Abstract

1. We prove Liouville’s Theorem for the order of approximation by rationals of real algebraic numbers.

2. We construct several transcendental numbers.

3. We define Poissonian Behaviour, and study the spacings between the ordered fractional parts of \( \{n^k \alpha \} \).

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Chapter 1

Liouville's Theorem Constructing Transcendentals

1.1 Review of Approximating by Rationals

Definition 1.1.1 (Approximated by rationals to order $n$). A real number $x$ is approximated by rationals to order $n$ if there exist a constant $k(x)$ (possibly depending on $x$) such that there are infinitely many rational $\frac{p}{q}$ with

$$\left| x - \frac{p}{q} \right| < \frac{k(x)}{q^n}. \quad (1.1)$$

Recall that Dirichlet’s Box Principle gives us:

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2} \quad (1.2)$$

for infinitely many fractions $\frac{p}{q}$. This was proved by choosing a large parameter $Q$, and considering the $Q + 1$ fractionary parts $\{qx\} \in [0, 1)$ for $q \in \{0, \ldots, Q\}$. The box principle ensures us that there must be two different $q$’s, say:

$$0 \leq q_1 < q_2 \leq Q \quad (1.3)$$

such that both $\{q_1x\}$ and $\{q_2x\}$ belong to the same interval $[\frac{a}{Q}, \frac{a+1}{Q} \}$, for some $0 \leq a \leq Q - 1$. Note that there are exactly $Q$ such intervals partitioning $[0, 1)$, and $Q + 1$ fractionary parts! Now, the length of such an interval is $\frac{1}{Q}$ so we get
There exist integers $p_1$ and $p_2$ such that
\[
\{q_1 x\} = q_1 x - p, \quad \{q_2 x\} = q_2 x - p.
\] (1.5)

Letting $p = p_2 - p_1$, we find
\[
\left| (q_2 - q_1) x - p \right| \leq \frac{1}{Q}.
\] (1.6)

Let $q = q_2 - q_1$, so $1 \leq q \leq Q$, and the previous equation can be rewritten as
\[
\left| x - \frac{p}{q} \right| < \frac{1}{qQ} \leq \frac{1}{q^2}.
\] (1.7)

Now, letting $Q \to \infty$, we get an infinite collection of rational fractions $\frac{p}{q}$ satisfying the above equation. If this collection contains only finitely many distinct fractions, then one of these fractions, say $\frac{p_0}{q_0}$, would occur for infinitely many choices $Q_k$ of $Q$, thus giving us:
\[
\left| x - \frac{p_0}{q_0} \right| < \frac{1}{qQ_k} \to 0,
\] (1.8)

as $k \to \infty$. This implies that $x = \frac{p_0}{q_0} \in \mathbb{Q}$. So, unless $x$ is a rational number, we can find infinitely many distinct rational numbers $\frac{p}{q}$ satisfying Equation 1.7. This means that any real, irrational number can be approximated to order $n = 2$ by rational numbers.

### 1.2 Liouville’s Theorem

**Theorem 1.2.1 (Liouville’s Theorem).** Let $x$ be a real algebraic number of degree $n$. Then $x$ is approximated by rationals to order at most $n$.

**Proof.** Let
\[
f(X) = a_n X^n + \cdots + a_1 X + a_0
\] (1.9)

be the polynomial with integer coefficients of smallest degree (minimal polynomial) such that $x$ satisfies
\[ f(x) = 0. \quad (1.10) \]

Note that \( \deg x = \deg f \) and the condition of minimality implies that \( f(X) \) is irreducible over \( \mathbb{Z} \). Further, a well known result from algebra states that a polynomial irreducible over \( \mathbb{Z} \) is also irreducible over \( \mathbb{Q} \).

In particular, as \( f(X) \) is irreducible over \( \mathbb{Q} \), \( f(X) \) does not have any rational roots. If it did, then \( f(X) \) would be divisible by a linear polynomial \( (X - \frac{a}{b}) \). Let \( G(X) = \frac{f(X)}{X - \frac{a}{b}} \). Clear denominators (multiply throughout by \( b \)), and let \( g(X) = bG(X) \). Then \( \deg g = \deg f - 1 \), and \( g(x) = 0 \). This contradicts the minimality of \( f \) (we choose \( f \) to be a polynomial of smallest degree such that \( f(x) = 0 \)). Therefore, \( f \) is non-zero at every rational.

Let

\[ M = \sup_{|z - x| < 1} |f'(z)|. \quad (1.11) \]

Let now \( \frac{p}{q} \) be a rational such that \( |x - \frac{p}{q}| < 1 \). The Mean Value Theorem gives us that

\[ \left| f \left( \frac{p}{q} \right) - f(x) \right| = \left| f'(c) \left( x - \frac{p}{q} \right) \right| \leq M \left| x - \frac{p}{q} \right| \quad (1.12) \]

where \( c \) is some real number between \( x \) and \( \frac{p}{q} \); \( |c - x| < 1 \) for \( \frac{p}{q} \) moderately close to \( x \).

