LEGAL DECOMPOSITIONS ARISING FROM NON-POSITIVE LINEAR RECURRENCES

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Abstract. Zeckendorf's theorem states that any positive integer can be written uniquely as a sum of non-adjacent Fibonacci numbers; this result has been generalized to many recurrence relations, especially those arising from linear recurrences with leading term positive. We investigate legal decompositions arising from two new sequences: the (s, b)-Generacci sequence and the Fibonacci Quilt sequence. Both satisfy recurrence relations with leading term zero, and thus previous results and techniques do not apply. These sequences exhibit drastically different behavior. We show that the (s, b)-Generacci sequence leads to unique legal decompositions, whereas not only do we have non-unique legal decompositions with the Fibonacci Quilt sequence, we also have that in this case the average number of legal decompositions grows exponentially. Another interesting difference is that while in the (s, b)-Generacci case the greedy algorithm always leads to a legal decomposition, in the Fibonacci Quilt setting the greedy algorithm leads to a legal decomposition (approximately) 93% of the time. In the (s, b)-Generacci case, we again have Gaussian behavior in the number of summands as well as for the Fibonacci Quilt sequence when we restrict to decompositions resulting from a modified greedy algorithm.

1. Introduction

A beautiful result of Zeckendorf describes the Fibonacci numbers as the unique sequence from which every natural number can be expressed uniquely as a sum of nonconsecutive terms in the sequence [18]. Zeckendorf's theorem inspired many questions about this decomposition, and generalizations of the notions of legal decompositions of natural numbers as sums of elements from an integer sequence has been a fruitful area of research [2, 3, 4, 5, 6, 9, 10, 11, 13, 14, 15, 16, 17].

Much of previous work has focused on sequences given by a *Positive Linear Recurrence (PLR)*, which are sequences where there is a fixed depth $L > 0$ and non-negative integers $c_1, \dots, c_L$ with c_1, c_L non-zero such that

$$a_{n+1} \;=\; c_1 a_n + \cdots + c_L a_{n+1-L}. \tag{1.1}$$

The restriction that the negative of the trace is positive, i.e. $c_1 > 0$, is required to gain needed control over roots of polynomials associated to the characteristic polynomials of the recurrence and related generating functions, though in the companion paper [8] we show how to bypass some of these technicalities through new combinatorial techniques. The motivation for this paper is to investigate whether the positivity of the first coefficient is needed solely to simplify the arguments, or if fundamentally different behavior can emerge if the said condition is not

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met. To this end, we investigate the legal decompositions arising from two different sequences which we introduce in this paper: the (s, b)-Generacci sequence and the Fibonacci Quilt sequence. Both satisfy recurrence relations with leading term zero, hence previous results and techniques are not applicable. Moreover, although both have non-positive linear recurrences (as their leading term is zero), they exhibit drastically different behavior: the (s, b)-Generacci sequence leads to unique legal decompositions, whereas not only do we have non-unique legal decompositions with the Fibonacci Quilt sequence, we also have that the average number of legal decompositions grows exponentially. Another interesting difference is that while in the (s, b)-Generacci case the greedy algorithm always leads to a legal decomposition, in the Fibonacci Quilt setting the greedy algorithm leads to a legal decomposition (approximately) 93% of the time.

We conclude the introduction by first describing the two sequences and their resulting decomposition rules and then stating our results. Then in §2 we determine the recurrence relations for the sequences, in §3 we prove our claims on the growth of the average number of decompositions from the Fibonacci Quilt sequence, and then analyze the greedy algorithm and a generalization (for the Fibonacci Quilt sequence) in §4.

1.1. The (s, b)-Generacci Sequence and the Fibonacci Quilt Sequence.

1.1.1. *The (s, b)-Generacci Sequence.*

One interpretation of Zeckendorf's Theorem [18] is that the Fibonacci sequence is the unique sequence from which all natural numbers can be expressed as a sum of nonconsecutive terms. Note there are two ingredients to the rendition: a sequence and a rule for determining what is a legal decomposition. An equivalent formulation for the Fibonacci numbers is to consider the sequence divided into bins of size one and decompositions can use the element in a bin at most once and cannot use elements from adjacent bins. A generalization of this bin idea was explored by the authors in [6], where bins of size 2 with the same non-adjacency condition were considered; the sequence that arose was called the Kentucky sequence. The Kentucky sequence is what we now refer to here as the $(1, 2)$-Generacci sequence. This leads to a natural extension where we consider bins of size b and any two summands of a decomposition must come from distinct bins with at least s bins between them. We now give the technical definitions of the (s, b)-Generacci sequences and their associated legal decompositions.

Definition 1.1 ((s, b)-Generacci legal decompositions)**.** *For fixed integers $s, b \geq 1$, let an increasing sequence of positive integers $\{a_i\}_{i=1}^{\infty}$ and a family of subsequences*

$$\mathcal{B}_n = \{a_{b(n-1)+1}, \dots, a_{bn}\}$$

be given (we call these subsequences bins*). We declare a decomposition of an integer $m = a_{\ell_1} + a_{\ell_2} + \cdots + a_{\ell_k}$ where $a_{\ell_i} > a_{\ell_{i+1}}$ to be an (s, b)-*Generacci legal decomposition *provided $\{a_{\ell_i}, a_{\ell_{i+1}}\} \not\subset \mathcal{B}_{j-s} \cup \mathcal{B}_{j-s+1} \cup \cdots \cup \mathcal{B}_j$ for all i, j. (We say $\mathcal{B}_j = \emptyset$ for $j \leq 0$.)*

Thus if we have a summand $a_{\ell_i} \in \mathcal{B}_j$ in a legal decomposition, we cannot have any other summands from that bin, nor any summands from any of the s bins preceding or any of the s bins following $\mathcal{B}_j$.

Definition 1.2 ((s, b)-Generacci sequence)**.** *For fixed integers $s, b \geq 1$, an increasing sequence of positive integers $\{a_i\}_{i=1}^{\infty}$ is the (s, b)-*Generacci sequence *if every a_i for $i \geq 1$ is the smallest positive integer that does not have an (s, b)-Generacci legal decomposition using the elements $\{a_1, \dots, a_{i-1}\}$.*

Using the above definition and Zeckendorf's theorem, we see that the $(1,1)$-Generacci sequence is the Fibonacci sequence (appropriately normalized). Some other known sequences arising from the (s,b)-Generacci sequences are Narayana's cow sequence, which is the $(2,1)$-Generacci sequence, and the Kentucky sequence, which is the $(1,2)$-Generacci sequence.

Theorem 1.3 (Recurrence Relation and Explicit Formula)**.** *Let $s, b \geq 1$ be fixed. If $n > (s+1)b+1$, then the n^{th} term of the (s,b)-Generacci sequence is given by the recurrence relation*

$$a_n \ = \ a_{n-b} + b a_{n-(s+1)b}. \tag{1.2}$$

We have a generalized Binet's formula, with

$$a_n \ = \ c_1 \lambda_1^n \left[1 + O\left((\lambda_2/\lambda_1)^n\right)\right] \tag{1.3}$$

where λ_1 is the largest root of $x^{(s+1)b} - x^{sb} - b = 0$, and c_1 and λ_2 are constants with $\lambda_1 > 1$, $c_1 > 0$ and $|\lambda_2| < \lambda_1$.

