CLOSED FORM CONTINUED FRACTION EXPANSIONS OF SPECIAL QUADRATIC IRRATIONALS

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ABSTRACT. We derive closed form expressions for the continued fractions of powers of certain quadratic surds. Specifically, consider the recurrence relation $G_{n+1} = mG_n + \ell G_{n-1}$ with $G_0 = 0$, $G_1 = 1$, m a positive integer and $\ell = \pm 1$ (note $m = \ell = 1$ gives the Fibonacci numbers). Let $\varphi_{m,\ell} = \lim_{n \to \infty} G_n/G_{n-1}$. We find simple closed form continued fraction expansions for $\varphi_{m,\ell}^k$ for any integer k by exploiting elementary properties of the recurrence relation and continued fractions.

This paper is dedicated to the memory of Alf van der Poorten.

1. INTRODUCTION

In [vdP4], Alf van der Poorten wrote: *The elementary nature and simplicity of the theory of continued fractions is mostly well disguised in the literature. This makes one reluctant to quote sources when making a remark on the subject and seems to necessitate redeveloping the theory* ab initio. As our work is an outgrowth of [vdP4], we happily refer the reader to that paper for some basic background information on continued fractions, and to the books [HW, MT-B] for proofs. Briefly, every real number α has a continued fraction expansion

$$\alpha \;=\; a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}, \tag{1.1}$$

where each a_i is an integer (and a positive integer unless $i = 0$). The a_i's are called the partial quotients. For brevity we often write

$$\alpha \;=\; [a_0, a_1, a_2, \dots]. \tag{1.2}$$

If we truncate the expansion at a_n, we obtain the nth partial quotient

$$\frac{p_n}{q_n} \;=\; [a_0, a_1, a_2, \dots, a_n]. \tag{1.3}$$

The p_n's and q_n's satisfy the very important relation

$$p_n q_{n-1} - p_{n-1} q_n \;=\; (-1)^{n-1}. \tag{1.4}$$

Continued fractions encode much useful information about the algebraic structure of a number, and frequently arise in approximation theory and dynamical systems. Clearly α is rational if and only if its continued fraction is finite, and a beautiful theorem of Lagrange states that α is a quadratic irrational if and only if the continued fraction expansion is periodic.

In this paper we explore the continued fraction expansions of powers of quadratic surds. Recall a quadratic surd is an irrational number of the form $(P \pm \sqrt{D})/Q$ where $P, Q \in \mathbb{Z}$ and D is a non-square integer. By Lagrange's theorem we know these numbers and their powers have periodic continued fractions, which suggests many questions, such as what is the length of the period as well as what are the entries. Some special cases were done by van der Poorten in [vdP4]. He studied solutions to Pell's Equation $X^2 - DY^2 = \pm 1$ (D a non-square integer). Using the solution, he derived expansions for the continued fraction of $\sqrt{D}$ and $\frac{1+\sqrt{D}}{2}$ (with $D \equiv 1 \bmod 4$), and then for the expansions of some simple functions of these numbers as well as numbers related to Diophantine equations similar to Pell's equation.

Another technique that shows promise in manipulating continued fractions comes from an unfinished paper of Bill Gosper [BG]. He develops a set of algorithms for finding closed form expressions of simple functions of a given quadratic irrational, as well as for more complicated functions combining quadratic irrationals. His paper addresses the need for finding a nice way of adding and multiplying the continued fractions of quadratic surds. The basic technique employed is an analog of Euclid's algorithm. This operation is carried out using two-dimensional arrays in the simple cases, and then requires added dimensions when considering functions of two or more quadratic irrationals. Unfortunately these algorithms, while useful for many applications, do not reveal the nature of the underlying structure in a closed form in an accessible manner.

The purpose of this paper is to continue these investigations for additional families. We derive closed form expressions for the continued fractions of powers of certain quadratic surds.