Now we use the fact that \( f(X) \) does not have any rational roots:

\[ 0 \neq f \left( \frac{p}{q} \right) = a_n \left( \frac{p}{q} \right)^n + \cdots + a_0 = \frac{a_n p^n + \cdots a_1 p^{n-1} q + a_0 q^n}{q^n} \quad (1.13) \]

The numerator of the last term is a nonzero integer, hence it has absolute value at least 1. Since we also know that \( f(x) = 0 \) it follows that

\[ \left| f \left( \frac{p}{q} \right) - f(x) \right| = \left| f \left( \frac{p}{q} \right) \right| = \frac{|a_n p^n + \cdots a_1 p^{n-1} q + a_0 q^n|}{q^n} \geq \frac{1}{q^n}. \quad (1.14) \]

Combining the equations 1.12 and 1.14, we get:

\[ \frac{1}{q^n} \leq M \left| x - \frac{p}{q} \right| \Rightarrow \frac{1}{MQ^n} \leq \left| x - \frac{p}{q} \right| \quad (1.15) \]
whenever \(|x - \frac{p}{q}| < 1\). This last equation shows us that \(x\) can be approximated by rationals to order at most \(n\). For assume it was otherwise, namely that \(x\) can be approximated to order \(n + \epsilon\). Then we would have an infinite sequence of distinct rational numbers \(\{\frac{p_i}{q_i}\}_{i \geq 1}\) and a constant \(k(x)\) depending only on \(x\) such that

\[
\left| x - \frac{p_i}{q_i} \right| < \frac{k(x)}{q_i^{n+\epsilon}}. \tag{1.16}
\]

Since the numbers \(\frac{p_i}{q_i}\) converge to \(x\) we can assume that they already are in the interval \((x - 1, x + 1)\). Hence they also satisfy Equation 1.15:

\[
\frac{1}{q_i^n} \leq M \left| x - \frac{p_i}{q_i} \right|. \tag{1.17}
\]

Combining the last two equations we get

\[
\frac{1}{Mq_i^n} \leq \left| x - \frac{p_i}{q_i} \right| < \frac{k(x)}{q_i^{n+\epsilon}}, \tag{1.18}
\]

hence

\[
q_i^\epsilon < M \tag{1.19}
\]

and this is clearly impossible for arbitrarily large \(q\) since \(\epsilon > 0\) and \(q_i \to \infty\).

\[\square\]

Exercise 1.2.2. Justify the fact that if \(\{\frac{p_i}{q_i}\}_{i \geq 1}\) is a rational approximation to order \(n \geq 1\) of \(x\), then \(q_i \to \infty\).

Remark 1.2.3. So far we have seen that the order to which an algebraic number can be approximated by rationals is bounded by its degree. Hence if a real, irrational number \(\alpha \notin \mathbb{Q}\) can be approximated by rationals to an arbitrary large order, then \(\alpha\) must be transcendental! This provides us with a recipe for constructing transcendental numbers.

1.3 Constructing Transcendental Numbers

1.3.1 \(\sum_m 10^{-m!}\)

The following construction of transcendental numbers is due to Liouville.
Theorem 1.3.1. The number

\[ x = \sum_{m=1}^{\infty} \frac{1}{10^m!} \]  

is transcendental.

Proof. The series defining \( x \) is convergent, since it is dominated by the geometric series \( \sum \frac{1}{10^m} \). In fact, the series converges very rapidly and it is this high rate of convergence that will yield \( x \) is transcendental.

Fix \( N \) large, and let \( n > N \). Write

\[ \frac{p_n}{q_n} = \sum_{m=1}^{n} \frac{1}{10^m!} \]  

(1.21)

with \( p_n, q_n > 0 \) and \( (p_n, q_n) = 1 \). Then \( \{\frac{p_n}{q_n}\}_{n \geq 1} \) is a monotone increasing sequence converging to \( x \). In particular, all these rational numbers are distinct. Not also that \( q_n \) must divide \( 10^n! \), which implies

\[ q_n \leq 10^n! \]  

(1.22)

Using this, we get

\[ 0 < x - \frac{p_n}{q_n} = \sum_{m>n} \frac{1}{10^m!} = \frac{1}{10^{(n+1)!}} \left( 1 + \frac{1}{10^{n+2}} + \frac{1}{10^{(n+2)(n+3)}} + \cdots \right) \]

\[ < \frac{2}{10^{(n+1)!}} = \frac{2}{(10^n!)^{n+1}} \]

\[ < \frac{2}{q_{n+1}^{n+1}} \leq \frac{2}{q_n^N} \]  

(1.23)

This gives an approximation by rationals of order \( N \) of \( x \). Since \( N \) can be chosen arbitrarily large, this implies that \( x \) can be approximated by rationals to arbitrary order. We can conclude, in view of our precious remark 1.2.3 that \( x \) is transcendental.

\[ 1.3.2 \quad [10^{1!}, 10^{2!}, \ldots] \]

Theorem 1.3.2. The number
\[ y = [10^1!, 10^2!, \ldots] \]  

is transcendental.

**Proof.** Let \( \frac{p_n}{q_n} \) be the continued fraction of \([10^1! \cdot 10^n!]\). Then

\[
\left| \frac{y - p_n}{q_n} \right| = \frac{1}{q_nq'_{n+1}} = \frac{1}{q_n(a'_{n+1}q_n + q_n-1)}
\]

\[
< \frac{1}{a_{n+1}} = \frac{1}{10(n+1)!}.
\]

Since \( q_k = a_nq_{k-1} + q_{k-2} \), it implies that \( q_k > q_{k-1} \) Also, \( q_{k+1} = a_{k+1}q_n + q_k-1 \), so we get

\[
\frac{q_{k+1}}{q_k} = a_{k+1} + \frac{q_k-1}{q_k} < a_{k+1} + 1.
\]

Hence writing this inequality for \( k = 1, \ldots, n \) we obtain

\[
q_n = \frac{q_1q_2q_3 \ldots q_n}{q_1q_2 \ldots q_{n-1}} < (a_1 + 1)(a_2 + 1) \cdots (a_n + 1)
\]

\[
= (1 + \frac{1}{a_1}) \cdots (1 + \frac{1}{a_n})a_1 \cdots a_n
\]

\[
< 2^n a_1 \cdots a_n = 2^n 10^1! \cdot \cdots \cdot n!
\]

\[
< 10^{2n!} = a_n^2
\]

Combining equations 1.25 and 1.27 we get:

\[
\left| \frac{y - p_n}{q_n} \right| < \frac{1}{a_{n+1}} = \frac{1}{a_n^{n+1}}
\]

\[
< \left( \frac{1}{a_n^2} \right)^{\frac{n}{2}} < \left( \frac{1}{q_n^2} \right)^{\frac{n}{2}}
\]

\[
= \frac{1}{q_n^{n/2}}.
\]

In this way we get, just as in the previous theorem, an approximation of \( y \) by rationals to arbitrary order. This proves that \( y \) is transcendental.