Remark 1.4. *The (s,b)-Generacci sequence also satisfies the recurrence*

$$a_n \ = \ a_{n-1} + a_{n-1-f(n-1)}, \tag{1.4}$$

where $f(kb+j) = sb + j - 1$ for $j = 1, \dots, b$. While this representation does have its leading coefficient positive, note the depth $L = f(n-1) + 1$ is not *independent of n, and thus this representation is not a Positive Linear Recurrence.*

The proof of Theorem 1.3 is given in §2.1. We note that the leading term in the recurrence in (1.2) is zero whenever $b \geq 2$, and hence this sequence falls out of the scope of the Positive Linear Recurrences results.

1.1.2. *Fibonacci Quilt Sequence.*

The Fibonacci Quilt sequence arose from the goal of finding a sequence coming from a 2-dimensional process. We begin by recalling the beautiful fact that the Fibonacci numbers tile the plane with squares spiraling to infinity, where the side length of the n^{th} square is F_n (see Figure 1; note that here we start the Fibonacci sequence with two 1's).

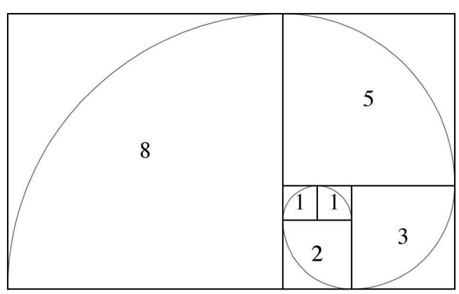

FIGURE 1. The (start of the) Fibonacci Spiral.

Inspired by Zeckendorf decomposition rules and by the Fibonacci spiral we define the following notion of legal decompositions and create the associated integer sequence which we call the Fibonacci Quilt sequence. The spiral depicted in Figure 1 can be viewed as a log cabin *quilt* pattern, such as that presented in Figure 2 (left). Hence we adopt the name Fibonacci Quilt sequence.

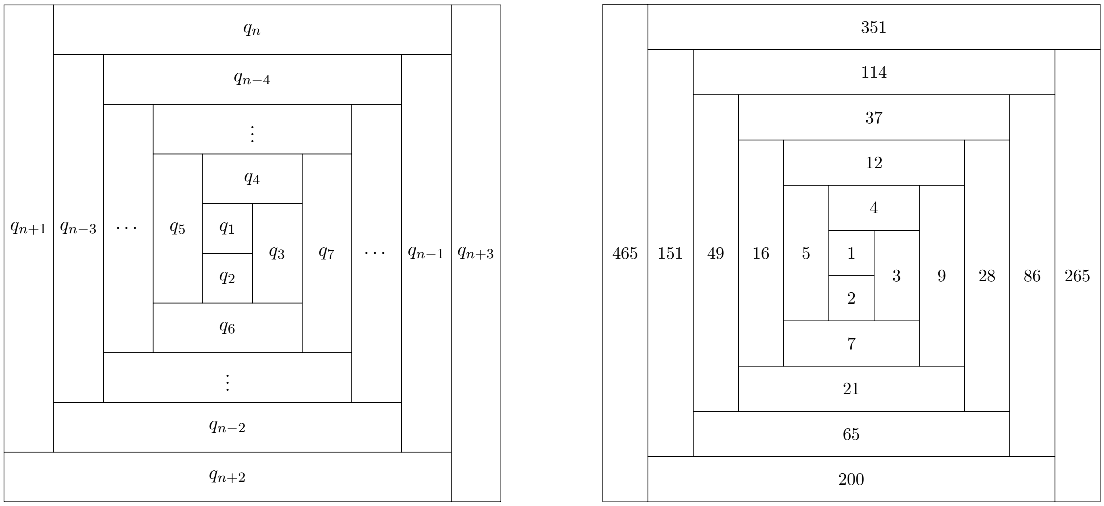

FIGURE 2. (Left) Log Cabin Quilt Pattern. (Right) First few terms of the Fibonacci Quilt sequence.

Definition 1.5 (FQ-legal decomposition). *Let an increasing sequence of positive integers* $\{q_i\}_{i=1}^{\infty}$ *be given. We declare a decomposition of an integer*

$$m \ = \ q_{\ell_1} + q_{\ell_2} + \cdots + q_{\ell_t} \tag{1.5}$$

(where $q_{\ell_i} > q_{\ell_{i+1}}$*) to be an* FQ-legal decomposition *if for all* i, j, $|\ell_i - \ell_j| \neq 0, 1, 3, 4$ *and* $\{1, 3\} \not\subset \{\ell_1, \ell_2, \ldots \ell_t\}$.

This means that if the terms of the sequence are arranged in a spiral in the rectangles of a log cabin quilt, we cannot use two terms if they share part of an edge. Figure 2 shows that $q_n + q_{n-1}$ is not legal, but $q_n + q_{n-2}$ is legal for $n \geq 4$. The starting pattern of the quilt forbids decompositions that contain $q_3 + q_1$.

We define a new sequence $\{q_n\}$, called the Fibonacci Quilt sequence, in the following way.

Definition 1.6 (Fibonacci Quilt sequence). *An increasing sequence of positive integers* $\{q_i\}_{i=1}^{\infty}$ *is called the* Fibonacci Quilt sequence *if every* q_i $(i \geq 1)$ *is the smallest positive integer that does not have an FQ-legal decomposition using the elements* $\{q_1, \ldots, q_{i-1}\}$.

From the definition of an FQ-legal decomposition, the reader can see that the first five terms of the sequence must be $\{1, 2, 3, 4, 5\}$. We have $q_6 \neq 6$ as $6 = q_4 + q_2 = 4 + 2$ is an FQ-legal decomposition. We must have $q_6 = 7$. Continuing we have the start of the Fibonacci Quilt sequence displayed in Figure 2 (right). Note that with the exception of a few initial terms, the Fibonacci Quilt sequence and the Padovan (see entry A000931 from the OEIS) sequence are eventually identical.

Theorem 1.7 (Recurrence Relations). *Let* q_n *denote the* n^{th} *term in the Fibonacci Quilt sequence. Then*

$$\text{for } n \geq 6,\ q_{n+1} \ = \ q_n + q_{n-4}, \tag{1.6}$$

$$\text{for } n \geq 5,\ q_{n+1} \ = \ q_{n-1} + q_{n-2}, \tag{1.7}$$

$$\sum_{i=1}^{n-5} q_i \ = \ q_n - 6. \tag{1.8}$$

The proof is given in §2.2.

Remark 1.8. *At first the above theorem seems to suggest that the Fibonacci Quilt sequence* is *a PLR, as* (1.6) *gives us a recurrence where the leading coefficient is positive and, unlike the alternative expression for the* (s, b)*-Generacci, this time the depth is fixed. The reason it is not a PLR is subtle, and has to do with the second part of the definition: the decomposition law. The decomposition law is* not *from using* (1.6) *to reduce summands, but from the geometry of the spiral. It is worth remarking that* (1.7) *is the minimal length recurrence for this sequence, and the characteristic polynomial arising from* (1.6) *is divisible by the polynomial from* (1.7)*.*

1.2. **Results.** Our theorems are for two sequences which satisfy recurrences with leading term zero. Prior results in the literature mostly considered Positive Linear Recurrences and results included the uniqueness of legal decompositions, Gaussian behavior of the number of summands, and exponential decay in the distribution of the gaps between summands [2, 4, 9, 10, 16, 17]. In [6], a first example of a non-positive linear recurrence appeared and the aforementioned results were proved using arguments technically similar to those already present in the literature. What is new in this paper are two extensions of the work presented in [6]. The first is the (s, b)-Generacci sequence, whose legal decompositions are unique but where new techniques are required to prove its various properties. The second is the more interesting newly discovered Fibonacci Quilt sequence, which displays drastically different behavior, one consequence being that the FQ-legal decompositions are not unique (for example, there are three distinct FQ-legal decompositions of 106: 86+16+4, 86+12+7+1, and 65+37+4).

