Eigenvalue Distribution of Highly Palindromic Toeplitz Matrices

SMALL 2009 Random Matrix Theory Group

Steven Jackson, Steven J. Miller, Thuy Pham

Background

Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix p and define $\text{Prob}(A) = \prod_{1 \le i \le j \le N} p(a_{ij})$, so

$$\text{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1 \le i \le j \le N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij})dx_{ij}.$$

Goal: understanding the eigenvalues of A.

Eigenvalue Distribution

To each A, attach a probability measure:

$$\mu_{A,N}(x) = \frac{1}{N}\sum_{i=1}^{N} \delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right)$$

$$\int_a^b \mu_{A,N}(x)dx = \frac{\#\left\{\lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a,b]\right\}}{N}$$

$$\begin{aligned} k^{\text{th}} \text{ moment} &= \int_{-\infty}^{\infty} x^k \mu_{A,N}(x)dx = \frac{\sum_{i=1}^N \lambda_i(A)^k}{2^k N^{\frac{k}{2}+1}} \\ &= \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}} \\ &= \frac{\sum_{i_1=1}^N \cdots \sum_{i_k=1}^N a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}}{2^k N^{\frac{k}{2}+1}}. \end{aligned}$$

We want to understand the eigenvalues of A, but it is the matrix elements that are chosen randomly and independently.

Eigenvalue Trace Lemma: Let A be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\text{Trace}(A^k) = \sum_{n=1}^{N} \lambda_i(A)^k,$$

where

$$\text{Trace}(A^k) = \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}.$$

The moments then determine the eigenvalue distribution provided they do not grow too quickly.

Questions:

- What is the average moment of A_N?
- How do these average moments behave as $N \to \infty$?

Toeplitz Matrices

Previous Results

$N \times N$ Toeplitz matrix:

$$\begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{N-1} \\ b_{-1} & b_0 & b_1 & \cdots & b_{N-2} \\ b_{-2} & b_{-1} & b_0 & \cdots & b_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{1-N} & b_{2-N} & b_{3-N} & \cdots & b_0 \end{pmatrix}$$

- Entries must match in pairs to contribute to moments
- Number of ways to match k objects in pairs equals k^{th} moment of standard normal distribution
- Moments are nearly, but not quite, those of standard normal

$N \times N$ Palindromic Toeplitz matrix:

$$\begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_2 & b_1 & b_0 \\ b_1 & b_0 & b_1 & \cdots & b_3 & b_2 & b_1 \\ b_2 & b_1 & b_0 & \cdots & b_4 & b_3 & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ b_2 & b_3 & b_4 & \cdots & b_0 & b_1 & b_2 \\ b_1 & b_2 & b_3 & \cdots & b_1 & b_0 & b_1 \\ b_0 & b_1 & b_2 & \cdots & b_2 & b_1 & b_0 \end{pmatrix}$$

- New symmetry removes obstructions: all configurations contribute 1
- Density of eigenvalues is the standard normal

More Palindromes?

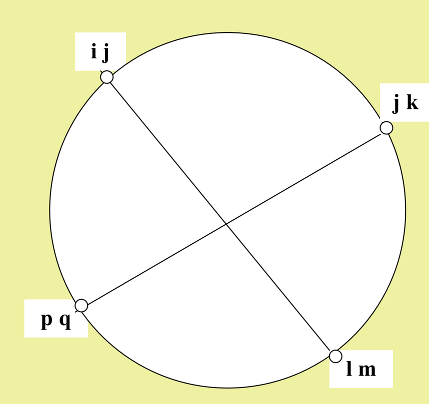
$N \times N$ Doubly Palindromic Toeplitz matrix:

$$\begin{pmatrix} b_0 & b_1 & \cdots & b_1 & b_0 & b_0 & b_1 & \cdots & b_1 & b_0 \\ b_1 & b_0 & \cdots & b_2 & b_1 & b_0 & b_0 & \cdots & b_2 & b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_1 & b_2 & \cdots & b_0 & b_1 & b_2 & b_3 & \cdots & b_0 & b_1 \\ b_0 & b_1 & \cdots & b_1 & b_0 & b_1 & b_2 & \cdots & b_0 & b_0 \\ b_0 & b_0 & \cdots & b_2 & b_1 & b_0 & b_1 & \cdots & b_1 & b_0 \\ b_1 & b_0 & \cdots & b_3 & b_2 & b_1 & b_0 & \cdots & b_2 & b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_1 & b_2 & \cdots & b_0 & b_0 & b_1 & b_2 & \cdots & b_0 & b_1 \\ b_0 & b_1 & \cdots & b_1 & b_0 & b_0 & b_1 & \cdots & b_1 & b_0 \end{pmatrix}$$

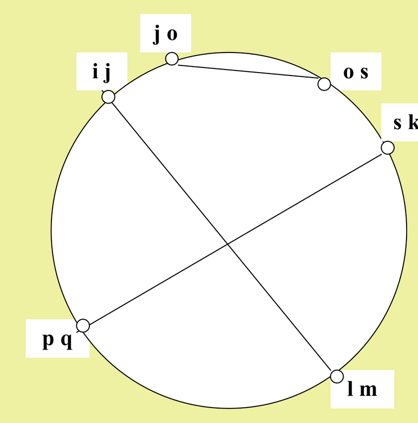
- All configurations contribute equally
- Moments grow faster than any normal distribution

Configuration Liftings and Equal Contributions

Lemma: Consider a configuration of the $2m^{\text{th}}$ moment. All configurations of the $(2m+2)^{\text{th}}$ moment obtained by adding a pair of adjacent entries to this configuration contribute equally to the $(2m+2)^{\text{th}}$ moment.

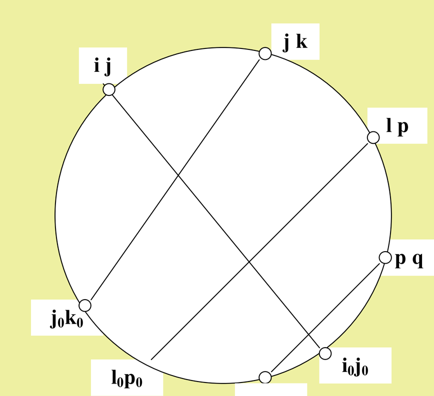


Configuration 1

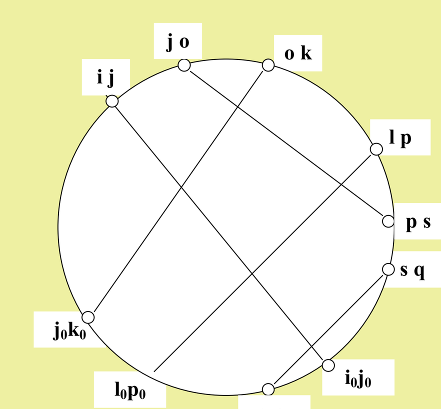


Configuration 2

Lemma: Consider a configuration of matchings for the $2m^{\text{th}}$ moment. All configurations at the $(2m+2)^{\text{th}}$ moment obtained by adding a pair of non-adjacent entries contribute equally to the $(2m+2)^{\text{th}}$ moment.



Configuration 1



Configuration 2

Theorem: If all configurations at the $2m^{\text{th}}$ moment contribute equally, then all configurations at the $(2m+2)^{\text{th}}$ moment also contribute equally.

Corollary: All even configurations contribute equally.