\( \square \)
1.3.3 Buffon’s Needle and $\pi$

Consider a collection of infinitely long parallel lines in the plane, where the spacing between any two adjacent lines is $d$. Let the lines be located at $x = 0, \pm d, \pm 2d, \ldots$. Consider a rod of length $l$, where for convenience we assume $l < d$.

If we were to randomly throw the rod on the plane, what is the probability it hits a line? This question was first asked by Buffon in 1733.

Because of the vertical symmetry, we may assume the center of the rod lies on the line $x = 0$, as shifting the rod (without rotating it) up or down will not alter the number of intersections. By the horizontal symmetry, we may assume $-\frac{d}{2} \leq x < \frac{d}{2}$. We posit that all values of $x$ are equally likely. As $x$ is continuous distributed, we may add in $x = \frac{d}{2}$ without changing the probability. The probability density function of $x$ is $\frac{dx}{d}$.

Let $\theta$ be the angle the rod makes with the $x$-axis. As each angle is equally likely, the probability density function of $\theta$ is $\frac{d\theta}{2\pi}$.

We assume that $x$ and $\theta$ are chosen independently. Thus, the probability density for $(x, \theta)$ is $\frac{dx\,d\theta}{d\,2\pi}$.

The projection of the rod (making an angle of $\theta$ with the $x$-axis) along the $x$-axis is $l \cdot |\cos \theta|$. If $|x| \leq l \cdot |\cos \theta|$, then the rod hits exactly one vertical line exactly once; if $x > l \cdot |\cos \theta|$, the rod does not hit a vertical line. Note that if $l > d$, a rod could hit multiple lines, making the arguments more involved.

Thus, the probability a rod hits a line is

$$p = \int_{\theta=0}^{2\pi} \int_{x=-l \cdot |\cos \theta|}^{l \cdot |\cos \theta|} \frac{dx\,d\theta}{d \cdot 2\pi}$$

$$= \int_{\theta=0}^{2\pi} \frac{l \cdot |\cos \theta|}{d} \frac{d\theta}{2\pi}$$

$$= \frac{2l}{\pi d}.$$  \hspace{1cm} (1.29)

**Exercise 1.3.3.** Show

$$\frac{1}{2\pi} \int_{0}^{2\pi} |\cos \theta| \, d\theta = \frac{2}{\pi}. \hspace{1cm} (1.30)$$

Let $A$ be the random variable which is the number of intersections of a rod of length $l$ thrown against parallel vertical lines separated by $d > l$ units. Then
\[
A = \begin{cases} 
1 & \text{with probability } \frac{2l}{\pi d}, \\
0 & \text{with probability } 1 - \frac{2l}{\pi d}.
\end{cases} \quad (1.31)
\]

If we were to throw \( N \) rods independently, since the expected value of a sum is the sum of the expected values (Lemma ??), we expect to observe

\[
I = N \cdot \frac{2l}{\pi d} \quad (1.32)
\]

intersections.

Turning this around, let us throw \( N \) rods, and let \( I \) be the number of observed intersections of the rods with the vertical lines. Then

\[
I \approx N \cdot \frac{2l}{\pi d} \quad \Rightarrow \quad \pi \approx \frac{N}{I} \cdot \frac{2l}{d}. \quad (1.33)
\]

The above is an experimental formula for \( \pi \)!
Chapter 2

Poissonian Behavior and \( \{n^k \alpha\} \)

2.1 Equidistribution

We say a sequence of number \( x_n \in [0, 1) \) is equidistributed if

\[
\lim_{N \to \infty} \frac{\#\{n : 1 \leq n \leq N \text{ and } x_n \in [a, b]\}}{N} = b - a \tag{2.1}
\]

for any subinterval \([a, b]\) of \([0, 1]\).

Recall Weyl’s Result, Theorem 2.1: If \( \alpha \not\in \mathbb{Q} \), then the fractional parts \( \{n\alpha\} \)
are equidistributed. Equivalently, \( n\alpha \mod 1 \) is equidistributed.

Similarly, one can show that for any integer \( k \), \( \{n^k \alpha\} \) is equidistributed. See Robert Lipshitz’s paper for more details.

2.2 Point Masses and Induced Probability Measures

Recall from physics the concept of a unit point mass located at \( x = a \). Such a point mass has no length (or, in higher dimensions, width or height), but finite mass. As mass is the integral of the density over space, a finite mass in zero volume (or zero length on the line) implies an infinite density.

We can make this more precise by the notion of an Approximation to the Identity. See also Theorem 2.2.

Definition 2.2.1 (Approximation to the Identity). A sequence of functions \( g_n(x) \) is an approximation to the identity (at the origin) if

1. \( g_n(x) \geq 0 \),
2. $\int g_n(x)dx = 1$.

3. Given $\epsilon, \delta > 0$ there exists $N > 0$ such that for all $n > N$, $\int_{|x|>\delta} g_n(x)dx < \epsilon$.

We represent the limit of any such family of $g_n(x)s$ by $\delta(x)$.

If $f(x)$ is a nice function (say near the origin its Taylor Series converges) then

$$\int f(x)\delta(x)dx = \lim_{n \to \infty} \int f(x)g_n(x) = f(0). \quad (2.2)$$

**Exercise 2.2.2.** Prove Equation 2.2.

Thus, in the limit the functions $g_n$ are acting like point masses. We can consider the probability densities $g_n(x)dx$ and $\delta(x)dx$. For $g_n(x)dx$, as $n \to \infty$, almost all the probability is concentrated in a narrower and narrower band about the origin; $\delta(x)dx$ is the limit with all the mass at one point. It is a discrete (as opposed to continuous) probability measure.

