such a typical x ? Write

$$x = [0; a_1(x), a_2(x), a_3(x), \dots]. \quad (12.24)$$

See in this symbolic representation of a number, but unlike a decimal, all numbers are now possible as digits. Via this connection, $x \in \mathbb{N}^{\mathbb{N}}$ (let's assume x is irrational). Consider the shift operator.

Analogous to decimals: let $x = \sum \frac{d_n(x)}{10^n}$. If we look at $10x \bmod 1$, we have $\sum \frac{d_{n+1}(x)}{10^n}$. Consider the shift on this space. Here, want to send $[a_1, a_2, a_3, \dots]$ to $[a_2, a_3, a_4, \dots]$. The operation is

$$Tx = \begin{cases} \{\frac{1}{x}\} & \text{if } x \in (0, 1) \\ 0 & \text{if } x = 0 \end{cases} \quad (12.25)$$

where $\{y\}$ is the fractional part of y . We can look at this on the first quadrant, getting hyperbolas. It is not Lebesgue measure preserving. Gauss discovered that the following measure is invariant:

$$\frac{1}{\log 2} \int_A \frac{dx}{1+x}. \quad (12.26)$$

Gauss didn't know how fast one approaches this measure, what is the speed of convergence? Not unrelated to speed of convergence of continued fractions. Easy to see this works for intervals, but how does one find this? Gauss found it from solving some differential equation. Kuzmin and Levy in the 1900s finished the work.

What is normality? We have an iterated function system. We should have a notion of normality. For decimal expansions (or, even simpler, binary expansions $x = \sum \frac{b_n(x)}{2^n}$, $b_n \in \{0, 1\}$), then any finite word of 0s and 1s should occur with the correct frequency. For continued fractions, we have infinitely many words of a given length (very different than the binary case). We expect almost all numbers are normal. Surprisingly, only proved in the 1970s.

For decimals, the number

$$.1234567891011121314151617181920 \dots \quad (12.27)$$

is normal. We also have

$$.12357111317192329 \dots \quad (12.28)$$

In general, for good functions f , we have

$$.f(1)f(2)f(3)f(4)f(5)\dots \quad (12.29)$$

will be decimal normal.

Theorem 12.5.1. *For almost all x ,*

$$\begin{aligned} \frac{a_1(x) + a_2(x) + \dots + a_n(x)}{n} &\rightarrow \infty \\ \sqrt[n]{a_1(x)a_2(x)\dots a_n(x)} &\rightarrow \text{Khinchin's Constant} \end{aligned} \quad (12.30)$$

Chapter 13

Thursday, February 19th, 2004: Euler and Prime Producing Quadratics

Lecturer: Scott Arms

13.1 Prime Generating Polynomials: Examples

References: *Prime Producing Polynomials and Principal Ideal Domains* (D. Fendel); *Prime-Producing Quadratics* (R. A. Mollin, American Mathematical Monthly, vol 104, 1997, 529-544); *Prime Generating Polynomial* (E. Weisstein, math-world.com).

Example 13.1.1. Consider

$$\left\{ x^2 - x + 41 : x \in \{0, 1, \dots, 40\} \right\}. \quad (13.1)$$

Note 0 and 1 give the same prime; however, every element in this range is a prime number! Another example is

$$\left\{ x^2 + x + 39 : x \in \{0, 1, \dots, 39\} \right\}. \quad (13.2)$$

This gives distinct primes (letter to Legendre). Both examples are due to Euler.

Consider now

$$\begin{aligned} & \left\{ x^2 + x + 3 : x \in \{0, 1\} \right\} \\ & \left\{ x^2 + x + 5 : x \in \{0, 1, \dots, 5 - 2\} \right\} \\ & \left\{ x^2 + x + 11 : x \in \{0, 1, \dots, 11 - 2\} \right\} \\ & \left\{ x^2 + x + 17 : x \in \{0, 1, \dots, 17 - 2\} \right\}. \end{aligned} \quad (13.3)$$

We want long strings of primes from a polynomial (ie, evaluating the polynomial at consecutive integers gives primes).

