This essentially builds upon the previous work but looks at from a different perspective that more easily generalizes for greater palindromicity (or whatever we'll call it). It also might be a cleaner approach for the doubly palindromic case. I was having trouble with this giving the right answer when I converted it into a formula, but I managed to figure out the problem (see below) while I was TeXing it up. Also, I realized the $\frac{N}{2}$ isn't the best specific example as $\frac{N}{2}$ and $N - \frac{N}{2}$ are the same.

The big idea here is to look at where we can have a specific $C$ value and calculating that "area," then the number of values $\ell$ can take on given the $C$ value, and then multiplying the two together to get the contribution for that specific $C$, and then summing over all possible $C$'s. There will be $O(1)$ terms all over the place here, but I'll suppress them as they're understood to be insignificant in the limit.

In order to demonstrate the method, I will use it for the $\frac{N}{2}$ and the $\frac{N}{2} - 1$ terms for the $4^{th}$ moment of the doubly palindromic, then how it generalizes.

**Specific Case.** We begin by considering when $k = i + \frac{N}{2}$ (note that the case where $k = i - \frac{N}{2}$ turns out the exact same). Clearly, this restricts $i$ to $\{1, 2, \dots, \frac{N}{2}\}$, so we are looking at the top half of the matrix. Moreover, we need $a_{kj}$ to be on the same side of the diagonal as $a_{ij}$ for it to be a good matching in this case, so we either need $a_{ij}$ to be below the diagonal, or $a_{ij}$ more than $\frac{N}{2}$ above the diagonal. For the first case, we have an $\frac{N}{2}$ by $\frac{N}{2}$ triangle defined by the diagonal and the "horizontal bisector" of the matrix, and for the second we also have an $\frac{N}{2}$ by $\frac{N}{2}$ triangle defined by the shifted diagonal boundary and the borders of the matrix. Conveniently, this gives a square overall.

Finally, we have to figure out how many values of $\ell$ can work. As they are on the same side of the main diagonal, column $i$ and row $k$ start out aligned, become unaligned when one has hit the diagonal but the other hasn't, then become realigned once both have hit the diagonal. As they will be unaligned for $\frac{N}{2}$ values of $\ell$, they will be aligned for $\frac{N}{2}$ values of $\ell$. Thus, we get $\left(\frac{N}{2}\right)^3 = \frac{N^3}{8}$ overall, so it contributes $\frac{1}{8}$ to the moment.

Now, we consider when $k = i + \frac{N}{2} - 1$. Unsurprisingly, this happens everywhere in the top half of the matrix where $k = i + \frac{N}{2}$ didn't work. One can calculate this by simply doing $\frac{N^2}{2} - \frac{N^2}{4}$, or by noting that the this area is a parallelogram of base $\frac{N}{2}$ and height $\frac{N}{2}$. Furthermore, we see that column $i$ and row $k$ will be aligned when only one has reached the main diagonal, so we again have $\frac{N}{2}$ values of $\ell$. Thus, we get $\left(\frac{N}{2}\right)^3 = \frac{N^3}{8}$, so it contributes $\frac{1}{8}$ to the moment.

We then multiply these by two to account for the minus terms, then add them together to get $\frac{1}{2}$. We multiply by another factor of two to account for both moments, yielding a contribution of 1 to the fourth moment, agreeing with our previous calculations.

**General Case for $2^n$ Palindromes.** Consider when $k = i + \frac{mN}{2^n}$, where $m \in \{0, 1, 2, \dots, 2^n\}$. This restricts $i$ to $\{1, 2, \dots, \frac{2^n - m}{2^n} \cdot N\}$. In calculating the area these cases occupy, we get $\left(\frac{2^n-m}{2^n}\right)^2 N$. Column $i$ and row $k$ are then aligned except for the $\frac{mN}{2^n}$ values of $\ell$ where only one has reached the main diagonal. Therefore, we get $\left(\frac{2^n-m}{2^n}\right)^3 N$, for a contribution of $\left(\frac{2^n-m}{2^n}\right)^3$ to the moment.

Similarly, when $k = i + \frac{mN}{2^n} - 1$, we again have a parallelogram of height $\frac{2^n-m}{2^n} \cdot N$ and width $\frac{m}{2^n}$. In this case, the alignment occurs when one has reached the main diagonal but the other hasn't, so we have $\frac{m}{2^n} \cdot N$ choices for that as well.

Thus, we have a contribution of $\frac{(2^n-m)^3+(2^n-m)m^2}{2^n}$ overall for this pair of constants. We then multiply by 4 to account for the minus signs and the two adjacent matching cases.

**The Adjacent Contribution to the Fourth Moment.** We now want to sum over all possible $m$ values here. The one catch is that when $m=0$ we only multiply by 2, as there is no negative case (that's what the mistake was). Thus, we want to evaluate

$$2+\frac{4}{2^{3n}}\sum_{m=1}^{2^n}\left[(2^n-m)^3+(2^n-m)m^2\right].$$

We begin by expanding out the second term in the sum:

$$2+\frac{4}{2^{3n}}\sum_{m=1}^{2^n}\left[(2^n-m)^3+2^n m^2-m^3\right].$$

Here, we note that the sums over $(2^n-m)^3$ and $-m^3$ almost cancel out exactly (they would cancel out exactly if we summed over 0 as well. Since we know the value inside the sum will be 1 for $m=0$, we can rewrite this as

$$-2+\frac{4}{2^{3n}}\sum_{m=0}^{2^n}\left[(2^n-m)^3+2^n m^2-m^3\right]=-2+\frac{4}{2^{3n}}\sum_{m=0}^{2^n}2^n m^2.$$

Now, we can pull the $2^n$ out and cancel, and get

$$-2+\frac{4}{2^{2n}}\sum_{m=0}^{2^n}m^2.$$

We can evaluate this out directly, and cancel a resulting $2^n$ to get

$$\begin{aligned}-2+\frac{4}{2^n}\cdot\frac{(2^n+1)(2\cdot 2^n+1)}{6} &= -2+\frac{2}{3}(1+\frac{1}{2^n})(2\cdot 2^n+1)\\ &= -2+\frac{2}{3}(3+2^{n+1}+\frac{1}{2^n}).\end{aligned}$$

Just as some numerics, the first few values are (moving from single palindrome and onwards): 2, 3, 5.5, 10.75, and 21.375. I tested out the quadruple and octuple cases using the trace method in Mathematica, and in each case I got something very nearly $\frac{3}{2}$ the value reported here.