Note that $\delta(x-a)$ acts like a point mass; however, instead of having its mass concentrated at the origin, it is now concentrated at $a$.

**Exercise 2.2.3.** Let

$$g_n(x) = \begin{cases} n & \text{if } |x| \leq \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases} \quad (2.3)$$

Prove $g_n(x)$ is an approximation to the identity at the origin.

**Exercise 2.2.4.** Let

$$g_n(x) = c \frac{\frac{1}{n}}{\frac{1}{n^2}+x^2}. \quad (2.4)$$

Find $c$ such that the above is an approximation to the identity at the origin.

Given $N$ point masses located at $x_1, x_2, \ldots, x_N$, we can form a probability measure

$$\mu_N(x)dx = \frac{1}{N} \sum_{n=1}^{N} \delta(x-x_n)dx. \quad (2.5)$$

Note $\int \mu_N(x)dx = 1$, and if $f(x)$ is a nice function,

$$\int f(x)\mu_N(x)dx = \frac{1}{N} \sum_{n=1}^{N} f(x_n). \quad (2.6)$$
Exercise 2.2.5. Prove Equation 2.6 for nice \( f(x) \).

Note the right hand side of Equation 2.6 looks like a Riemann sum. Or it would look like a Riemann sum if the \( x_n \)'s were equidistributed. In general the \( x_n \)'s will not be equidistributed, but assume for any interval \([a, b]\) that as \( N \to \infty \), the fraction of \( x_n \)’s \((1 \leq n \leq N)\) in \([a, b]\) goes to \( \int_a^b p(x)dx \) for some nice function \( p(x) \):

\[
\lim_{N \to \infty} \frac{\#\{n: 1 \leq n \leq N \text{ and } x_n \in [a, b]\}}{N} \to \int_a^b p(x)dx. \tag{2.7}
\]

In this case, if \( f(x) \) is nice (say twice differentiable, with first derivative uniformly bounded), then

\[
\int f(x)\mu_N(x)dx = \frac{1}{N} \sum_{n=1}^N f(x_n)
\approx \sum_{k=-\infty}^{\infty} f\left(\frac{k}{N}\right) \frac{\#\{n: 1 \leq n \leq N \text{ and } x_n \in \left[\frac{k}{N}, \frac{k+1}{N}\right]\}}{N}
\to \int f(x)p(x)dx. \tag{2.8}
\]

Definition 2.2.6 (Convergence to \( p(x) \)). If the sequence of points \( x_n \) satisfies Equation 2.7 for some nice function \( p(x) \), we say the probability measures \( \mu_N(x)dx \) converge to \( p(x)dx \).

2.3 Neighbor Spacings

We now consider finer questions. Let \( \alpha_n \) be a collection of points in \([0, 1)\). We order them by size:

\[
0 \leq \alpha_{\sigma(1)} \leq \alpha_{\sigma(2)} \leq \cdots \leq \alpha_{\sigma(N)}, \tag{2.9}
\]

where \( \sigma \) is a permutation of \( 123 \cdots N \). Note the ordering depends crucially on \( N \). Let \( \beta_j = \alpha_{\sigma(j)} \).

We consider how the differences \( \beta_{j+1} - \beta_j \) are distributed. We will use a slightly different definition of distance, however.
Recall $[0, 1)$ is equivalent to the unit circle under the map $x \to e^{2\pi ix}$. Thus, the numbers .999 and .001 are actually very close; however, if we used the standard definition of distance, then $|.999 - .001| = .998$, which is quite large. Wrapping $[0, 1)$ on itself (identifying 0 and 1), we see that .999 and .001 are separated by .002.

**Definition 2.3.1 (mod 1 distance).** Let $x, y \in [0, 1)$. We define the mod 1 distance from $x$ to $y$, $||x - y||$, by

$$||x - y|| = \min \{ |x - y|, 1 - |x - y| \}.$$  \hspace{1cm} (2.10)

**Exercise 2.3.2.** Show that the mod 1 distance between any two numbers in $[0, 1)$ is at most $\frac{1}{2}$.

In looking at spacings between the $\beta_j$s, we have $N - 1$ pairs of neighbors:

$$(\beta_2, \beta_1), (\beta_3, \beta_2), \ldots, (\beta_N, \beta_{N-1}).$$ \hspace{1cm} (2.11)

These pairs give rise to spacings $\beta_{j+1} - \beta_j \in [0, 1)$. We can also consider the pair $(\beta_1, \beta_N)$. This gives rise to the spacing $\beta_1 - \beta_N \in [-1, 0)$; however, as we are studying this sequence mod 1, this is equivalent to $\beta_1 - \beta_N + 1 \in [0, 1)$.

Henceforth, whenever we perform any arithmetic operation, we always mean mod 1; thus, our answers always live in $[0, 1)$

**Definition 2.3.3 (Neighbor Spacings).** Given a sequence of numbers $\alpha_n$ in $[0, 1)$, fix an $N$ and arrange the numbers $\alpha_n$ ($n \leq N$) in increasing order. Label the new sequence $\beta_j$; note the ordering will depend on $N$. Let $\beta_{-j} = \beta_{N-j}$ and $\beta_{N+j} = \beta_j$.

1. The nearest neighbor spacings are the numbers $\beta_{j+1} - \beta_j$, $j = 1$ to $N$.
2. The $k^{th}$-neighbor spacings are the numbers $\beta_{j+k} - \beta_j$, $j = 1$ to $N$.

Remember to take the differences $\beta_{j+k} - \beta_j$ mod 1.

**Exercise 2.3.4.** Let $\alpha = \sqrt{2}$, and let $\alpha_n = \{n\alpha\}$ or $\{n^2 \alpha\}$. Calculate the nearest neighbor and the next-nearest neighbor spacings in each case for $N = 10$.

