ON THE ASYMPTOTIC BEHAVIOR OF VARIANCE OF PLRS DECOMPOSITIONS

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ABSTRACT. A positive linear recurrence sequence is of the form $H_{n+1} = c_1 H_n + \cdots + c_L H_{n+1-L}$ with each $c_i \geq 0$ and $c_1 c_L > 0$, with appropriately chosen initial conditions. There is a notion of a legal decomposition (roughly, given a sum of terms in the sequence we cannot use the recurrence relation to reduce it) such that every positive integer has a unique legal decomposition using terms in the sequence; this generalizes the Zeckendorf decomposition, which states any positive integer can be written uniquely as a sum of non-adjacent Fibonacci numbers. Previous work proved not only that a decomposition exists, but that the number of summands $K_n(m)$ in legal decompositions of $m \in [H_n, H_{n+1})$ converges to a Gaussian. Using partial fractions and generating functions it is easy to show the mean and variance grow linearly in n: $an + b + o(1)$ and $Cn + d + o(1)$, respectively; the difficulty is proving a and C are positive. Previous approaches relied on delicate analysis of polynomials related to the generating functions and characteristic polynomials, and is algebraically cumbersome. We introduce new, elementary techniques that bypass these issues. The key insight is to use induction and bootstrap bounds through conditional probability expansions to show the variance is unbounded, and hence $C > 0$ (the mean is handled easily through a simple counting argument).

1. INTRODUCTION

There are many ways to define the Fibonacci numbers. An equivalent approach to the standard recurrence relation, where $F_{n+1} = F_n + F_{n-1}$ and $F_1 = 1$ and $F_2 = 2$, is that they are the unique sequence of integers such that every positive number can be written uniquely as a sum of non-adjacent terms. This expansion is called the Zeckendorf decomposition [25], and much is known about it. In particular, the distribution of the number of summands of $m \in [F_n, F_{n+1})$ converges to a Gaussian as $n \to \infty$, with mean and variance growing linearly with n. Similar results hold for a large class of sequences which have a notion of legal decomposition leading to unique decomposition; see [1, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24].

Given a sequence $\{H_n\}$, one can frequently prove that the mean and the variance of the number of summands of $m \in [H_n, H_{n+1})$ grows linearly with n. Explicitly, there are constants a, b, C and d such that the mean is $an + b + o(1)$ and the variance is $Cn + d + o(1)$. The difficulty is proving that a and C are positive, which is needed for the proofs of Gaussian behavior. Until recently, the only approaches have been technical and involved generating functions, partial fraction expansions and generalized Binet formulas applied to polynomials associated to the characteristic polynomials of the sequence, which have required a lot of work to show the leading terms are positive for such recurrences. The point of this work is to bypass

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these arguments through elementary counting. We concentrate on positive linear recurrence sequences (defined below) to highlight the main ideas of the method; with additional work these arguments can be extended to more general sequences (see [6]). In addition to the arguments below, one can also obtain similar results (though not as elementarily) through Markov chains [2] or through an analysis of two dimensional recurrences [19].

Definition 1.1. *A sequence $\{H_n\}_{n=1}^{\infty}$ of positive integers is a* ***Positive Linear Recurrence Sequence (PLRS)*** *if the following properties hold.*

(1) Recurrence relation: *There are non-negative integers $L, c_1, \dots, c_L$ such that*

$$H_{n+1} \ = \ c_1 H_n + \cdots + c_L H_{n+1-L},$$

with L, c_1 and c_L positive.

(2) Initial conditions: *$H_1 = 1$, and for $1 \leq n < L$ we have*

$$H_{n+1} \ = \ c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1.$$

We define the ***size*** *of $\{H_n\}$ to be $c_1 + \cdots + c_L$ and the* ***length*** *of $\{H_n\}$ to be L.*

Definition 1.2. *Let $\{H_n\}$ be a PLRS. A decomposition $\sum_{i=1}^{m} a_i H_{m+1-i}$ of a positive integer ω (and the sequence $\{a_i\}_{i=1}^{m}$) is* ***legal*** *if $a_1 > 0$, the other $a_i \geq 0$, and one of the following two conditions holds.*

- *Condition 1: We have $m < L$ and $a_i = c_i$ for $1 \leq i \leq m$.*
- *Condition 2: There exists $s \in \{1, \dots, L\}$ such that*

$$a_1 \ = \ c_1,\ a_2 \ = \ c_2,\ \dots,\ a_{s-1} \ = \ c_{s-1} \text{ and } a_s < c_s,$$

  and $\{b_i\}_{i=1}^{m-s}$ (with $b_i = a_{s+i}$) is legal.