1.2.1. *Decomposition results.*

Theorem 1.9 (Uniqueness of Decompositions for (s, b)-Generacci)**.** *For each pair of integers* $s, b \geq 1$*, a unique* (s, b)*-Generacci sequence exists. Consequently, for a given pair of integers* $s, b \geq 1$*, every positive integer can be written uniquely as a sum of distinct terms of the* (s, b)*-Generacci sequence where no two summands are in the same bin, and between any two summands there are at least* s *bins between them.*

As Theorem 1.9 follows from a similar argument to that in the appendix of [6], we omit it in this paper.

Remark 1.10. *We could also prove this result by showing that our sequence and legal decomposition rule give rise to an* f*-decomposition. These were defined and studied in* [9]*, and briefly a valid* f*-decomposition means that for each summand chosen a block of consecutive summands before are not available for use, and that number depends solely on* n*. The methods of* [9] *are applicable and yield that each positive integer has a unique legal decomposition.*

These results are not available for the Fibonacci Quilt sequence, as the FQ-legal decomposition is not an f*-decomposition. The reason is that in an* f*-decomposition there is a function* f *such that if we have* q_n *then we cannot have any of the* $f(n)$ *terms of the sequence immediately prior to* q_n*. There is no such* f *for the Fibonacci Quilt sequence, as for* $n \geq 8$ *if we have* q_n *we cannot have* q_{n-1} *and* q_{n-3} *but we can have* q_{n-2}*.*

We have already seen that the Fibonacci Quilt sequence exhibits non-unique decompositions; this is just the beginning of the difference in behavior. The first result concerns the exponential number of FQ-legal decompositions as we decompose larger integers. First we need to introduce some notation. Let $\{q_n\}$ denote the Fibonacci Quilt sequence. For each positive integer m let $d_{\rm FQ}(m)$ denote the number of FQ-legal decomposition of m, and $d_{\rm FQ;ave}(n)$ the average number of FQ-legal decompositions of integers in $I_n := [0, q_{n+1})$; thus

$$d_{\rm FQ;ave}(n) \ := \ \frac{1}{q_{n+1}} \sum_{m=0}^{q_{n+1}-1} d_{\rm FQ}(m). \tag{1.9}$$

In §3 we prove the following.

Theorem 1.11 (Growth Rate of Average Number of Decompositions)**.** *Let r_1 be the largest root of $r^7 - r^6 - r^2 - 1 = 0$ (so $r_1 \approx 1.39704$) and let λ_1 be the largest root of $x^3 - x - 1 = 0$ (so $\lambda_1 = \frac{1}{3}\left(\frac{27}{2} - \frac{3\sqrt{69}}{2}\right)^{1/3} + 3^{-2/3}\left(\frac{1}{2}\left(9+\sqrt{69}\right)\right)^{1/3} \approx 1.32472$), and set $\lambda = r_1/\lambda_1 \approx 1.05459$. There exist computable constants $C_2 > C_1 > 0$ such that for all n sufficiently large,*

$$C_1\lambda^n \ \le \ d_{\rm FQ;ave}(n) \ \le \ C_2\lambda^n. \tag{1.10}$$

Thus the average number of FQ-legal decompositions of integers in $[0, q_{n+1})$ tends to infinity exponentially fast.

Remark 1.12. *At the cost of additional algebra one could prove the existence of a constant C such that $d_{\rm FQ;ave}(n) \sim C\lambda^n$; however, as the interesting part of the above theorem is the exponential growth and not the multiplicative factor, we prefer to give the simpler proof which captures the correct growth rate.*

We end with another new behavior. For many of the previous recurrences, the greedy algorithm successfully terminates in a legal decomposition; that is *not* the case for the Fibonacci Quilt sequence. In §4 we prove the following.

Theorem 1.13. *There is a computable constant $\rho \in (0,1)$ such that, as $n \to \infty$, the percentage of positive integers in $[1, q_n)$ where the greedy algorithm terminates in a Fibonacci Quilt sequence legal decomposition converges to ρ. This constant is approximately .92627.*

Interestingly, a simple modification of the greedy algorithm *does* always terminate in a legal decomposition, and this decomposition yields a minimal number of summands.

Definition 1.14 (Greedy-6 Decomposition)**.** *The Greedy-6 Decomposition writes m as a sum of Fibonacci Quilt numbers as follows:*

- *if there is an n with $m = q_n$ then we are done,*
- *if $m = 6$ then we decompose m as $q_4 + q_2$ and we are done, and*
- *if $m \ge q_6$ and $m \ne q_n$ for all $n \ge 1$, then we write $m = q_{\ell_1} + x$ where $q_{\ell_1} < m < q_{\ell_1+1}$ and $x > 0$, and then iterate the process with input $m := x$.*

We denote the decomposition that results from the Greedy-6 Algorithm by $\mathcal{G}(m)$.

Theorem 1.15. *For all $m > 0$, the Greedy-6 Algorithm results in a FQ-legal decomposition. Moreover, if $\mathcal{G}(m) = q_{\ell_1} + q_{\ell_2} + \cdots + q_{\ell_{t-1}} + q_{\ell_t}$ with $q_{\ell_1} > q_{\ell_2} > \cdots > q_{\ell_t}$, then the decomposition satisfies exactly one of the following conditions:*

(1) $\ell_i - \ell_{i+1} \ge 5$ *for all* i *or*
(2) $\ell_i - \ell_{i+1} \ge 5$ *for* $i \le t-3$ *and* $\ell_{t-2} \ge 10$, $\ell_{t-1} = 4$, $\ell_t = 2$.

Further, if $m = q_{\ell_1} + q_{\ell_2} + \cdots + q_{\ell_{t-1}} + q_{\ell_t}$ with $q_{\ell_1} > q_{\ell_2} > \cdots > q_{\ell_t}$ denotes a decomposition of m where either

(1) $\ell_i - \ell_{i+1} \ge 5$ *for all* i *or*
(2) $\ell_i - \ell_{i+1} \ge 5$ *for* $i \le t-3$ *and* $\ell_{t-2} \ge 10$, $\ell_{t-1} = 4$, $\ell_t = 2$,

then $q_{\ell_1} + q_{\ell_2} + \cdots + q_{\ell_{t-1}} + q_{\ell_t} = \mathcal{G}(m)$. That is, the decomposition of m is the Greedy-6 decomposition.