**Definition 2.3.5 (wrapped unit interval).** We call $[0, 1)$, when all arithmetic operations are done mod 1, the wrapped unit interval.
2.4 Poissonian Behavior

Let $\alpha \notin \mathbb{Q}$. Fix a positive integer $k$, and let $\alpha_n = \{nk\alpha\}$. As $N \to \infty$, look at the ordered $\alpha_n$s, denoted by $\beta_n$. How are the nearest neighbor spacings of $\beta_n$ distributed? How does this depend on $k$? On $\alpha$? On $N$?

Before discussing this problem, we consider a simpler case. Fix $N$, and consider $N$ independent random variables $x_n$. Each random variable is chosen from the uniform distribution on $[0, 1)$; thus, the probability that $x_n \in [a, b)$ is $b - a$.

Let $y_n$ be the $x_n$s arranged in increasing order. How do the neighbor spacings behave?

First, we need to decide what is the correct scale to use for our investigations. As we have $N$ objects on the wrapped unit interval, we have $N$ nearest neighbor spacings. Thus, we expect the average spacing to be $\frac{1}{N}$.

**Definition 2.4.1 (Unfolding).** Let $z_n = Ny_n$. The numbers $z_n = Ny_n$ have unit mean spacing. Thus, while we expect the average spacing between adjacent $y_n$s to be $\frac{1}{N}$ units, we expect the average spacing between adjacent $z_n$s to be 1 unit.

So, the probability of observing a spacing as large as $\frac{1}{2}$ between adjacent $y_n$s becomes negligible as $N \to \infty$. What we should ask is what is the probability of observing a nearest neighbor spacing of adjacent $y_n$s that is half the average spacing. In terms of the $z_n$s, this will correspond to a spacing between adjacent $z_n$s of $\frac{1}{2}$ a unit.

2.4.1 Nearest Neighbor Spacings

By symmetry, on the wrapped unit interval the expected nearest neighbor spacing is independent of $j$. Explicitly, we expect $\beta_{j+1} - \beta_j$ to have the same distribution as $\beta_{i+1} - \beta_i$.

What is the probability that, when we order the $x_n$s in increasing order, the next $x_n$ after $x_1$ is located between $\frac{t}{N}$ and $\frac{t+\Delta t}{N}$? Let the $x_n$s in increasing order be labeled $y_1 \leq y_2 \leq \cdots \leq y_N$, $y_N = x_{\sigma(N)}$.

As we are choosing the $x_n$s independently, there are $(N-1)$ choices of subscript $n$ such that $x_n$ is nearest to $x_1$. This can also be seen by symmetry, as each $x_n$ is equally likely to be the first to the right of $x_1$ (where, of course,.001 is just a little to the right of .999), and we have $N - 1$ choices left for $x_n$.

The probability that $x_n \in \left[\frac{t}{N}, \frac{t+\Delta t}{N}\right]$ is $\frac{\Delta t}{N}$. 


For the remaining $N - 2$ of the $x_n$s, each must be further than $\frac{t + \Delta t}{N}$ from $x_n$. Thus, they must all lie in an interval (or possibly two intervals if we wrap around) of length $1 - \frac{t + \Delta t}{N}$. The probability that they all lie in this region is $\left(1 - \frac{t + \Delta t}{N}\right)^{N-2}$.

Thus, if $x_1 = y_l$, we want to calculate the probability that $||y_{l+1} - y_l|| \in \left[\frac{t}{N}, \frac{t + \Delta t}{N}\right]$. This is

$$\text{Prob}\left(||y_{l+1} - y_l|| \in \left[\frac{t}{N}, \frac{t + \Delta t}{N}\right]\right) = \binom{N-1}{1} \cdot \frac{\Delta t}{N} \cdot \left(1 - \frac{t + \Delta t}{N}\right)^{N-2}$$

$$= \left(1 - \frac{1}{N}\right) \cdot \left(1 - \frac{t + \Delta t}{N}\right)^{N-2} \Delta t.$$  

(2.12)

For $N$ enormous and $\Delta t$ small,

$$\left(1 - \frac{1}{N}\right) \approx 1$$

$$\left(1 - \frac{t + \Delta t}{N}\right)^{N-2} \approx e^{-(t+\Delta t)} \approx e^{-t}. \quad (2.13)$$

Thus

$$\text{Prob}\left(||y_{l+1} - y_l|| \in \left[\frac{t}{N}, \frac{t + \Delta t}{N}\right]\right) \to e^{-t} \Delta t. \quad (2.14)$$

**Remark 2.4.2.** The above argument is infinitesimally wrong. Once we’ve located $y_{l+1}$, the remaining $x_n$s do not need to be more than $\frac{t + \Delta t}{N}$ units to the right of $x_1 = y_l$; they only need to be further to the right than $y_{l+1}$. As the incremental gain in probabilities for the locations of the remaining $x_n$s is of order $\Delta t$, these contributions will not influence the large $N$, small $\Delta t$ limits. Thus, we ignore these effects.

To rigorously derive the limiting behavior of the nearest neighbor spacings using the above arguments, one would integrate over $x_m$ ranging from $\frac{t}{N}$ to $\frac{t + \Delta t}{N}$, and the remaining events $x_n$ would be in the a segment of length $1 - x_m$. As

$$\left|\left(1 - x_m\right) - \left(1 - \frac{t + \Delta t}{N}\right)\right| \leq \frac{\Delta t}{N},$$

(2.15)

this will lead to corrections of higher order in $\Delta t$, hence negligible. We can rigorously avoid this by instead considering the following:
1. Calculate the probability that all the other $x_n$s are at least $\frac{t}{N}$ units to the right of $x_1$. This is

$$p_t = \left(1 - \frac{t}{N}\right)^{N-1} \to e^{-t}. \quad (2.16)$$

2. Calculate the probability that all the other $x_n$s are at least $\frac{t + \Delta t}{N}$ units to the right of $x_1$. This is

$$p_{t+\Delta t} = \left(1 - \frac{t + \Delta t}{N}\right)^{N-1} \to e^{-(t + \Delta t)}. \quad (2.17)$$