If $\sum_{i=1}^{m} a_i H_{m+1-i}$ is a legal decomposition of ω, we define the ***number of summands*** *(of this decomposition of ω) to be $a_1 + \cdots + a_m$.*

Furthermore, we define two types of blocks, where a ***block*** *is a nonempty ordered subset of the coefficients $[a_i, a_{i+1}, \dots, a_{i+j}]$ inclusive:*

- *a* ***Type 1 block*** *corresponds to Condition 1, and has length $m < L$ and size $a_i + \cdots + a_{i+m-1}$,*
- *a* ***Type 2 block*** *corresponds to Condition 2, and has length $s \leq L$ and size $a_i + \cdots + a_{i+s-1}$.*

Remark 1.3. *A Type 2 block has three key properties.*

- *A legal decomposition of ω stays legal if a Type 2 block is inserted (between Type 1 and/or 2 blocks) or removed and indices are shifted appropriately.*
- *If we know the size of a Type 2 block, the block's content and its length are uniquely determined. So we can define a* ***length function*** *$\ell(t)$ to be the length of a Type 2 block with size t.*
- *A Type 2 block always has nonnegative size and strictly positive length. Specifically, consider a Type 2 block with size 0. Then, in Condition 2, we always have $a_1 = 0 < c_1$, so $s = 1$. Thus a Type 2 block with size 0 has length 1. In other words, $\ell(0) = 1$ holds for all PLRS.*

If a legal decomposition contains a Type 1 block, then it must be the last block. Thus any legal decomposition contains at most one Type 1 block. A Type 1 block, according to Condition 1, always has positive size and positive length.

The following two examples clarify the above.

Example 1.4. ***The Fibonacci Sequence*** *(size 2 and length 2).*
Type 1 block: [1].
Type 2 blocks: [0], [1 0].
An example of a legal decomposition: $F_5 + F_3 + F_1$ *with block representation: [1 0] [1 0] [1].*
After removing the second to last block, the new block representation is [1 0] [1].
The resulting legal decomposition is $F_3 + F_1$.

Example 1.5. ***PLRS sequence*** $H_n = 2H_{n-1} + 2H_{n-2} + 0 + 2H_{n-4}$ *(size 6 and length 4).*
Type 1 blocks: [2], [2 2], [2 2 0].
Type 2 blocks: [0], [1], [2 0], [2 1], [2 2 0 0], [2 2 0 1].
An example of a legal decomposition: $H_7 + 2H_4 + H_1$ *with block representation: [1] [0] [0] [2 0] [0] [1].*
After removing the second to last block, the new block representation is [1] [0] [0] [2 0] [1].
The resulting legal decomposition is $H_6 + 2H_3 + H_1$.

Before we state our main result we first set some notation.

Definition 1.6. *Let* $\{H_n\}$ *be a Positive Linear Recurrence Sequence. For each* n*, let the discrete outcome space* Ω_n *be the set of legal decompositions of integers in* $[H_n, H_{n+1})$*. By the Generalized Zeckendorf Theorem (see for example* [22]*) every integer has a unique legal decomposition, so* $|\Omega_n| = H_{n+1} - H_n$*. Define the probability measure on subsets of* Ω_n *by*

$$\mathbb{P}_n(A) \ = \ \sum_{\substack{\omega \in A \\ \omega \in \Omega_n}} \frac{1}{H_{n+1} - H_n}, \quad A \subset \Omega_n;$$

thus each of the $H_{n+1} - H_n$ *legal decompositions is weighted equally. We define the random variable* K_n *by setting* $K_n(\omega)$ *equal to the number of summands of* $\omega \in \Omega_n$*. When* $n > 2L$ *(so there are at least three blocks) we define the random variable* Z_n *by setting* $Z_n(\omega)$ *equal to the size of the second to last block of* $\omega \in \Omega_n$*. Note that the second to last block must be a Type 2 block. Finally, we define the random variable* L_n *by setting* $L_n(\omega)$ *equal to the length of the second to last block of* $\omega \in \Omega_n$*; i.e.,* $L_n(\omega) = \ell(Z_n(\omega))$.