A lofty goal: find all primes p with quadratic of this form "working" (ie, $x^2 + x + p$ is prime for $x \in \{0, \dots, p - 2\}$; we'll formalize later what working means).

13.2 Prime Production Length

Definition 13.2.1 (Prime Production Length). For $F_\Delta(x) = x^2 + x + A$ (with discriminant $\Delta = 1 - 4A$) has prime production length l if $l \geq 0$ is the least integer such that $F_\Delta(x)$ is prime for $x \in \{0, 1, \dots, l - 1\}$ and either $F_\Delta(l)$ is composite or $F_\Delta(l) = 1$ or $F_\Delta(l) = F_\Delta(x)$ for some integer $x \in \{0, 1, \dots, l - 1\}$.

Theorem 13.2.2. If $l \geq 1$ is the prime production length of $F(x) = F_\Delta(x)$, then $l \leq A - 1$. If p is the smallest odd prime such that Δ is a quadratic residue modulo p , then $l < p$. Moreover, if $l \geq \frac{A-1}{2}$ and $A \neq 2$, then $A = p$.

Proof.

$$\begin{aligned} F(A - 1) &= (A - 1)^2 + (A - 1) + A \\ &= A^2 - 2A + 1 - 1 + A + A \\ &= A^2, \end{aligned} \quad (13.4)$$

which is composite. Thus, $l \leq A - 1$. Without loss of generality, let $x \in \{0, 1, \dots, p - 1\}$. We break into two cases ($x = 0$ and $x \neq 0$).

If $x = 0$, then p divides the discriminant $\Delta = 1 - 4A$. Thus, $F(\frac{p-1}{2}) = \frac{p^2 - \Delta}{4} \equiv 0 \pmod{p}$. If $l > \frac{p-1}{2}$, then $\frac{p^2 - \Delta}{4} = p$. So $0 > \Delta = p^2 - 4p$. SO $p = 3$ (since p is odd), $\Delta = -3$, $A = 1$, and $l = 0$ (contradiction); thus, $l \leq \frac{p-1}{2} < p$.

Suppose $\Delta \neq -7$ and $l \geq \frac{A-1}{2}$. Now $\Delta = 1 - 4A \equiv 1 \equiv 1^2 \pmod{A}$. So either $A = 2$ or $p \leq A$ (as 1^2 is a quadratic residue). As p is odd, this implies that

$p \leq A$. If $p < A$, then $l \leq \frac{p-1}{2} < \frac{A-1}{2}$, a contradiction. This completes the proof in the case $x = 0$.

Suppose now $x > 0$. Without loss of generality, we may assume x is odd (otherwise we can take $p - x$, which squares to the same value mod p). So $\Delta \equiv (2n + 1)^2 \pmod{p}$, and $F(n) = n^2 + n + A$. This gives (simple algebra) that $\frac{(2n+1)^2 - \Delta}{4} \equiv 0 \pmod{p}$. Also, $F(p - 1 - n) \equiv F(n) \pmod{p}$. Therefore, $l \leq p - 1 - n < p$ (p dividing two different things, can't both be prime). Suppose $A \neq 2$ and $l > \frac{A-1}{2}$. Thus, $0 \leq n \leq \frac{p-1}{2}$; by a similar argument as before we find that for $A \neq 2$, since p is the minimal p making it a quadratic residue, that $0 \leq n < \frac{p-1}{2} \leq \frac{A-1}{2} \leq l$. So $F(n) = p$ is a prime (since $n < l$, and l is the prime production length). This yields that $F(0) = A \leq F(n) = p$ (since F is increasing). Thus, $A \leq p$, which gives $A = p$. This proof is from Mollin 1997. \square

Theorem 13.2.3. *Assuming the Hardy-Littlewood Prime k -tuple conjecture, for all $B \in \mathbb{N}$ there exists an $A \in \mathbb{N}$ such that $x^2 + x + A$ has prime producing length B .*

For $B = 41$, $A > 10^{18}$ for quadratics of this form (Lukes, Patterson, Williams).