Let $\mathcal{D}(m)$ be a given decomposition of m as a sum of Fibonacci Quilt numbers (not necessarily legal):

$$m \;=\; c_1q_1 + c_2q_2 + \cdots + c_nq_n, \quad c_i \in \{0, 1, 2, \dots\}. \tag{1.11}$$

We define the number of summands by

$$\#\text{summands}(\mathcal{D}(m)) \;:=\; c_1 + c_2 + \cdots + c_n. \tag{1.12}$$

Theorem 1.16. *If $\mathcal{D}(m)$ is any decomposition of m as a sum of Fibonacci Quilt numbers, then*

$$\#\text{summands}(\mathcal{G}(m)) \;\le\; \#\text{summands}(\mathcal{D}(m)). \tag{1.13}$$

1.2.2. *Gaussian Behavior of Number of Summands in (s, b)-Generacci legal decompositions.* Below we report on the distribution of the number of summands in the (s, b)-Generacci legal decompositions. In attacking this problem we developed a new technique similar to ones used before but critically different in that we are able to bypass technical assumptions that other papers needed to prove a Gaussian distribution. We elaborate on this method in [8], where we also determine the distribution of gaps between summands. We have chosen to concentrate on the Fibonacci Quilt sequence results in this paper, and just state many of the (s, b)-Generacci sequence outcomes, as we see the same behavior as in other systems for the (s, b)-Generacci numbers, but see fundamentally new behavior for the Fibonacci Quilt sequence.

Theorem 1.17 (Gaussian Behavior of Summands for (s, b)-Generacci)**.** *Let the random variable Y_n denote the number of summands in the (unique) (s, b)-Generacci legal decomposition of an integer picked at random from $[0, a_{bn+1})$ with uniform probability.*[1] *Then for μ_n and σ_n^2, the mean and variance of Y_n, we have*

$$\mu_n \;=\; An + B + o(1) \tag{1.14}$$

$$\sigma_n^2 \;=\; Cn + D + o(1) \tag{1.15}$$

for some positive constants A, B, C, D. Moreover if we normalize Y_n to $Y_n' = (Y_n - \mu_n)/\sigma_n$, then Y_n' converges in distribution to the standard normal distribution as $n \to \infty$.

Unfortunately, the above methods do not directly generalize to Gaussian results for the Fibonacci Quilt sequence. Interestingly and fortunately there is a strong connection between the two sequences, and in [8] we show how to interpret many questions concerning the Fibonacci Quilt sequence to a weighted average of several copies of the $(4, 1)$-Generacci sequence. This correspondence is not available for questions on unique decomposition, but does immediately yield Gaussian behavior and determines the limiting behavior of the individual gap measures.

2. Recurrence Relations

2.1. **Recurrence Relations for the (s, b)-Generacci Sequence.** Recall that for $s, b \ge 1$, an (s, b)-Generacci decomposition of a positive integer is legal if the following conditions hold.

(1) No term a_i is used more than once.
(2) No two distinct terms a_i, a_j in a decomposition can have indices i, j from the same bin.
(3) If a_i and a_j are summands in a legal decomposition, then there are at least s bins between them.

[1] Using the methods of [3], these results can be extended to hold almost surely for sufficiently large subinterval of $[a_{(n-1)b+1}, a_{bn+1})$.

The terms of the (s,b)-Generacci sequence can be pictured as follows:

$$\underbrace{a_1,\dots,a_b}_{\mathcal{B}_1},\underbrace{a_{1+b},\dots,a_{2b}}_{\mathcal{B}_2},\dots,\underbrace{a_{1+nb},\dots,a_{(n+1)b}}_{\mathcal{B}_{n+1}},\underbrace{a_{1+(n+1)b},\dots,a_{(n+2)b}}_{\mathcal{B}_{n+2}},\dots. \tag{2.1}$$

We now prove the following results related to the elements of the (s,b)-Generacci sequence.

Lemma 2.1. *If $s,b \geq 1$, then $a_i = i$ for all $1 \leq i \leq (s+1)b+1$, where a_i is the i^{th} term in the (s,b)-Generacci sequence.*

Proof. This follows directly from the definition of the (s,b)-Generacci sequence. That is, we note that at the $(s+1)^{\text{th}}$-bin, we clearly have s bins to the left, yet we are unable to use any elements from those bins to decompose any new integers. Thus $a_i = i$, for all $1 \leq i \leq (s+1)b+1$. □

Lemma 2.2. *If k can be decomposed using summands $\{a_1,\dots,a_p\}$, then so can $k-1$.*

Proof. Let $k = a_{\ell_1} + a_{\ell_2} + \cdots + a_{\ell_t}$ with $\ell_1 > \ell_2 > \cdots > \ell_t$ be a legal decomposition of k. So $k-1 = a_{\ell_1} + a_{\ell_2} + \cdots + (a_{\ell_t} - 1)$.

It must be the case that either $a_{\ell_t} - 1$ is zero or it has a legal decomposition with summands indexed smaller than ℓ_t, as a_{ℓ_t} was added because it was the smallest integer that could not be legally decomposed with summands indexed smaller than ℓ_t. If ℓ_t was sufficiently distant from ℓ_{t-1} for the decomposition of k to be legal, using summands with even smaller indices does not create an illegal interaction with the remaining summands $a_{\ell_1},\dots,a_{\ell_{t-1}}$. □

This lemma allows us to conclude that the smallest integer that does not have a legal decomposition using $\{a_1,\dots,a_n\}$ is one more than the largest integer that does have a legal decomposition using $\{a_1,\dots,a_n\}$.

Lemma 2.3. *If $s,b,n \geq 1$ and $1 \leq j \leq b+1$, then*

$$a_{j+nb} \;=\; a_{1+nb} + (j-1)a_{1+(n-s)b}. \tag{2.2}$$

Proof. The term a_{1+nb} is the first entry in the $(n+1)^{\text{st}}$ bin and trivially satisfies the recursion relation for $j=1$.

Recall a legal decomposition containing a member of the $(n+1)^{\text{st}}$ bin would not have other addends from any of bins $\{\mathcal{B}_{n-s+1},\mathcal{B}_{n-s+2},\dots,\mathcal{B}_n,\mathcal{B}_{n+1}\}$. So by construction we have $a_{2+nb} = a_{1+nb} + a_{1+(n-s)b}$, as the largest integer that can be legally decomposed using addends from bins $\mathcal{B}_1,\mathcal{B}_2,\dots,\mathcal{B}_{n-s}$ is $a_{1+(n-s)b} - 1$.

Using the same argument we have

$$a_{3+nb} \;=\; a_{2+nb} + a_{1+(n-s)b} \;=\; a_{1+nb} + 2a_{1+(n-s)b}. \tag{2.3}$$

We proceed similarly for $j = 4,\dots,b$. For $j = b+1$, the term $a_{b+1+nb} = a_{1+(n+1)b}$ is the first entry in the $(n+2)^{\text{nd}}$ bin. By construction $a_{1+(n+1)b} = a_{(n+1)b} + a_{1+(n-s)b}$. Using Equation (2.2) with $j = b$ we have

$$a_{1+(n+1)b} \;=\; a_{(n+1)b} + a_{1+(n-s)b} = a_{1+nb} + (b-1)a_{1+(n-s)b} + a_{1+(n-s)b} \;=\; a_{1+nb} + ba_{1+(n-s)b}. \tag{2.4}$$

□

Proof of Theorem 1.3. Fix $s, b \geq 1$. We proceed by considering i of the form $j + nb$, $j \in \{1, \ldots, b\}$, so $a_i = a_{j+nb}$ is the j^{th} entry in the $(n+1)^{\text{st}}$ bin. Using Lemma 2.3,

$$\begin{aligned} a_{j+nb} &= a_{1+nb} + (j-1)a_{1+(n-s)b} \\ &= a_{1+(n-1)b} + ba_{1+(n-s-1)b} + (j-1)a_{1+(n-s)b} \\ &= a_{1+(n-1)b} + (j-1)a_{1+(n-s-1)b} + (b-j+1)a_{1+(n-s-1)b} + (j-1)a_{1+(n-s)b} \\ &= a_{j+(n-1)b} + (b-j+1)a_{1+(n-s-1)b} + (j-1)a_{1+(n-s)b}. \end{aligned} \tag{2.5}$$