As remarked above, previous work has shown that $\mathbb{E}[K_n] = an + b + f(n)$ where $a > 0$ and $f(n) = o(1)$; this can be proved through very simple counting arguments (see [6]). While it is also known that $\mathrm{Var}[K_n] = Cn + d + o(1)$, previous approaches could not easily show $C \neq 0$. We elementarily prove $C > 0$ by giving a positive lower bound c for it.

Theorem 1.7. *Let* $\{H_n\}$ *be a positive linear recurrence sequence with size* S *and length* L*. Then there is a* $c > 0$ *such that* $\mathrm{Var}[K_n] \geq cn$ *for all* $n > L$.

We sketch the proof. We can remove the second to last block of a legal decomposition to get a shorter legal decomposition, forming relations between longer legal decompositions and shorter legal decompositions. We then use strong induction and conditional probabilities to prove the theorem.

Remark 1.8. *As it is known that* $\mathrm{Var}[K_n] = Cn + d + o(1)$*, to prove that* $C > 0$ *it would suffice to show* $\lim_{n\to\infty} \mathrm{Var}[K_n]$ *diverges to infinity. Unfortunately the only elementary proofs we could find of this also establish the correct growth rate; we would be very interested in seeing an approach that yielded (for example)* $\mathrm{Var}[K_n] \gg \log n$ *(which would then immediately improve to implying* $C > 0$*).*

2. Lemmas derived from Expectation

We first determine a relationship between K_n and Z_n. Then, with the help of $\mathbb{E}[K_n] = an + b + f(n)$, we explain how to explicitly determine the positive lower bound c. In the arguments below note $\ell(t)$ and $H_{n+1} - H_n$ are increasing respectively with t and n.

Lemma 2.1. *Let $n > 2L$. For all $0 \leq t < S$, we define $S_t := \{\omega \in \Omega_n | Z_n(\omega) = t\}$, and $h_t(\omega)$ to be the decomposition after removing the second to last block of ω. (When we remove the second to last block with size t, we completely remove that block from ω and shift all the indices to the left of that block by $\ell(t)$.) When we remove the second to last block (a Type 2 block) from ω, then $h_t(\omega)$ is legal and h_t is a bijection between S_t and $\Omega_{n-\ell(t)}$.*

Proof. Let $\omega \in \Omega_n$ be arbitrary and consider $h_t(\omega)$. Since the block we remove has size t and thus length $\ell(t)$, $h_t(\omega)$ must be in $\Omega_{n-\ell(t)}$.

Next, consider $\omega, \omega' \in S_t$, such that $h_t(\omega) = h_t(\omega')$. As the size determines the composition for Type 2 blocks, we are removing the same block at the same position for ω, ω'. This implies $\omega = \omega'$.

Finally, for any $\omega \in \Omega_{n-\ell(t)}$, if we insert the size t type 2 block before its last block, we get a legal decomposition in Ω_n. Thus h_t is surjective.

Therefore, h_t is a bijection between S_t and $\Omega_{n-\ell(t)}$. □

Corollary 2.2. *We have*

$$\mathbb{P}[Z_n = t] \;=\; \frac{|S_t|}{|\Omega_n|} \;=\; \frac{|\Omega_{n-\ell(t)}|}{|\Omega_n|} \;=\; \frac{H_{n-\ell(t)+1} - H_{n-\ell(t)}}{H_{n+1} - H_n}.$$

Remark 2.3. *As*

$$\mathbb{P}[Z_n = 0] \;\geq\; \mathbb{P}[Z_n = 1] \geq \cdots \geq \mathbb{P}[Z_n = S-1] \tag{2.1}$$

and the sum of these S terms is 1, we have

$$\mathbb{P}[Z_n = 0] \;\geq\; \frac{1}{S}, \tag{2.2}$$

(which is the consequence we need below).