13.3 Optimality of Euler's Quadratic Polynomial

Let $x^2 + x + A = (x + \alpha)(x + \bar{\alpha})$, with $\alpha = \frac{1 + \sqrt{1-4A}}{2}$. Let $K = K_{\alpha(A)} = \mathbb{Q}(\alpha(A)) = \mathbb{Q}(\sqrt{1-4A})$. We let \mathcal{O}_K be all $x \in K$ such that the minimal polynomial of x is monic and has integer coefficients. This is a Dedekind ring ring.

Lemma 13.3.1. *The set of fractional ideals modulo the principal ideals is finite; we call its order $h = h_{\Delta}$ (the class number).*

Theorem 13.3.2. *If \mathcal{O}_K is a unique factorization domain (UFD), α as above, then $F(x) = F_{\Delta}(x) = x^2 + x + A$ has prime producing length $A - 1$.*

Theorem 13.3.3. *If $F(x)$ as above has prime values for $0 \leq x \leq \lfloor \frac{1}{2} \sqrt{\frac{4A-1}{3}} \rfloor$, then \mathcal{O}_K (as above) is a principal ideal domain (PID).*

Corollary 13.3.4. *\mathcal{O}_K (as above) is a PID if and only if it is a UFD.*

Theorem 13.3.5 (Stark 1967). *\mathcal{O}_K (as above) is a PID if and only if $4A - 1 \in \{3, 7, 11, 19, 43, 67, 163\}$.*

This gives that the Euler Polynomial has the optimal prime producing length for quadratics of this form.

13.4 Other Polynomial Forms

Euler also looked at non-monic polynomials, such as (letter to Legendre) $2x^2 + p$, p prime. For example, for $p = 29$, this is prime for $x \in \{0, 1, \dots, 28\}$. If you try $3x^2 + 3x + 23$, this has prime production length of 21. If one tries $6x^2 + 6x + 31$, this has prime production length of 29.

These polynomials are related. We have

$$\begin{aligned} 2x^2 + 29 &= 2x^2 - \frac{0^2 - 4 \cdot 2 \cdot 29}{4 \cdot 2} \\ 3x^2 + 3x + 23 &= 3x^2 + 3x + \frac{3^2 - 4 \cdot 2 \cdot 23}{4 \cdot 3} \\ 6x^2 + 6x + 31 &= 6x^2 + 6x + \frac{6^2 - 4 \cdot 6 \cdot 31}{4 \cdot 6}. \end{aligned} \quad (13.5)$$

These all have prime producing lengths equal to $\lfloor \frac{|\text{discriminant}|}{4q} \rfloor$, with q the leading coefficient.

Definition 13.4.1 (Fundamental Discriminant). *If $D \neq 1$ is a square-free integer and*

$$\Delta = \begin{cases} 4D & \text{if } D \not\equiv 1 \pmod{4} \\ D & \text{if } D \equiv 1 \pmod{4} \end{cases} \quad (13.6)$$

then Δ is a Fundamental discriminant.

Definition 13.4.2. *For Δ a Fundamental Discriminant, $q \geq 1$, $q|\Delta$ (q -square-free), then*

$$F_{\Delta,q}(x) = \begin{cases} qx^2 - \frac{\Delta}{4q} & \text{if } 4q|\Delta \\ qx^2 + qx + \frac{q^2 - \Delta}{4q} & \text{otherwise} \end{cases} \quad (13.7)$$

Definition 13.4.3. *$F(\Delta, q)$ is the maximum number of primes dividing any $F(x) = F_{\Delta,q}(x)$ for $x \in \{0, 1, \dots, \lfloor \frac{|\Delta|}{4q} - 1 \rfloor\}$.*

The examples we've listed earlier in the talk fit in this formulation. For example, the discriminant of $2x^2 + 29$ is $0 - 4 \cdot 2 \cdot 29$ has two prime factors (2, 29). The discriminant of $3x^2 + 3x + 23$ is $-3 \cdot 89$ which has two prime factors (3, 89). The discriminant of $6x^2 + 6x + 31$ is $4(-3 \cdot 59)$, which has three prime factors (2,3,59).