Again using the construction of our sequence we have $a_{1+(n-s)b} = a_{(n-s)b} + a_{1+(n-2s-1)b}$. This substitution gives

$$\begin{aligned} a_{j+nb} &= a_{j+(n-1)b} + (b-j+1)a_{1+(n-s-1)b} + (j-1)a_{(n-s)b} + (j-1)a_{1+(n-2s-1)b} \\ &= a_{j+(n-1)b} + a_{1+(n-s-1)b} + (j-1)a_{1+(n-2s-1)b} + (b-j)a_{1+(n-s-1)b} + (j-1)a_{(n-s)b} \\ &= a_{j+(n-1)b} + a_{j+(n-s-1)b} + (b-j)a_{1+(n-s-1)b} + (j-1)a_{(n-s)b}. \end{aligned} \tag{2.6}$$

Note that by Lemma 2.3, $a_{(n-s)b} = a_{1+(n-s-1)b} + (b-1)a_{1+(n-2s-1)b}$, so the last two terms in (2.6) may be simplified as

$$\begin{aligned} (b-j)a_{1+(n-s-1)b} &+ (j-1)a_{1+(n-s-1)b} + (j-1)(b-1)a_{1+(n-2s-1)b} \\ &= (b-1)\left[a_{1+(n-s-1)b} + (j-1)a_{1+(n-2s-1)b}\right] \\ &= (b-1)a_{j+(n-s-1)b}. \end{aligned} \tag{2.7}$$

Substituting (2.7) into Equation (2.6) yields

$$\begin{aligned} a_{j+nb} &= a_{j+(n-1)b} + a_{j+(n-s-1)b} + (b-1)a_{j+(n-s-1)b} \\ &= a_{j+(n-1)b} + ba_{j+(n-s-1)b}, \end{aligned} \tag{2.8}$$

which completes the proof of the first part of Theorem 1.3.

For the proof of the second part, we have from Lemma 2.3

$$a_{j+nb} = a_{j-1+nb} + a_{1+(n-s)b}, \tag{2.9}$$

thus

$$a_{j+nb} = a_{j-1+nb} + a_{j-1+nb-(sb+j-2)} \tag{2.10}$$

for $j = 2, \ldots, b+1$. The result now follows if we define $f(j+nb) = sb+j-1$, for $j = 1, \ldots, b$.

We prove the Generalized Binet Formula and the approximation in Appendix A of [7]. □

2.2. **Recurrence Relations for the Fibonacci Quilt Sequence.**

Proof of Theorem 1.7. The proof is by induction. The basis cases for $n \leq 11$ can be checked by brute force. We now turn to the inductive step. We assume (1.6), (1.7), and (1.8) hold for all $n = 1, 2, 3, \ldots, k-1$ where $k \geq 12$. Next we must show they hold for $n = k$.

We first note that by construction we can legally decompose all numbers in the interval $[1, q_{k-4} - 1]$ using terms in $\{q_1, \ldots, q_{k-5}\}$; q_{k-4} was added to the sequence because it was the first number that could not be decomposed using those terms. So, using q_k and any subset of $\{q_1, \ldots, q_{k-5}\}$ as summands, we can legally decompose all numbers in the interval, $[q_k, q_k + q_{k-4} - 1]$. We can decompose all numbers in the interval $[q_{k-4}, q_k - 1]$ using summands from $\{q_1, \ldots, q_{k-1}\}$ and thus we can decompose all numbers in the interval $[1, q_k + q_{k-4} - 1]$ using $\{q_1, \ldots, q_k\}$. The term q_{k+1} will be the smallest number that we cannot legally decompose using $\{q_1, \ldots, q_k\}$. The argument above shows that $q_{k+1} \geq q_k + q_{k-4}$.

Notice

$$\begin{aligned} q_k + q_{k-4} &= (q_{k-1} + q_{k-5}) + q_{k-4} \\ &= q_{k-1} + (q_{k-4} + q_{k-5}) \\ &= q_{k-1} + q_{k-2}. \end{aligned} \tag{2.11}$$

It remains to show that there is no legal decomposition of $m = q_k + q_{k-4} = q_{k-1} + q_{k-2}$. If q_k were in the decomposition of m, the remaining summands would have to add to q_{k-4}. But that is a contradiction as q_{k-4} was added to the sequence because it had no legal decompositions as sums of other terms. Similarly, we can see that any legal decomposition of m does not use $q_{k-1}, q_{k-2}, q_{k-4}$.

Notice that q_{k-3} must be part of any possible legal decomposition of m: if it were not, then $m < \sum_{i=1}^{k-5} q_i = q_k - 6 < q_k < q_k + q_{k-4} = m$. Hence any legal decomposition would have $m = q_{k-3} + x$, where the largest possible summand in the decomposition of x is q_{k-5}.

Now assume we have a legal decomposition of $m = q_{k-3} + x$. There are two cases.

Case 1: The legal decomposition of x uses q_{k-5} as a summand. So

$$m = q_{k-3} + x = q_{k-3} + q_{k-5} + y \tag{2.12}$$

and y can be legally decomposed using summands from $\{q_1, q_2, \dots, q_{k-10}\}$. Then using Equation (1.8), $y < \sum_{i=1}^{n-10} q_i = q_{k-5} - 6$. This leads us to the following:

$$\begin{aligned} q_k + q_{k-4} = m &< q_{k-3} + q_{k-5} + q_{k-5} - 6 \\ &< q_{k-3} + q_{k-4} + q_{k-5} - 6 \\ &= q_{k-1} + q_{k-5} - 6 \\ &= q_k - 6 \\ &< q_k, \end{aligned} \tag{2.13}$$

a contradiction.

Case 2: The largest possible summand used in the legal decomposition of x is q_{k-8}. Thus

$$\begin{aligned} q_k + q_{k-4} = m &< q_{k-3} + \sum_{i=1}^{k-8} q_i \\ &= q_{k-3} + q_{k-3} - 6 \\ &< q_{k-2} + q_{k-3} \\ &< q_k, \end{aligned} \tag{2.14}$$

another contradiction.

So m cannot be legally decomposed using $\{q_1, \dots, q_k\}$ and $q_{k+1} = q_k + q_{k-4}$. The proof of Equation (1.7) follows from the work done in Equation (2.11). To prove Equation (1.8), note

$$\sum_{i=1}^{k-5} q_i = q_{k-5} + \sum_{i=1}^{k-6} q_i = q_{k-5} + q_{k-1} - 6 = q_k - 6. \tag{2.15}$$

Thus equations (1.6), (1.7), and (1.8) hold for all $n \geq 5$. □

Proposition 2.4 (Explicit Formula)**.** *Let q_n denote the n^{th} term in the Fibonacci Quilt sequence. Then*

$$q_n = \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \alpha_3 \overline{\lambda_2}^n, \tag{2.16}$$

where $\alpha_1 \approx 1.26724$*,*

$$\lambda_1 \;=\; \frac{1}{3}\left(\frac{27}{2}-\frac{3\sqrt{69}}{2}\right)^{1/3}+\frac{\left(\frac{1}{2}\left(9+\sqrt{69}\right)\right)^{1/3}}{3^{2/3}} \;\approx\; 1.32472 \tag{2.17}$$

and $\lambda_2 \approx -0.662359 - 0.56228i$ *(which has absolute value approximately 0.8688).*

Proof. Using the recurrence relation from Equation (1.6) in Theorem 1.7, we have the characteristic equation

$$x^3 \;=\; x+1. \tag{2.18}$$

Hence $q_n \;=\; \alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \alpha_3\overline{\lambda_2}^n$, where λ_1, λ_2 and $\overline{\lambda_2}$ are the three distinct solutions to the characteristic equation, which are easily found by the cubic formula.