For an arbitrary $\omega \in S_t$, the second to last block has size $Z_n = t$, and the remaining blocks form a legal decomposition in $\Omega_{n-\ell(t)}$ with size $K_{n-\ell(t)}(h_t(\omega))$, so $K_n(\omega) = K_{n-\ell(t)}(h_t(\omega)) + t$. Since h is a bijection, we have the following two equations:

$$\begin{aligned} \mathbb{E}[K_n | Z_n = t] &= \mathbb{E}[K_{n-\ell(t)} + t] \\ &= a(n - \ell(t)) + b + f(n - \ell(t)) + t, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \mathbb{E}[K_n^2 | Z_n = t] &= \mathbb{E}[(K_{n-\ell(t)} + t)^2] \\ &= \mathbb{E}[K_{n-\ell(t)}^2 + 2tK_{n-\ell(t)} + t^2] \\ &= \mathbb{E}[K_{n-\ell(t)}^2] + 2t\mathbb{E}[K_{n-\ell(t)}] + t^2 \\ &= \mathbb{E}[K_{n-\ell(t)}^2] + 2t[a(n-\ell(t)) + b + f(n-\ell(t))] + t^2. \end{aligned} \tag{2.4}$$

Furthermore, by (2.3) we have

$$\begin{aligned}
\mathbb{E}[K_n] &= \sum_{t=0}^{S-1} \mathbb{P}[Z_n = t] \cdot \mathbb{E}[K_n | Z_n = t] \\
&= \sum_{t=0}^{S-1} \mathbb{P}[Z_n = t] \cdot [a(n - \ell(t)) + b + f(n - \ell(t)) + t] \\
&= an + b + \sum_{t=0}^{S-1} \mathbb{P}[Z_n = t] \cdot [t + f(n - \ell(t)) - a\ell(t)] \\
&= an + b + f(n),
\end{aligned} \tag{2.5}$$

where the last equality comes from the definition of $f(n)$.

If we set $Y_n(\omega) := Z_n(\omega) + f(n - L_n(\omega)) - aL_n(\omega)$, then we have

$$\mathbb{E}[Y_n] = \sum_{t=0}^{S-1} \mathbb{P}[Z_n = t] \cdot [t + f(n - \ell(t)) - a\ell(t)] = f(n). \tag{2.6}$$

Now that we have $\mathbb{E}[Y_n]$, we use it to estimate $\mathrm{Var}[Y_n]$.

Lemma 2.4. *For n sufficiently large we have*

$$\mathrm{Var}[Y_n] > \frac{a^2}{2S}. \tag{2.7}$$

Proof. First, for all $n > 2L$ we have

$$\begin{aligned}
\mathrm{Var}[Y_n] &= \mathbb{E}[Y_n^2] - (\mathbb{E}[Y_n])^2 \\
&= \left(\mathbb{E}[(Z_n - aL_n + f(n - L_n))^2]\right) - (f(n))^2 \\
&= \left(\mathbb{E}[(Z_n - aL_n)^2] + \mathbb{E}[2(Z_n - aL_n) \cdot f(n - L_n)] + \mathbb{E}[f(n - L_n)^2]\right) - (f(n))^2 .
\end{aligned}$$

Note that $Z_n - aL_n$ is bounded since $-aL \le Z_n - aL_n \le S$ for all $n > 2L$. Also we know $f(n) = o(1)$, so $f(n - L_n) = o(1)$ since $L_n \le L$. Hence the following three limits are all zero:

$$\lim_{n\to\infty} \mathbb{E}[2(Z_n - aL_n) \cdot f(n - L_n)] = \lim_{n\to\infty} \mathbb{E}[f(n - L_n)^2] = \lim_{n\to\infty} (f(n))^2 = 0. \tag{2.8}$$

Further, we know

$$\mathrm{Var}[Y_n] - \mathbb{E}[(Z_n - aL_n)^2] = \mathbb{E}[2(Z_n - aL_n) \cdot f(n - L_n)] + \mathbb{E}[f(n - L_n)^2] - (f(n))^2 , \tag{2.9}$$

so

$$\lim_{n\to\infty} \left(\mathrm{Var}[Y_n] - \mathbb{E}[(Z_n - aL_n)^2]\right) = 0. \tag{2.10}$$

On the other hand, for all $n > 2L$ we have

$$\begin{aligned}
\mathbb{E}[(Z_n - aL_n)^2] &= \sum_{t=0}^{S-1} \mathbb{P}[Z_n = t] \cdot (t - a\ell(t))^2 \\
&\ge \mathbb{P}[Z_n = 0] \cdot (0 - a\ell(0))^2 \\
&\ge \frac{a^2}{S},
\end{aligned} \tag{2.11}$$

where the last inequality follows from (2.2).