Theorem 13.4.4 (Gauss). *Let $\Delta < 0$ be a Fundamental Discriminant with $N + 1$ distinct prime factors. Then the class number of the field corresponding to the polynomial with discriminant Δ , $h(\Delta)$, satisfies $h(\Delta) = 2^N$ if and only if the exponent of the class group is less than or equal to 2.*

Theorem 13.4.5 (Mollin 1995 or 1997). *Let $\Delta < -4$ be a Fundamental Discriminant with $N + 1$ distinct prime factors, p being the largest. Suppose $q \geq 1$ divides Δ , q is square-free, and q has m distinct prime factors. Then $(\Delta, q) = N + 1 - m$ and $h(\Delta) = 2^{F(\Delta, q)}$ if and only if the exponent of the class group is less than or equal to 2.*

Corollary 13.4.6. *For $\Delta < -4$ a Fundamental Discriminant, $h(\Delta) = 1$ if and only if $F(\Delta, 1) = 1$.*

Corollary 13.4.7. *For $\Delta < -4$ a Fundamental Discriminant, $h(\Delta) = 2$ if and only if $F(\Delta, 2) = 2$.*

We have that $2x^2 + p$ has prime production length equal to p if and only if $\mathbb{Q}(\sqrt{-2p})$ has class number 2 if and only if $p \in \{3, 5, 11, 29\}$.

Chapter 14

Thursday, February 26th, 2004: Eulerian Integrals: Γ and β -Functions

Lecturer: M.C.

A good reference (now available in English) is V. A. Zovich's *Introduction to Mathematical Analysis*.

14.1 Γ -Function

Definition 14.1.1 (Γ -Function).

$$\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1} dx = \int_0^{\infty} e^{-x} x^{\alpha} \frac{dx}{x}. \quad (14.1)$$

This integral makes sense for $\alpha > 0$. If we restrict α to be a positive integer, we have $\Gamma(n+1) = n!$. This follows from integration by parts; thus, the Γ -function is a generalization of the factorial function. The Γ -function satisfies $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. We have $\Gamma(0) = 1$, or $0! = 1$.

14.2 β -Function

Definition 14.2.1 (β -Function).

$$\beta(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx. \quad (14.2)$$

Converges for $a > 0$ and $b > 0$. Notice that $\beta(a, b) = \beta(b, a)$. Very important in future applications is

Lemma 14.2.2 (Lowering Formula). For $\alpha > 1$,

$$\beta(a, b) = \frac{a-1}{a+b-1} \beta(a-1, b). \quad (14.3)$$

Proof.

$$\beta(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx. \quad (14.4)$$

We integrate by parts, with $x^{a-1} = u$ and $(1-x)^{b-1} dx = dv$. We find

$$\begin{aligned} \beta(a, b) &= -\frac{1}{b} x^{a-1} (1-x)^b \Big|_0^1 + \frac{1}{b} \int_0^1 (a-1) x^{a-2} (1-x)^b dx \\ &= \frac{a-1}{b} \int_0^1 x^{a-2} (1-x)^{b-1} \cdot (1-x) dx \\ &= \frac{a-1}{b} \left[\int_0^1 x^{a-2} (1-x)^{b-1} dx - \int_0^1 x^{a-2} (1-x)^{b-1} dx \right] \\ &= \frac{a-1}{b} [\beta(a-1, b) - \beta(a, b)] \\ \frac{b}{a-1} \beta(a, b) &= \beta(a-1, b) - \beta(a, b) \\ \frac{a+b-1}{a-1} \beta(a, b) &= \beta(a-1, b) \\ \beta(a, b) &= \frac{a-1}{a+b-1} \beta(a-1, b). \end{aligned} \quad (14.5)$$

□

If we switch a and b we immediately obtain

Corollary 14.2.3. For $\beta > 1$,

$$\beta(a, b) = \frac{b-1}{a+b-1} \beta(a, b-1). \quad (14.6)$$

If we keep applying formulas like the above, we eventually obtain

$$\beta(a, b) = \frac{(a-1)(a-2)\cdots(2)}{(a+b-1)(a+b-2)\cdots(b+1)}\beta(1, b) \quad (14.7)$$