3. The probability that no $x_n$s are within $\frac{t}{N}$ units to the right of $x_1$ but at least one $x_n$ is between $\frac{t}{N}$ and $\frac{t + \Delta t}{N}$ units to the right is $p_{t+\Delta t} - p_t$:

$$p_t - p_{t+\Delta t} \to e^{-t} - e^{-(t + \Delta t)}$$

$$= e^{-t} \left(1 - e^{-\Delta t}\right)$$

$$= e^{-t} \left(1 - 1 + \Delta t + O\left((\Delta t)^2\right)\right)$$

$$\to e^{-t} \Delta t. \quad (2.18)$$

**Definition 2.4.3 (Unfolding Spacings).** If $y_{i+1} - y_i \in \left[\frac{t}{N}, \frac{t + \Delta t}{N}\right]$, then $N(y_{i+1} - y_i) \in [t, t + \Delta t]$. The new spacings $z_{i+1} - z_i$ have unit mean spacing. Thus, while we expect the average spacing between adjacent $y_n$s to be $\frac{1}{N}$ units, we expect the average spacing between adjacent $z_n$s to be 1 unit.

### 2.4.2 $k^{\text{th}}$ Neighbor Spacings

Similarly, one can easily analyze the distribution of the $k^{\text{th}}$ neighbor spacings when each $x_n$ is chosen independently from the uniform distribution on $[0, 1]$.

Again, consider $x_1 = y_i$. Now we want to calculate the probability that $y_{i+k}$ is between $\frac{t}{N}$ and $\frac{t + \Delta t}{N}$ units to the right of $y_i$.

Therefore, we need exactly $k - 1$ of the $x_n$s to lie between 0 and $\frac{t}{N}$ units to the right of $x_1$, exactly one $x_n$ (which will be $y_{i+k}$) to lie between $\frac{t}{N}$ and $\frac{t + \Delta t}{N}$ units to the right of $x_1$, and the remaining $x_n$s to lie at least $\frac{t + \Delta t}{N}$ units to the right of $y_{i+k}$. 


Remark 2.4.4. We face the same problem discussed in Remark 2.4.2; a similar argument will show that ignoring these affects will not alter the limiting behavior. Therefore, we will make these simplifications.

There are \( \binom{N-1}{k-1} \) ways to choose the \( x_n \)s that are at most \( \frac{t}{N} \) units to the right of \( x_1 \); there is then \( \binom{N-1-(k-1)}{1} \) ways to choose the \( x_n \) between \( \frac{t}{N} \) and \( \frac{t+\Delta t}{N} \) units to the right of \( x_1 \).

Thus,

\[
\text{Prob}\left( ||y_{t+k} - y_t|| \in \left[ \frac{t}{N}, \frac{t+\Delta t}{N} \right] \right) =
\]

\[
= \frac{N-1}{(k-1)} \left( \frac{t}{N} \right)^{k-1} \cdot \frac{(N-1)-(k-1)}{N} \frac{\Delta t}{N} \cdot \left( 1 - \frac{t+\Delta t}{N} \right)^{N-(k+1)}
\]

\[
= \frac{(N-1) \cdots (N-1-(k-2))}{N^{k-1}} \frac{(N-1)-(k-1)}{N} \frac{t^{k-1}}{(k-1)!} \left( 1 - \frac{t+\Delta t}{N} \right)^{N-(k+1)} \Delta t
\]

\[
\rightarrow \frac{t^{k-1}}{(k-1)!} e^{-t} \Delta t.
\] (2.19)

Again, one way to avoid the complications is to integrate over \( x_m \) ranging from \( \frac{t}{N} \) to \( \frac{t+\Delta t}{N} \).

Or, similar to before, we can proceed more rigorously as follows:

1. Calculate the probability that exactly \( k - 1 \) of the other \( x_n \)s are at most \( \frac{t}{N} \) units to the right of \( x_1 \), and the remaining \( (N-1)-(k-1) \) of the \( x_n \)s are at least \( \frac{t}{N} \) units to the right of \( x_1 \). As there are \( \binom{N-1}{k-1} \) ways to choose \( k - 1 \) of the \( x_n \)s to be at most \( \frac{t}{N} \) units to the right of \( x_1 \), this probability is

\[
p_t = \frac{(N-1)}{(k-1)} \left( \frac{t}{N} \right)^{k-1} \left( 1 - \frac{t}{N} \right)^{(N-1)-(k-1)}
\]

\[
\rightarrow \frac{N^{k-1}}{(k-1)!} \left( \frac{t}{N} \right)^{k-1} e^{-t}
\]

\[
\rightarrow \frac{t^{k-1}}{(k-1)!} e^{-t}.
\] (2.20)
2. Calculate the probability that exactly \( k - 1 \) of the other \( x_n \)'s are at most \( \frac{t}{N} \) units to the right of \( x_1 \), and the remaining \( (N - 1) - (k - 1) \) of the \( x_n \)'s are at least \( \frac{t + \Delta t}{N} \) units to the right of \( x_1 \). Similar to the above, this gives

\[
p_t = \binom{N - 1}{k - 1} \left( \frac{t}{N} \right)^{k-1} \left( 1 - \frac{t + \Delta t}{N} \right)^{(N-1)-(k-1)}
\]

\[
\rightarrow \frac{N^{k-1}}{(k-1)!} \frac{t^{k-1} e^{-(t+\Delta t)}}{N^{k-1}}
\]

\[
\rightarrow \frac{t^{k-1}}{(k-1)!} e^{-(t+\Delta t)}. \tag{2.21}
\]

3. The probability that exactly \( k - 1 \) of the \( x_n \)'s are within \( \frac{t}{N} \) units to the right of \( x_1 \) and at least one \( x_n \) is between \( \frac{t}{N} \) and \( \frac{t + \Delta t}{N} \) units to the right is \( p_{t + \Delta t} - p_t \):

\[
p_t - p_{t + \Delta t} \rightarrow \frac{t^{k-1}}{(k-1)!} e^{-t} - \frac{t^{k-1}}{(k-1)!} e^{-(t+\Delta t)} \rightarrow \frac{t^{k-1}}{(k-1)!} e^{-t} \Delta t. \tag{2.22}
\]

Note that when \( k = 1 \), we recover the nearest neighbor spacings.