By (2.10), we know there must exist $N > 2L$ such that for all $n > N$,

$$\left|\mathrm{Var}[Y_n] - \mathbb{E}[(Z_n - aL_n)^2]\right| < \frac{a^2}{2S}, \tag{2.12}$$

so $\mathrm{Var}[Y_n] - \mathbb{E}[(Z_n - aL_n)^2] > -\frac{a^2}{2S}$. Then, by (2.11), we get $\mathrm{Var}[Y_n] > \frac{a^2}{2S}$ for all $n > N$. $\square$

Finally, we choose c. Let

$$c \;=\; \min\left\{\frac{\mathrm{Var}[K_{L+1}]}{L+1},\ \frac{\mathrm{Var}[K_{L+2}]}{L+2},\ \dots,\ \frac{\mathrm{Var}[K_N]}{N},\ \frac{a^2}{2SL}\right\}, \tag{2.13}$$

Where N is as determined in Lemma 2.4. For all $n > L$, $H_{n+1} - H_n > 1$, so there are at least two integers in $[H_n, H_{n+1})$. Since the legal decomposition of H_n has only one summand while that of $H_n + 1$ has two summands, $\mathrm{Var}[K_n]$ is nonzero when $n > L$. Hence, $c > 0$. In the next section we show $\mathrm{Var}[K_n] \geq cn$ for all $n > L$.

3. A lower bound for the Variance

We prove Theorem 1.7 by strong induction. While the algebra is long, the main idea is easily stated: we condition based on how many summands are in the second to last block, which must be a type 2 block, and then use conditional probability arguments (inputting results for the mean and smaller cases) to compute the desired quantities

Proof. The base cases $n = L+1, L+2, \dots, N$ are automatically true by the way we choose c. Hence, we only need to consider the cases when $n > N$. In the induction hypothesis, we assume $\mathrm{Var}[K_r] \geq cr$ for $L < r < n$. In the inductive step, we prove $\mathrm{Var}[K_n] \geq cn$ where $n > N$.

For $L < r < n$, we have $\mathrm{Var}[K_r] \geq cr$ and $\mathbb{E}[K_r] = ar + b + f(r)$, hence

$$\begin{aligned}\mathbb{E}[K_r^2] &= \mathrm{Var}[K_r] + (\mathbb{E}[K_r])^2 \\ &\geq cr + (ar + b + f(r))^2 \\ &= cr + a^2r^2 + b^2 + (f(r))^2 + 2arb + 2arf(r) + 2bf(r).\end{aligned} \tag{3.1}$$

By (2.4), we have

$$\begin{aligned}\mathbb{E}[K_n^2] &= \sum_{t=0}^{S-1} \mathbb{P}[Z_n = t] \cdot \mathbb{E}[K_n^2 | Z_n = t] \\ &= \sum_{t=0}^{S-1} \mathbb{P}[Z_n = t] \cdot \left(\mathbb{E}[K_{n-\ell(t)}^2] + 2t[a(n-\ell(t)) + b + f(n-\ell(t))] + t^2\right).\end{aligned}$$

Note we only need to consider $n > N > 2L$, so $n > n - \ell(t) \geq n - L > L$ for all $0 \leq t \leq S-1$. Hence, by (3.1),

$$\begin{aligned}\mathbb{E}[K_{n-\ell(t)}^2] \;\geq\; & c(n-\ell(t)) + a^2(n-\ell(t))^2 + b^2 + [f(n-\ell(t))]^2 + 2a(n-\ell(t))b \\ & + 2a(n-\ell(t))f(n-\ell(t)) + 2bf(n-\ell(t)).\end{aligned}$$