However, as $\beta(1, b) = \frac{1}{b}$, yielding

$$\beta(a, b) = \frac{(a-1)!(b-1)!}{(a+b-1)!}. \quad (14.8)$$

Multiplying both sides by a (where we are assuming a, b are integers) gives us that

$$\begin{aligned} a\beta(a, b) &= \frac{a!(b-1)!}{(a+b-1)!} \\ &= \frac{1}{\binom{a+b-1}{a}}. \end{aligned} \quad (14.9)$$

This gives a generalization of the binomial coefficients:

$$\binom{x}{y} = \frac{1}{yB(y, x-y+1)}. \quad (14.10)$$

We can now generalize Pascal's Triangle:

$$\begin{array}{ccccccc} & & & & & & 1 \\ & & & & & & 1 & 1 \\ & & & & & & 1 & 2 & 1 \\ & & & & & & 1 & 3 & 3 & 1 \\ & & & & & & 1 & 4 & 6 & 4 & 6 & 1 \end{array} \quad (14.11)$$

or

$$\begin{array}{ccc} \binom{0}{0} \\ \binom{1}{0} & \binom{1}{1} \\ \binom{2}{0} & \binom{2}{1} & \binom{2}{2} \end{array} \quad (14.12)$$

Note that the first diagonal of Pascal's triangle is all 1s. The next diagonal is 1, 2, 3, 4, 5, and so on. Thus, it looks like x . Using the generalization of the

Binomial coefficients to $\binom{x}{y} = \frac{1}{y\beta(y, x-y+1)}$, we see that $\binom{x}{1} = x$. Now looking at the third diagonal, which starts 1, 3, 6, 10, and so on. This is $\frac{x(x+1)}{2}$, and when we study $\binom{x}{2}$ we see something similar. Looking at $\binom{x}{b}$ where b is a half-integer yields interesting patterns. For example, b equals 2 has one zero, b equals 3 has two zeros; if we take $b = \frac{5}{2}$, we get 2 zeros, and if we take $b = \frac{7}{2}$ we get three zeros.

We can rewrite, using the relation

$$\beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \quad (14.13)$$

Using this, we can re-write the binomial coefficient generalization as

$$\binom{x}{y} = \frac{\Gamma(x+1)}{\Gamma(y+1)}\Gamma(x-y+1). \quad (14.14)$$

We don't have Pascal's Triangle if we don't have addition between entries. The great result is that the addition holds for the continuous analogue as well!

14.3 Catalan Numbers

Start with

$$11 \quad (14.15)$$

We then write again underneath

$$\begin{array}{c} 11 \\ 11 \end{array} \quad (14.16)$$

And continue in some sense. A better definition (one we remember) is

$$c_n = \frac{1}{n+1} \binom{2n}{n}. \quad (14.17)$$

Definition 14.3.1 (Non-Crossing Partitions). We define a non-crossing partition P of a set S to be a partition into pairs $S_j = \{s_{j_1}, s_{j_2}\}$ such that $s_{j_1} < s_{k_1} < s_{j_2}$ iff $s_{j_1} < s_{k_2} < s_{j_2}$.

Lemma 14.3.2. *The number s_k of non-crossing partitions of $[2k]$ is the k^{th} Catalan number $c_k = \frac{1}{k+1} \binom{2k}{k}$.*

Any non-crossing partition pairs 1 with some even element $2m$, since any element s_{j_1} between 1 and its pair partner must also have s_{j_2} between 1 and its pair partner. The number of pair partitions containing $\{1, 2m\}$ is $s_{m-1}s_{k-m}$: it is determined by a non-crossing partition of the numbers inside $(1, 2m)$ and one of those outside $(1, 2m)$. This gives us the recursion relation $s_k = \sum_{i=0}^{k-1} s_i s_{k-1-i}$ for $k \geq 2$.

Another definition is in terms of legal arrangements of parentheses.

Chapter 15

Thursday, March 4th, 2004: Continued Fractions related to Elliptic Functions

Lecturer: Eric Conrad

15.1 Elliptic Functions

Lots of ways to tackle elliptic functions – we will follow an approach from the early 19th century. What is an elliptic function? One definition is a trigonometric functions of Jacobi's amplitude function. Equivalent to a definition by Liouville (meromorphic doubly periodic single valued). Jacobi showed equivalent.