### 2.5 Induced Probability Measures

We have proven the following:

**Theorem 2.5.1.** Consider \( N \) independent random variables \( x_n \) chosen from the uniform distribution on the wrapped unit interval \([0, 1)\). For fixed \( N \), arrange the \( x_n \)'s in increase order, labeled \( y_1 \leq y_2 \leq \cdots \leq y_N \).

Form the induced probability measure \( \mu_{N,1} \) from the nearest neighbor spacings. Then as \( N \to \infty \) we have

\[
\mu_{N,1}(t) dt = \frac{1}{N} \sum_{n=1}^{N} \delta \left( t - N (y_n - y_{n-1}) \right) dt \to e^{-t} dt. \tag{2.23}
\]

Equivalently, using \( z_n = Ny_n \):

\[
\mu_{N,1}(t) dt = \frac{1}{N} \sum_{n=1}^{N} \delta \left( t - (z_n - z_{n-1}) \right) dt \to e^{-t} dt. \tag{2.24}
\]
More generally, form the probability measure from the $k^{th}$ nearest neighbor spacings. Then as $N \to \infty$ we have

$$
\mu_{N,k}(t) \, dt = \frac{1}{N} \sum_{n=1}^{N} \delta \left( t - N(y_n - y_{n-1}) \right) dt \to \frac{t^{k-1}}{(k-1)!} e^{-t} dt. \tag{2.25}
$$

Equivalently, using $z_n = Ny_n$:

$$
\mu_{N,k}(t) \, dt = \frac{1}{N} \sum_{n=1}^{N} \delta \left( t - (z_n - z_{n-1}) \right) dt \to \frac{t^{k-1}}{(k-1)!} e^{-t} dt. \tag{2.26}
$$

**Definition 2.5.2 (Poissonian Behavior).** We say a sequence of points $x_n$ has Poissonian Behavior if in the limit as $N \to \infty$ the induced probability measures $\mu_{N,k}(t) \, dt$ converge to $\frac{t^{k-1}}{(k-1)!} e^{-t} dt$.

**Exercise 2.5.3.** Let $\alpha \in \mathbb{Q}$, and define $\alpha_n = \{n^m\alpha\}$ for some positive integer $m$. Show the sequence of points $\alpha_n$ does not have Poissonian Behavior.

**Exercise 2.5.4.** Let $\alpha \notin \mathbb{Q}$, and define $\alpha_n = \{n\alpha\}$. Show the sequence of points $\alpha_n$ does not have Poissonian Behavior. Hint: for each $N$, show the nearest neighbor spacings take on at most three distinct values (the three values depend on $N$). As only three values are ever assumed for a fixed $N$, $\mu_{N,1}(t) \, dt$ cannot converge to $e^{-t} dt$.

2.6 Non-Poissonian Behavior

**Conjecture 2.6.1.** With probability one (with respect to Lebesgue Measure, see Definition ??), if $\alpha \notin \mathbb{Q}$, if $\alpha_n = \{n^2\alpha\}$ then the sequence of points $\alpha_n$ is Poissonian.

There are constructions which show certain irrationals give rise to non-Poissonian behavior.

**Theorem 2.6.2.** Let $\alpha \in \mathbb{Q}$ such that $\left| \alpha - \frac{p_n}{q_n} \right| < \frac{a_n}{q_n^2}$ holds infinitely often, with $a_n \to 0$. Then there exist integers $N_j \to \infty$ such that $\mu_{N_j,1}(t)$ does not converge to $e^{-t} dt$. 
As $a_n \to 0$, eventually $a_n < \frac{1}{10}$ for all $n$ large. Let $N_n = q_n$, where $\frac{p_n}{q_n}$ is a good rational approximation to $\alpha$:

$$|\alpha - \frac{p_n}{q_n}| < \frac{a_n}{q_n^2}.$$  
(2.27)

Remember that all subtractions are performed on the wrapped unit interval. Thus, $||.999 - .001|| = .002$.

We look at $\alpha_k = \{k^2 \alpha\}$, $1 \leq k \leq N_n = q_n$. Let the $\beta_k$s be the $\alpha_k$s arranged in increasing order, and let the $\gamma_k$s be the numbers $\{k^2 \frac{p_n}{q_n}\}$ arranged in increasing order:

$$\beta_1 \leq \beta_2 \leq \cdots \leq \beta_N$$
$$\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_N.$$  
(2.28)

### 2.6.1 Preliminaries

#### Lemma 2.6.3

If $\beta_l = \alpha_k = \{k^2 \alpha\}$, then $\gamma_l = \{k^2 \frac{p_n}{q_n}\}$. Thus, the same permutation orders both the $\alpha_k$s and the $\gamma_k$s.

**Proof.** Multiplying both sides of Equation 2.27 by $\frac{k^2}{q_n^2}$ yields

$$\left|\frac{k^2 \alpha - k^2 \frac{p_n}{q_n}}{q_n^2}\right| < \frac{k^2 a_n}{q_n^2} \leq \frac{a_n}{q_n} < \frac{1}{2q_n}.$$  
(2.29)

Thus, $k^2 \alpha$ and $k^2 \frac{p_n}{q_n}$ differ by at most $\frac{1}{2q_n}$. Therefore

$$\left|\left\{k^2 \alpha\right\} - \left\{k^2 \frac{p_n}{q_n}\right\}\right| < \frac{1}{2q_n}.$$  
(2.30)

As the numbers $\{m^2 \frac{p_n}{q_n}\}$ all have denominators of size at most $\frac{1}{q_n}$, we see that $\{k^2 \frac{p_n}{q_n}\}$ is the closest of the $\{m^2 \frac{p_n}{q_n}\}$ to $\{k^2 \alpha\}$.