Adjacent Matchings

- Easier to calculate contributions of adjacent case
- Multiply together areas (figure 1) with corresponding matchings (figure 2) to get number of matchings.
- Can be extended to higher moments of doubly palindromic Toeplitz matrix

Adjacent Matching Figures

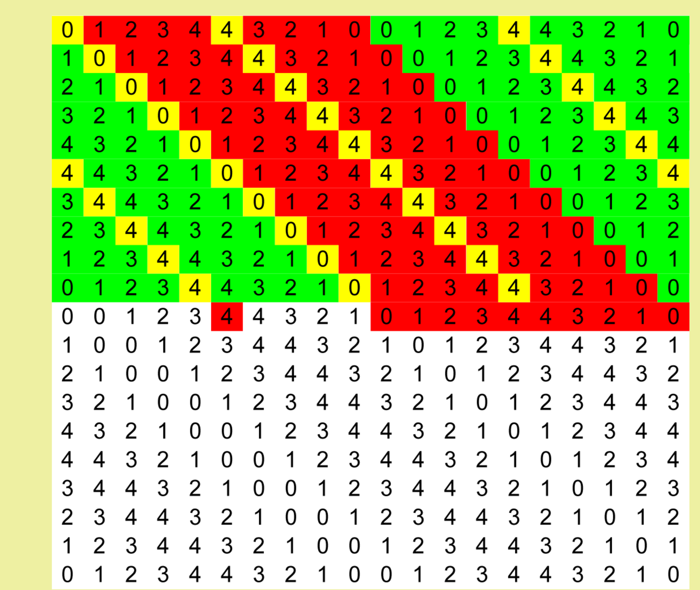


FIGURE 1. For $N = 20$, green areas indicate locations where $k = i + \frac{N}{2}$ gives a matching, red areas where $k = i + \frac{N}{2}$ works, and yellow where both work. Note that the yellow areas are one-dimensional and won't matter in the limit.

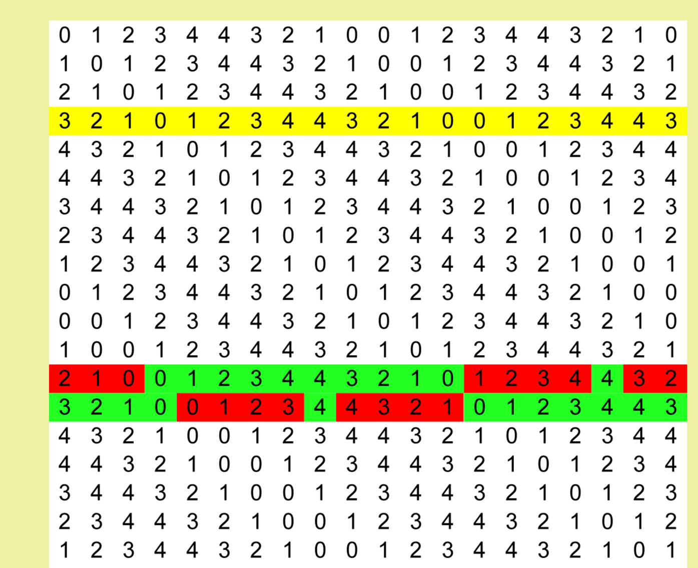


FIGURE 2. We use the symmetry of the matrix and compare the i^{th} and k^{th} rows for matchings. We multiply the number of good l values by the corresponding areas to get the contribution to the 4^{th} moment.

Results

Theorem: For a Toeplitz matrix with 2^n palindromes, sum over 2^n similar cases to find fourth moment is

$$M_{4,n} = 2^{n+1} + 2^{-n} \sim 2^{n+1}$$

Theorem: For a Doubly Palindromic Toeplitz matrix ($n = 1$) the even moments are given by

$$M_{2m,1} = (2m-1)!!\left(-2 + 2^{-m}\left(\sum_{b=1}^{3} b^m\right)\right).$$

- The $2k^{\text{th}}$ moment is bounded below by the Gaussian's moment, $(2k-1)!!$.
- The $2k^{\text{th}}$ moment is bounded above by $(2k-1)!! \cdot (4 \cdot 2^n - 1)^{k-1}$.

Theorem: Moments grow sufficiently slowly to determine a unique probability distribution, has 'fattest' tails of any ensemble studied to date.

Acknowledgements

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