As the Fibonacci case is perhaps the most interesting, and some generalizations can be reduced to this case, we state the results there first. Let f_k denote the kth Fibonacci number, and F_n denote the nth Fibonacci number. The reason for using both upper and lower case letters for the Fibonacci numbers is to make the algebra in §2 visually easier to read by using upper case for the Fibonacci numbers whose subscript is variable. Set $\varphi = \lim_{n\to\infty} F_n/F_{n-1}$, which is the golden mean; note $\varphi^k = \lim_{n\to\infty} F_n/F_{n-k}$.

Theorem 1.1. *We have*

$$\frac{F_n}{F_{n-k}} = \begin{cases} [f_{k+1} + f_{k-1}, \frac{F_{n-k}}{F_{n-2k}}] & \text{if } k \text{ is even} \\ [f_{k+1} + f_{k-1} - 1,\ 1,\ \frac{F_{n-k}}{F_{n-2k}} - 1] & \text{if } k \text{ is odd,} \end{cases} \tag{1.5}$$

which yields

$$\varphi^k = \begin{cases} [\overline{f_{k+1} + f_{k-1}}] & \text{if } k \text{ is odd} \\ [f_{k+1} + f_{k-1} - 1, \overline{1, f_{k+1} + f_{k-1} - 2}] & \text{if } k \text{ is even.} \end{cases} \tag{1.6}$$

The techniques used to prove the above theorem can be extended to certain recurrence relations. We let $g_k = G_k$ for the same reason as before.

Theorem 1.2. *Consider the recurrence relation $G_{n+1} = mG_n + \ell G_{n-1}$ with $G_0 = 0$, $G_1 = 1$, m a positive integer and $\ell = \pm 1$, again letting $g_k = G_k$. Let*

$$\varphi_{m,\ell} = \lim_{n\to\infty} \frac{G_n}{G_{n-1}} = \frac{m \pm \sqrt{m^2 + 4\ell}}{2}. \tag{1.7}$$

Then for any positive integer k, if $\ell = 1$ we have

$$\varphi_{m,1}^k = \begin{cases} [\overline{g_{k+1} + g_{k-1}}] & \text{if } k \text{ is odd} \\ [g_{k+1} + g_{k-1} - 1, \overline{1, g_{k+1} + g_{k-1} - 2}] & \text{if } k \text{ is even,} \end{cases} \tag{1.8}$$

while if $\ell = -1$ we have

$$\varphi_{m,-1}^k = [g_{k+1} - g_{k-1} - 1, \overline{1, (g_{k+1} - g_{k-1} - 2)}]. \tag{1.9}$$

The numerical data that led us to these results, as well as some additional experimental observations which, as of now, we have not been able to isolate into general theorems, are available at [Fi].

On a personal note, this paper is an outgrowth of two undergraduate research classes taken by the first named author under the instruction of the second named author (and others) at Princeton University in 2002 and 2003. One of the topics covered was continued fractions, where many of Alf van der Poorten's papers [BvdP, BvdPR, BvdP, vdP1, vdP2, vdP3, vdP4, vdP5, vdP6, vdPS1, vdPS2] were read and enjoyed. These types of questions are ideally suited to introduce students to research. The results here, while later discovered to have been proved through other techniques (see for example [LMW]), were originally found through numerical exploration, which suggested the proof strategy.

The second named author had the fortune of being at Brown University with Alf a few years later, and remembers fondly numerous conversations on continued fractions and other topics in general, and the expansions in this paper in particular. Alf's constant enthusiasm, knowledge of the field and helpful comments greatly improved the exposition of continued fractions in [MT-B], a textbook developed from the Princeton course.

2. CONTINUED FRACTIONS OF POWERS OF THE GOLDEN MEAN

As we'll see later that the proof of Theorem 1.2 reduces to the special case of the Fibonacci numbers, we study this case first. For notational convenience, we write F_n for the nth Fibonacci number when n varies, and write f_k for the kth Fibonacci number when k is a fixed input. This notation helps us visually parse the equations that follow.