Elliptic functions were functions related to what is now called elliptic integrals. Legendre studied these functions extensively. He showed that any elliptic integral (won't say exactly what this is: an integral of a rational function of a cubic or quartic) can be written in terms of elementary integrals plus three kinds of elliptic integrals. One kind, which he called the first kind, $F(x, k)$:

$$F(x, k) = \int_0^x \frac{dx}{\sqrt{(1-s^2)(1-k^2s^2)}}, \quad (15.1)$$

(technically, an incomplete integral of the first kind) is the most important type. Normally $0 < k < 1$ (if $k > 1$, change $s \rightarrow \frac{s}{k}$; thus it suffices to take $k < 1$).

More specifically, we can consider

$$F(x, a, b) = \int_0^x \frac{dx}{\sqrt{(1 - a^2 s^2)(1 - b^2 s^2)}}, \quad (15.2)$$

and by simply changing s we can reduce to $F(x, k)$.

He had two students in correspondence (Jacobi, Abel). If we cover up the second factor, it is an arcsin. If we make $k = \pm 1$, it is a perfect square, and simple to evaluate (inverse hyperbolic tangent, or can change variables a bit and get an inverse tangent). Would you prefer to work with tangent or inverse tangents, or sine or arcsine? Easier to work with the formers than the latters. For these, we work with the inverses of these integrals.

Jacobi defines the amplitude function, does a trig substitution in this, Some technicalities in inverting. What happens when we invert? Jacobi obtained three functions; we'll use slightly different notation.

$$s = \operatorname{sn}(u, a, b), \quad (15.3)$$

where sn is the sinus amplitudinus (sine of the amplitude); Jacobi's notation was $\operatorname{sn} \operatorname{am}(u, k)$.

It is helpful to define two more functions. Remember we have classically that $1 - \sin^2 \theta = \cos^2 \theta$. Thus, analogously, we let

$$c = \operatorname{cn}(u, a, b) \quad (15.4)$$

and

$$d = \operatorname{dn}(u, a, b). \quad (15.5)$$

We have

$$\operatorname{dn} = \Delta \operatorname{am}, \quad \Delta(s) = \sqrt{1 - k^2 s^2}. \quad (15.6)$$

These are related through a birth-death process. We have variable u and parameters a, b :

$$\begin{aligned} \frac{d \operatorname{sn}(u, a, b)}{du} &= c \cdot \operatorname{dn}(u, a, b) \\ \frac{d \operatorname{cn}(u, a, b)}{du} &= -a^2 s \cdot \operatorname{dn}(u, a, b) \\ \frac{d \operatorname{dn}(u, a, b)}{du} &= -b^2 s \cdot \operatorname{cn}(u, a, b). \end{aligned} \quad (15.7)$$

Jacobi uses k , which corresponds to $a = 1, b = k$.

We need initial conditions for our differential equations. We have

$$\operatorname{sn}(0) = 0, \quad \operatorname{cn}(0) = 1, \quad \operatorname{dn}(0) = 1. \quad (15.8)$$

Using the initial conditions and the differential equations, we can expand in a Maclaurin series.

The functions have quarter-periods K, iK' given by

$$K(k) = F(1, k); \quad (15.9)$$

this is called a complete elliptic integral of the first kind. Further,

$$K'(k) = K(k') = K(\sqrt{1 - k^2}), \quad (15.10)$$

where k' is the complementary modulus to k .

The name of these integrals come from finding arc lengths of curves (ellipses, lemniscates, and so on).

15.2 Maclaurin - Taylor Series

Now that we have Maclaurin series, we go to our toolbox which includes the Laplace Transform:

$$\mathcal{L}\{f(t), s\} = \int_0^{\infty} f(t)e^{-st} dt. \quad (15.11)$$

We will use a variant: $\mathcal{L}\{f(u), x^{-1}\}$. Formal integration gives

$$\mathcal{L}\left\{\sum_{n=0}^{\infty} \frac{a_n u^n}{n!}, x^{-1}\right\} = x \sum_{n=0}^{\infty} a_n x^n. \quad (15.12)$$

Issues of convergence: there is a norm which justifies this (the formal power series norm).