After we replace $\mathbb{E}[K_{n-\ell(t)}^2]$ in the conditional expectation $\mathbb{E}[K_n^2|Z_n = t]$ with this lower bound, any term either does not depend on t or can be combined with other terms to form $(t + f(n - \ell(t)) - a\ell(t))$. The final equation will then have two parts, one of which does not depend on

t, while the other can be written in the form of $Z_n + f(n - L_n) - aL_n$, which is exactly Y_n. We find

$$\begin{aligned}
\mathbb{E}[K_n^2] \ \geq\ & \sum_{t=0}^{S-1} \mathbb{P}[Z_n = t] \cdot \Big[c(n - \ell(t)) + a^2(n - \ell(t))^2 + b^2 + [f(n - \ell(t))]^2 + 2a(n - \ell(t))b \\
&+\ 2a(n - \ell(t))f(n - \ell(t)) + 2bf(n - \ell(t)) + 2t[a(n - \ell(t)) + b + f(n - \ell(t))] + t^2 \Big] \\
=\ & \sum_{t=0}^{S-1} \mathbb{P}[Z_n = t] \cdot \Big[c(n - \ell(t)) + a^2(n - \ell(t))^2 + b^2 + [f(n - \ell(t))]^2 + 2a(n - \ell(t))b \\
&+\ 2a(n - \ell(t))f(n - \ell(t)) + 2bf(n - l(t)) + [2tan - 2ta\ell(t) + 2tb + 2tf(n - \ell(t))] + t^2 \Big] \\
=\ & (an + b)^2 + cn + \sum_{t=0}^{S-1} \mathbb{P}[Z_n = t] \cdot \Big[- c\ell(t) - 2a^2 n\ell(t) + a^2(\ell(t))^2 + [f(n - \ell(t))]^2 \\
&-\ 2a\ell(t)b + 2anf(n - \ell(t)) - 2a\ell(t)f(n - \ell(t)) + 2bf(n - \ell(t)) + 2tan - 2ta\ell(t) + 2tb \\
&+\ 2tf(n - \ell(t)) + t^2 \Big] \\
=\ & (an + b)^2 + cn + \sum_{t=0}^{S-1} \mathbb{P}[Z_n = t] \cdot \Big[[a^2(\ell(t))^2 + [f(n - \ell(t))]^2 + t^2 - 2a\ell(t)f(n - \ell(t)) - 2ta\ell(t) \\
&+ 2tf(n - \ell(t))] + 2an\,[t + f(n - \ell(t)) - a\ell(t)] + 2b\,(t + f(n - \ell(t)) - a\ell(t)) - c\ell(t) \Big] \\
=\ & (an + b)^2 + cn + \sum_{t=0}^{S-1} \mathbb{P}[Z_n = t] \cdot \Big[[t + f(n - \ell(t)) - a\ell(t)]^2 \\
&+\ 2(an + b)\,(t + f(n - \ell(t)) - a\ell(t)) - c\ell(t) \Big] \\
=\ & (an + b)^2 + cn + \sum_{t=0}^{S-1} \mathbb{P}[Z_n = t] \cdot (t + f(n - \ell(t)) - a\ell(t))^2 \\
&+\ 2(an + b) \sum_{t=0}^{S-1} \mathbb{P}[Z_n = t] \cdot (t + f(n - \ell(t)) - a\ell(t)) - c \sum_{t=0}^{S-1} \mathbb{P}[Z_n = t] \cdot \ell(t) \\
=\ & (an + b)^2 + cn + \mathbb{E}[(Z_n + f(n - L_n) - aL_n)^2] + 2(an + b)f(n) - c\mathbb{E}[L_n],
\end{aligned}$$

where the last equality comes from (2.6).

We already know $(\mathbb{E}[K_n])^2 = (an + b + f(n))^2 = (an + b)^2 + 2(an + b)f(n) + (f(n))^2$, hence

$$\begin{aligned}
\mathrm{Var}[K_n] - cn\ &=\ \mathbb{E}[K_n^2] - (\mathbb{E}[K_n])^2 - cn \\
&\geq\ \mathbb{E}[(Z_n + f(n - L_n) - aL_n)^2] - c\mathbb{E}[L_n] - (f(n))^2 \\
&=\ \mathbb{E}[Y_n^2] - c\mathbb{E}[L_n] - (\mathbb{E}[Y_n])^2 \\
&=\ \mathrm{Var}[Y_n] - c\mathbb{E}[L_n] \\
&\geq\ \mathrm{Var}[Y_n] - cL \\
&\geq\ 0,
\end{aligned}$$

where the last inequality comes from our definition of c and (2.7).

Therefore, $\mathrm{Var}[K_n] \geq cn$ for all $n > L$. In other words, if $\mathrm{Var}[K_n] = Cn + d + o(1)$, then $C \geq c > 0$. □

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