We solve for the α_i using the first few terms of the sequence. Straightforward calculations reveal

$$\begin{aligned}\alpha_1 &\approx 1.26724\\ \alpha_2 &\approx -0.13362 + 0.128277i\\ \alpha_3 &\approx -0.13362 - 0.128277i,\end{aligned} \tag{2.19}$$

completing the proof. □

3. Growth Rate of Number of Decompositions for the Fibonacci Quilt Sequence

We prove Theorem 1.11 by deriving a recurrence relation for the number of FQ-legal decompositions. Specifically, consider the following definitions.

- d_n: the number of FQ-legal decompositions using only elements of $\{q_1, q_2, \dots, q_n\}$. Note we include one empty decomposition of 0 in this count. Further, some of the decompositions are of numbers larger than q_{n+1} (for example, for n large $q_n + q_{n-2} + q_{n-20} > q_{n+1}$). We set $d_0 = 1$.
- c_n: the number of FQ-legal decompositions using only elements of $\{q_1, q_2, \dots, q_n\}$ *and* q_n is one of the summands. We set $c_0 = 1$.
- b_n: the number of FQ-legal decompositions using only elements of $\{q_1, q_2, \dots, q_n\}$ *and* both q_n and q_{n-2} are used.

By brute force one can compute the first few values of these sequences; see Table 1.

We first find three recurrence relations interlacing our three unknowns.

Lemma 3.1. *For* $n \geq 7$ *we have*

$$\begin{aligned} d_n &= c_n + c_{n-1} + \cdots + c_0 \;=\; c_n + d_{n-1} && (3.1)\\ c_n &= d_{n-5} + c_{n-2} - b_{n-2} && (3.2)\\ b_n &= d_{n-7}, && (3.3)\end{aligned}$$

which implies

$$d_n \;=\; d_{n-1} + d_{n-2} - d_{n-3} + d_{n-5} - d_{n-9}. \tag{3.4}$$

Proof. The relation for d_n in (3.1) is the simplest to see. The left hand side counts the number of FQ-legal decompositions where the largest element used is q_n, which may or may not be used. The right hand side counts the same quantity, partitioning based on the largest index used. It is important to note that c_0 is included and equals 1, as otherwise we would not have

n	d_n	c_n	b_n	q_n
1	2	1	0	1
2	3	1	0	2
3	4	1	0	3
4	6	2	1	4
5	8	2	1	5
6	11	3	1	7
7	15	4	1	9
8	21	6	2	12
9	30	9	3	16
10	42	12	4	21
11	59	17	6	28
12	82	23	8	37
13	114	32	11	49

TABLE 1. Values of the first few terms of d_n, c_n and b_n; for ease of comparison we have included q_n as well.

the empty decomposition (corresponding to an FQ-legal decomposition of 0). We immediately use this relation with $n-1$ for n to replace $c_{n-1}+\cdots+c_0$ with d_{n-1}.

Our second relation (3.2) comes from counting the number of FQ-legal decompositions where q_n is used and no larger index occurs, which is just c_n. Since q_n occurs in all such numbers we cannot use q_{n-1}, q_{n-3} or q_{n-4}, but q_{n-2} may or may not be used. If we do not use q_{n-2} then we are left with choosing FQ-legal decompositions where the largest index used is at most $n-5$; by definition this is d_{n-5}. We must add back all the numbers arising from decompositions using q_n and q_{n-2}. Note that if $n-2$ was the largest index used then the number of valid decompositions is c_{n-2}; however, this includes b_{n-2} decompositions where we use both q_{n-2} and q_{n-4}. As we *must* use q_n, we cannot use q_{n-4} and thus these b_{n-2} decompositions should not have been included; thus c_n equals $d_{n-5}+c_{n-2}-b_{n-2}$. (Note: alternatively one could prove the relation $c_n = d_{n-5}+b_n$.)

Finally, consider the relation for b_n (3.3). This counts the times we use q_n (which forbids us from using q_{n-1}, q_{n-3} and q_{n-4}) and q_{n-2} (which forbids us from using q_{n-3}, q_{n-5} and q_{n-6}). Note all other indices at most $n-7$ may or may not be used, and no other larger index can be chosen. By definition the number of valid choices is d_{n-7}.

We now easily derive a recurrence involving just the d's. The first relation yields $c_n = d_n - d_{n-1}$ while the third gives $b_n = d_{n-7}$. We can thus rewrite the second relation involving only d's, which immediately gives (3.4). □

Armed with the above, we solve the recurrence for d_n.

Lemma 3.2. *We have*

$$d_n \;=\; \beta_1 r_1^n \left[1 + O\left((r_2/r_1)^n\right)\right], \tag{3.5}$$

where $\beta_1 > 0$, $|r_1| \approx 1.39704$ *and* $|r_2| \approx 1.07378$ *are the two largest (in absolute value) roots of* $r^7 - r^6 - r^2 - 1 = 0$.

Proof. The characteristic polynomial associated to the recurrence for d_n in (3.4) factors as

$$r^9 - r^8 - r^7 + r^6 - r^4 + 1 \;=\; (r-1)(r+1)(r^7 - r^6 - r^2 - 1). \tag{3.6}$$

The roots of the septic are all distinct, with the largest $|r_1|$ approximately 1.39704 and the next two largest being complex conjugate pairs of size $|r_2| = |\overline{r_2}| \approx 1.07378$; the remaining roots are at most 1 in absolute value. Thus by standard techniques for solving recurrence relations [12] (as the roots are distinct) there are constants such that

$$d_n \;=\; \beta_1 r_1^n + \beta_2 r_2^n + \cdots + \beta_7 r_7^n + \beta_8 1^n + \beta_9 (-1)^n. \tag{3.7}$$

To complete the proof, we need only show that $\beta_1 > 0$ (if it vanished, then d_n would grow slower than one would expect). As the roots come from a degree 7 polynomial, it is not surprising that we do not have a closed form expression for them. Fortunately a simple comparison proves that $\beta_1 > 0$. Since d_n counts the number of FQ-legal decompositions using indices no more than q_n, we must have $d_n \geq q_n$. As q_n grows like λ_1^n with $\lambda_1 \approx 1.3247$, if $\beta_1 = 0$ then $d_n < q_n$ for large n, a contradiction. Thus $\beta_1 > 0$. □

We can now determine the average behavior of $d_{\rm FQ}(m)$, the number of FQ-legal decompositions of m.