This implies that if $\beta_l = \{k^2 \alpha\}$, then $\gamma_l = \{k^2 \frac{p_n}{q_n}\}$, completing the proof.

**Exercise 2.6.4.** Prove the ordering is as claimed. Hint: about each $\beta_l = \{k^2 \alpha\}$, the closest number of the form $\{c^2 \frac{p_n}{q_n}\}$ is $\{k^2 \frac{p_n}{q_n}\}$. 

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2.6.2 Proof of Theorem 2.6.2

Exercise 2.6.5. Assume \(||a - b||, ||c - d|| < \frac{1}{10}\). Show
\[
|| (a - b) - (c - d) || < ||a - b|| + ||c - d||. 
\]
(2.31)

Proof of Theorem 2.6.2: We have shown
\[
|| \beta_l - \gamma_l || < \frac{a_n}{q_n}. 
\]
(2.32)
Thus, as \(N_n = q_n\):
\[
\left| \left| N_n(\beta_l - \gamma_l) \right| \right| < a_n, 
\]
(2.33)
and the same result holds with \(l\) replaced by \(l - 1\).
By Exercise 2.6.5,
\[
\left| \left| N_n(\beta_l - \gamma_l) - N_n(\beta_{l-1} - \gamma_{l-1}) \right| \right| < 2a_n. 
\]
(2.34)
Rearranging gives
\[
\left| \left| N_n(\beta_l - \beta_{l-1}) - N_n(\gamma_l - \gamma_{l-1}) \right| \right| < 2a_n. 
\]
(2.35)
As \(a_n \to 0\), this implies the difference between \(\left| \left| N_n(\beta_l - \beta_{l-1}) \right| \right|\) and \(\left| \left| N_n(\gamma_l - \gamma_{l-1}) \right| \right|\) goes to zero.
The above distance calculations were done mod 1. The actual differences will differ by an integer. Thus,
\[
\mu_{N_n,\lambda}^S(t)dt = \frac{1}{N_n} \sum_{l=1}^{N_n} \delta \left( t - N_n(\beta_l - \beta_{l-1}) \right) 
\]
(2.36)
and
\[
\mu_{N_n,\lambda}^R(t)dt = \frac{1}{N_n} \sum_{l=1}^{N_n} \delta \left( t - N_n(\gamma_l - \gamma_{l-1}) \right) 
\]
(2.37)
are extremely close to one another; each point mass from the difference between adjacent \(\beta_l\)s is either within \(a_n\) units of a point mass from the difference between adjacent \(\gamma_l\)s, or is within \(a_n\) units of a point mass an integer number of units from a point mass from the difference between adjacent \(\gamma_l\)s. Further, \(a_n \to 0\).
Note, however, that if \( \gamma_l = \{k^2 p_n \over q_n\} \), then
\[
N_n \gamma_l = q_n \left\{k^2 p_n \over q_n\right\} \in \mathbb{N}.
\] (2.38)

Thus, the induced probability measure \( \mu_{p_n q_n N_n,1}(t)dt \) formed from the \( \gamma_l \)s is supported on the integers! Thus, it is impossible for \( \mu_{p_n q_n N_n,1}(t)dt \) to converge to \( e^{-t}dt \).

As \( \mu_{p_n q_n N_n,1}(t)dt \), modulo some possible integer shifts, is arbitrarily close to \( \mu_{p_n q_n N_n,1}(t)dt \), the sequence \( \{k^2 \alpha\} \) is not Poissonian along the subsequence of \( N \)s given by \( N_n \), where \( N_n = q_n, q_n \) is a denominator in a good rational approximation to \( \alpha \). \( \Box \)

### 2.6.3 Measure of \( \alpha \notin \mathbb{Q} \) with Non-Poissonian Behavior along a sequence \( N_n \)

What is the (Lebesgue) measure of \( \alpha \notin \mathbb{Q} \) such that there are infinitely many \( n \) with
\[
\left|\alpha - {p_n \over q_n}\right| < {a_n \over q_n}, \quad a_n \to 0.
\] (2.39)

If the above holds, then for any constant \( k(\alpha) \), for \( n \) large (large depends on both \( \alpha \) and \( k(\alpha) \)) we have
\[
\left|\alpha - {p_n \over q_n}\right| < {k(\alpha) \over q_n^{2+\epsilon}}.
\] (2.40)

By Theorem 2.6.2, this set has (Lebesgue) measure or size 0. Thus, almost no irrational numbers satisfy the conditions of Theorem 2.6.2, where almost no is relative to the (Lebesgue) measure.

**Exercise 2.6.6.** In a topological sense, how many algebraic numbers satisfy the conditions of Theorem 2.6.2? How many transcendental numbers satisfy the conditions?

**Exercise 2.6.7.** Let \( \alpha \) satisfy the conditions of Theorem 2.6.2. Consider the sequence \( N_n \), where \( N_n = q_n, q_n \) the denominator of a good approximation to \( \alpha \). We know the induced probability measures \( \mu_{p_n q_n N_n,1}(t)dt \) and \( \mu_{p_n q_n N_n,1}(t)dt \) do not converge to \( e^{-t}dt \). Do these measures converge to anything?
Remark 2.6.8. In The Distribution of Spacings Between the Fractional Parts of \( \{n^2 \alpha \} \) (Z. Rudnick, P. Sarnak, A. Zaharescu), it is shown that for most \( \alpha \) satisfying the conditions of Theorem 2.6.2, there is a sequence \( N_j \) along which \( \mu_{N_{n,1}}(t)dt \) does converge to \( e^{-t}dt \).