15.3 Example One: From IVP to Pythagorean Theorem

it is very useful to have a Pythagorean Theorem analogue. We have

$$\operatorname{cn}(u, a, b)^2 = 1 - a^2 \cdot \operatorname{sn}(u, a, b)^2. \quad (15.13)$$

Start by showing $c^2 + a^2s^2$ is constant (differentiate). Then using the initial values show the constant is one. Can get two more Pythagorean theorems. Thus, higher powers of $\text{cn}(u, a, b)$ can be replaced with $\text{sn}(u, a, b)$. The other important one is

$$\text{dn}(u, a, b)^2 = 1 - b^2 \cdot \text{sn}(u, a, b)^2. \quad (15.14)$$

Lots of Laplace transforms of elliptic functions. Consider $\mathcal{L} \{ \text{cn}(u, a, b), x^{-1} \}$: what is this equal to? We establish recurrences.

Let

$$\begin{aligned} C_n &= \mathcal{L} \{ \text{cn}(u, a, b) \text{sn}(u, a, b)^n, x^{-1} \} \\ D_n &= \mathcal{L} \{ \text{dn}(u, a, b) \text{sn}(u, a, b)^n, x^{-1} \}. \end{aligned} \quad (15.15)$$

We want $C_0 = \mathcal{L} \{ \text{cn}(u, a, b), x^{-1} \}$. If we integrate by parts, we obtain

$$C_0 = x - a^2 x D_1. \quad (15.16)$$

Look at the recurrences. Each time we get a new letter, we get an x . What does the x do? Pushes us further along in the Laplace transform power series. The formal power series norm: find the first coefficient with a disagreement between two terms, and take 2 to the negative of that power. We are pushes the x s off to infinity one at a time, and we get convergence in the sense of our norm.

Thus, C_0 is our start on the Laplace transform. The next thing we need is D_1 , which can be defined in terms of C_0 and C_2 . We already have a relation for C_0 ; C_2 can be obtained through D_1 and D_3 . We have three term recurrences. We find

$$\begin{aligned} C_0 &= x + a^2 x D_1 \\ C_n &= n x D_{n-1} + (n+1) a^2 x D_{n+1} \\ D_n &= n x C_{n-1} + (n+1) b^2 x C_{n+1}. \end{aligned} \quad (15.17)$$

Three term recurrences: should think continued fractions. We have

$$\begin{aligned} C_1 &= x D_0 - 2a^2 x D_2 \\ x D_0 &= C_1 + 2a^2 x D_2 \\ \frac{x D_0}{C_1} &= 1 + 2a^2 x \frac{D_2}{C_1} \\ \frac{C_1}{x D_0} &= \frac{1}{1 + 2a^2 x D_2 / C_1} \\ \frac{C_1}{D_0} &= \frac{x}{1 + 2a^2 x D_2 / C_1}, \end{aligned} \quad (15.18)$$

which sets us up for a continued fraction solution. Similar recurrence for $\frac{D_2}{C_1}$, and so on. We can write as

$$\frac{C_n}{D_{n-1}} = \frac{nx}{1 - (n+1)a^2x D_{n+1}/C_n}, \dots \quad (15.19)$$

We end up with the continued fraction

$$C_0 = \frac{x}{1 + \frac{1a^2x^2}{1 + \frac{4b^2x^2}{1 + \frac{9a^2x^2}{1 + \frac{16b^2x^2}{1 + \dots}}}}} \quad (15.20)$$

Stieltjes had formulas for $\mathcal{L}\{\text{sn}(u, k), x^{-1}\}$, $\mathcal{L}\{\text{cn}(u, k), x^{-1}\}$, $\mathcal{L}\{\text{dn}(u, k), x^{-1}\}$. Can also do Laplace transforms on

$$\text{sc}(u, k) = \frac{\text{sn}(u, k)}{\text{cn}(u, k)}, \quad (15.21)$$

which is an analogue of tangent (notation due to Glaisher). Using a modular transformation, one finds after taking the Laplace transform that we have something nice. In fact, it is related to sn at another lattice.

Chapter 16

Thursday, March th, 2004:

Lecturer:

$$\sqrt{2} = \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}} \quad (16.1)$$

Bibliography

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