Proof of Theorem 1.11. We have

$$d_{\rm FQ;ave}(n) \;=\; \frac{1}{q_{n+1}} \sum_{m=0}^{q_{n+1}-1} d_{\rm FQ}(m). \tag{3.8}$$

We first deal with the upper bound. The summation on the right hand side of Equation (3.8) is less than d_n, because d_n counts some FQ-legal decompositions that exceed q_{n+1}. Thus

$$d_{\rm FQ;ave}(n) \;\leq\; \frac{d_n}{q_{n+1}}. \tag{3.9}$$

For n large by Lemma 3.2 we have

$$d_n \;=\; \beta_1 r_1^n \left[1 + O\left((r_2/r_1)^n\right)\right] \tag{3.10}$$

with $\beta_1 > 0$ and $r_1 \approx 1.39704$, and from Proposition 2.4

$$q_n \;=\; \alpha_1 \lambda_1^n \left[1 + O\left((\lambda_2/\lambda_1)^n\right)\right] \tag{3.11}$$

where $\alpha_1 \approx 1.26724$,

$$\lambda_1 \;=\; \frac{1}{3}\left(\frac{27}{2} - \frac{3\sqrt{69}}{2}\right)^{1/3} + \frac{\left(\frac{1}{2}\left(9+\sqrt{69}\right)\right)^{1/3}}{3^{2/3}} \;\approx\; 1.32472 \tag{3.12}$$

and $\lambda_2 \approx -0.662359 - 0.56228i$ (which has absolute value approximately 0.8688). Thus there is a $C_2 > 0$ such that for n large we have $d_{\rm FQ;ave}(n) \leq C_2 (r_1/\lambda_1)^n$.

We now turn to the lower bound for $d_{\rm FQ;ave}(n)$. As we are primarily interested in the growth rate of $d_{\rm FQ;ave}(n)$ and not on optimal values for the constants C_1 and C_2, we can give a simple argument which suffices to prove the exponential growth rate, though at a cost of a poor choice of C_1. Note that for large n the sum on the right side of Equation (3.8) is clearly at least d_{n-2016}. To see this, note d_{n-2016} counts the number of FQ-legal decompositions using no summand larger than q_{n-2016}, and if q_{n-2016} is our largest summand then by (1.8) our number cannot exceed

$$\sum_{i=1}^{n-2016} q_i \;=\; q_{n-2011} - 6 \;\leq\; q_n. \tag{3.13}$$

Thus

$$d_{\rm FQ;ave}(n) \;\geq\; \frac{d_{n-2016}}{q_{n+1}}. \tag{3.14}$$

We now argue as we did for the upper bound, noting that for large n we have

$$d_{n-2016} \;=\; r_1^{-2016} \cdot \beta_1 r_1^n \left[1 + O\left((r_2/r_1)^n\right)\right]. \tag{3.15}$$

Thus for n sufficiently large

$$d_{\text{FQ;ave}}(n) \;\geq\; C_1(r_1/\lambda_1)^n, \tag{3.16}$$

completing the proof. □

4. Greedy Algorithms for the Fibonacci Quilt Sequence

4.1. **Greedy Decomposition.** Let h_n denote the number of integers from 1 to $q_{n+1}-1$ where the greedy algorithm successfully terminates in a legal decomposition. We have already seen that the first number where the greedy algorithm fails is 6; the others less than 200 are 27, 34, 43, 55, 71, 92, 113, 120, 141, 148, 157, 178, 185 and 194.

Table 2 lists h_n for the first few values of n, as well as ρ_n the percentage of integers in $[1, q_{n+1})$ where the greedy algorithm yields a legal decomposition.

n	q_n	h_n	ρ_n
1	1	1	100.0000
2	2	2	100.0000
3	3	3	100.0000
4	4	4	100.0000
5	5	5	83.3333
6	7	7	87.5000
7	9	10	90.9091
8	12	14	93.3333
9	16	19	95.0000
10	21	25	92.5926
11	28	33	91.6667
12	37	44	91.6667
13	49	59	92.1875
14	65	79	92.9412
15	86	105	92.9204
16	114	139	92.6667
17	151	184	92.4623

Table 2. Values of the first few terms of q_n, h_n and ρ_n.

We start by determining a recurrence relation for h_n.

Lemma 4.1. *For h_n as above,*

$$h_n \;=\; h_{n-1} + h_{n-5} + 1, \tag{4.1}$$

with initial values $h_k = k$ for $1 \leq k \leq 5$.

Proof. We can determine the number integers in $[1, q_{n+1})$ for which the greedy algorithm is successful by counting the same thing in $[1, q_n)$ and in $[q_n, q_{n+1})$. The number of integers in $[1, q_n)$ for which the greedy algorithm is successful is just h_{n-1}.

Integers $m \in [q_n, q_{n+1})$ for which the greedy algorithm is successful must have largest summand q_n. So $m = q_n + x$. We claim $x \in [0, q_{n-4})$. Otherwise $m = q_n + x \geq q_n + q_{n-4} = q_{n+1}$, which is a contradiction. If $x = 0$, then $m = q_n$ can be legally decomposed using the greedy algorithm and we must add 1 to our count. If m is to have a successful legal greedy decomposition then so must x. Hence it remains to count how many $x \in [1, q_{n-4})$ have successful legal greedy decompositions, but this is just h_{n-5}. Combining these counts finishes the proof. □

We now prove the greedy algorithm successfully terminates for a positive percentage of integers, as well as fails for a positive percentage of integers.

Proof of Theorem 1.13. Instead of solving the recurrence in (4.1), it is easier to let $g_n = h_n + 1$ and first solve

$$g_n \ = \ g_{n-1} + g_{n-5}, \quad g_k \ = \ k+1 \text{ for } 1 \le k \le 5. \tag{4.2}$$

The characteristic polynomial for this is

$$r^5 - r^4 - 1 \ = \ 0, \quad \text{or} \quad (r^3 - r - 1)(r^2 - r + 1). \tag{4.3}$$

By standard recurrence relation techniques, we have

$$g_n \ = \ c_1\lambda_1^n + c_2\lambda_2^n + \cdots + c_5\lambda_5^n, \tag{4.4}$$

where

$$\lambda_1 \ = \ \frac{1}{3}\left(\frac{27}{2} - \frac{3\sqrt{69}}{2}\right)^{1/3} + \frac{\left(\frac{1}{2}\left(9+\sqrt{69}\right)\right)^{1/3}}{3^{2/3}} \ \approx \ 1.32472 \tag{4.5}$$

is the largest root of the recurrence for g_n (the other roots are at most 1 in absolute value).

By Proposition 2.4 we have

$$q_n \ = \ \alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \alpha_3\lambda_3^n, \tag{4.6}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the same as in Equation (4.4) and $\alpha_1 \approx 1.26724$.

We must show that $c_1\alpha_1 \neq 0$, as this will imply that g_n and q_n both grow at the same exponential rate. As $g_n \geq 2g_{n-5}$ implies $g_n \geq c2^{n/5}$ we have that g_n is growing exponentially, thus $c_1 \neq 0$.

Unfortunately writing c_1 in closed form requires solving a fifth order equation, but this can easily be done numerically and the limiting ratio $\rho_n = h_n/(q_{n+1} - 1)$ can be approximated well. That ratio converges to $\frac{c_1}{\alpha_1}\frac{1}{\lambda_1} \approx 0.92627$. □

4.2. **Greedy-6 Decomposition.**

Lemma 4.2. *For $\ell \geq 1 + 5k$ and $k \geq 0$, we have $q_\ell + q_{\ell-5} + \cdots + q_{\ell-5k} < q_{\ell+1}$.*

Proof. We proceed by induction on k. For the Basis Step, note

$$q_\ell + q_{\ell-5} \ < \ q_\ell + q_{\ell-4} \ = \ q_{\ell+1}. \tag{4.7}$$

For the Inductive Step: By inductive hypothesis and the recurrence relation stated in Theorem 1.7,

$$q_\ell + (q_{\ell-5} + \cdots + q_{\ell-5k}) < q_\ell + q_{\ell-4} \ = \ q_{\ell+1}, \tag{4.8}$$

completing the proof. □

Proof of Theorem 1.15. For the first part, we verify that if $m \leq 151 = q_{17}$ the theorem holds. Define $I_n := [q_n, q_{n+1}) = [q_n, q_{n+1} - 1]$. Assume for all $m \in \cup_{\ell=1}^{n-1} I_\ell$, m satisfies the theorem. Now consider $m \in I_n$. If $m = q_n$ then we add done. Assume $m = q_n + x$ with $x > 0$. Since $q_{n+1} = q_n + q_{n-4}$, we know $x < q_{n-4}$. Then by the inductive hypothesis we know the x satisfies the theorem. Namely, $\mathcal{G}(x) = q_{k_1} + q_{k_2} + \cdots + q_{k_s}$ is a FQ-legal decomposition which satisfies either Condition (1) or (2) but not both. Then $\mathcal{G}(m) = q_n + q_{k_1} + q_{k_2} + \cdots + q_{k_s}$ and lastly $n - k_1 \geq 5$.

For the second part, let $m = q_{\ell_1} + q_{\ell_2} + \cdots + q_{\ell_{t-1}} + q_{\ell_t}$ be a decomposition that satisfies either Condition (1) or (2) but not both. Note that in both cases, this decomposition is legal. If $t = 1$, then m is a Fibonacci Quilt number and the theorem is trivial. So we assume $t \geq 2$. Hence by construction of the sequence, m is not a Fibonacci Quilt number.

Let $\mathcal{G}(m) = q_{k_1} + q_{k_2} + \cdots + q_{k_s}$. Note that $s \geq 2$. For contradiction we assume the given decomposition is not the Greedy-6 decomposition. Without loss of generality we may assume $q_{\ell_1} \neq q_{k_1}$. Since q_{k_1} was chosen according to the Greedy-6 algorithm, $q_{\ell_1} < q_{k_1}$.

Case 1: Using Lemma 4.2,

$$m = q_{\ell_1} + q_{\ell_2} + \cdots + q_{\ell_{t-1}} + q_{\ell_t} \leq q_{\ell_1} + q_{\ell_1 - 5} + \cdots + q_{\ell_1 - 5(t-1)} < q_{\ell_1 + 1} \leq q_{k_1} < m \quad (4.9)$$

which is a contradiction.

Case 2: Again using Lemma 4.2,

$$\begin{aligned} m &= q_{\ell_1} + q_{\ell_2} + \cdots + q_{\ell_{t-2}} + q_4 + q_2 = q_{\ell_1} + q_{\ell_2} + \cdots + q_{\ell_{t-2}} + q_5 + q_1 \\ &\leq q_{\ell_1} + q_{\ell_1 - 5} + \cdots + q_{\ell_1 - 5(t-2)} + q_1 \\ &< q_{\ell_1 + 1} + q_1 \\ &\leq q_{k_1} + q_1 \\ &\leq m \end{aligned} \quad (4.10)$$

which is a contradiction. □

In order to prove Theorem 1.16 we will need several relationships between the terms in the Fibonacci Quilt sequence. The following lemma describes those relationships.

Lemma 4.3. *The following hold.*

1. *If $n \geq 7$, then $2q_n = q_{n+2} + q_{n-5}$.*
2. *If $n \geq 8$, then $q_n + q_{n-2} = q_{n+1} + q_{n-5}$.*
3. *If $n \geq 10$, then $q_n + q_{n-3} = q_{n+1} + q_{n-8}$.*

Proof. The proof follows from repeated uses of the recurrence relations stated in Theorem 1.7:

$$\begin{aligned} 2q_n &= q_n + q_{n-1} + q_{n-5} = q_{n+2} + q_{n-5} \\ q_n + q_{n-2} &= q_n + q_{n-4} + q_{n-5} = q_{n+1} + q_{n-5} \\ q_n + q_{n-3} &= q_n + q_{n-4} + q_{n-3} - q_{n-4} = q_{n+1} + q_{n-8}. \end{aligned} \quad (4.11)$$

□

Proof of Theorem 1.16. The proof follows by showing that we can move from $\mathcal{D}(m)$ to $\mathcal{G}(m)$ without increasing the number of summands by doing five types of moves. That the summation remains unchanged after each move follows from Lemma 4.3 and Theorem 1.7.

1. Replace $2q_n$ with $q_{n+2} + q_{n-5}$ (for $n \geq 7$). (If $n \leq 6$, replace $2q_6$ with $q_8 + q_2$, replace $2q_5$ with $q_7 + q_1$, replace $2q_4$ with $q_6 + q_1$, replace $2q_3$ with $q_5 + q_1$, replace $2q_2$ with q_4, and replace $2q_1$ with q_2.)

2. Replace $q_{n-1} + q_{n-2}$ with q_{n+1} (for $n \geq 5$). In other words, if we have two adjacent terms, use the recurrence relation to replace. (If $n \leq 4$, replace $q_3 + q_2$ with q_5 and replace $q_2 + q_1$ with q_3.)

(3) Replace $q_n + q_{n-2}$ with $q_{n+1} + q_{n-5}$ (for $n \geq 8$). (If $n \leq 7$, replace $q_7 + q_5$ with $q_8 + q_2$, $q_6 + q_4$ with $q_7 + q_2$, $q_5 + q_3$ with $q_7 + q_1$, $q_4 + q_2$ with $q_5 + q_1$, and $q_3 + q_1$ with q_4.)

(4) Replace $q_n + q_{n-3}$ with $q_{n+1} + q_{n-8}$ (for $n \geq 10$). (If $n \leq 9$, replace $q_9 + q_6$ with $q_{10} + q_2$, $q_8 + q_5$ with $q_9 + q_1$, $q_7 + q_4$ with $q_8 + q_1$, $q_6 + q_3$ with $q_7 + q_1$, $q_5 + q_2$ with q_6, and $q_4 + q_1$ with q_5.)

(5) Replace $q_n + q_{n-4}$ with q_{n+1} (for $n \geq 6$). In other words, if we have two adjacent terms, use the recurrence relation to replace.

Notice that in all moves, the number of summands either decreases by one or remains unchanged. In addition, the sum of the indices either decreases or remains unchanged. There are three situations where neither the index sum nor the number of summands decreases; $q_5 + q_3 = q_7 + q_1$, $q_4 + q_2 = q_5 + q_1$, and $2q_3 = q_5 + q_1$. But in these situations, the number of q_5, q_4, q_3, q_2 decrease. Therefore this process eventually terminates because the index sum and the number of summands cannot decrease indefinitely.

Let $m = q_{\ell_1} + q_{\ell_2} + \cdots + q_{\ell_{t-1}} + q_{\ell_t}$ be the decomposition obtained after all possible moves. Each move either decreases the number of summands or replaces two summands with two that are farther apart in the sequence. In fact, closer examination of the moves reveals $\ell_i - \ell_{i-1} \geq 5$ except maybe $\ell_{t-1} = 5$ and $\ell_t = 1$.

If $\ell_{t-1} = 5$ and $\ell_t = 1$, replace $q_5 + q_1$ with $q_4 + q_2$. By Theorem 1.15 this is the Greedy-6 decomposition of m. □

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