The following lemma collects some well known properties of the Fibonacci numbers which will be useful below. The proofs are standard (see for instance [Kos], especially Chapter 5).

Lemma 2.1. *The Fibonacci numbers satisfy the following properties:*
   (1) $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$.
   (2) $\varphi = \lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \frac{1+\sqrt{5}}{2}$.
   (3) *Binet's Formula:*

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1+\sqrt{5}}{2}\right)^n\right].$$

   (4) $F_{n-2}F_n - (F_{n-1})^2 = (-1)^{n-1}$.
   (5) $F_n = f_{k+1}F_{n-k} + f_k F_{n-k-1}$.

The relations discussed below were initially found by numerically exploring the finite continued fraction expansions of F_n/F_{n-k}. Note that for the golden mean $\varphi = \lim_{n\to\infty} F_n/F_{n-1}$, we have

$$\varphi = \frac{1+\sqrt{5}}{2} = [1,1,1,1,1,\dots], \tag{2.1}$$

and

$$\lim_{n\to\infty} \frac{F_n}{F_{n-k}} = \lim_{n\to\infty} \frac{F_n}{F_{n-1}}\frac{F_{n-1}}{F_{n-2}}\cdots\frac{F_{n-(k-1)}}{F_{n-k}} = \varphi^k. \tag{2.2}$$

For $k = 2$, it's trivial to find the continued fraction of φ^2 since $F_n = F_{n-1} + F_{n-2}$ implies

$$\frac{F_n}{F_{n-2}} = 1 + \frac{F_{n-1}}{F_{n-2}}. \tag{2.3}$$

Taking the limit as $n \to \infty$, we find

$$\varphi^2 = \lim_{n \to \infty} \frac{F_n}{F_{n-2}} = 1 + \varphi = [2, 1, 1, 1, 1, \ldots]. \tag{2.4}$$

With $k = 3$, however, the result is not as obvious. From Lemma 2.1(5) we note that $F_n = f_3 F_{n-2} + f_2 F_{n-3} = 2F_{n-2} + F_{n-3}$, which implies

$$\frac{F_n}{F_{n-3}} = 1 + 2\frac{F_{n-2}}{F_{n-3}}. \tag{2.5}$$

Unfortunately, in general there is no simple expression for the continued fraction of 2α given the expansion of α. Numerical computations of the continued fraction expansions of F_n/F_{n-3} were found to have the form $[4,4, \ldots, \rho]$ where ρ is 3 if $n \equiv 1 \bmod 3$, 5 if $n \equiv 2 \bmod 3$, and 4 if $n \equiv 0 \bmod 3$. This is easily proved, as after some algebra we find

$$\frac{F_n}{F_{n-3}} = 4 + \frac{1}{F_{n-3}/F_{n-6}}, \tag{2.6}$$

and the claim now follows from knowing the first few ratios. Further, we get the continued fraction expansion for φ^3 simply by taking limits.

Theorem 1.1 states that this algebra can be generalized to any k, as we now show.

Proof of Theorem 1.1. We have

$$
\begin{aligned}
\frac{F_n}{F_{n-k}} &= \frac{f_{k+1}F_{n-k} + f_k F_{n-k-1}}{F_{n-k}} \\
&= f_{k+1} + f_k \frac{F_{n-k-1}}{F_{n-k}} \\
&= f_{k+1} + \frac{f_{k-1}F_{n-k-1} + f_{k-2}F_{n-k-1}}{F_{n-k}} \\
&= f_{k+1} + \frac{(f_{k-1}F_{n-k-1} + f_{k-1}F_{n-k-2}) + f_{k-2}F_{n-k-1} - f_{k-1}F_{n-k-2}}{F_{n-k}} \\
&= f_{k+1} + \frac{f_{k-1}F_{n-k}}{F_{n-k}} + \frac{f_{k-2}F_{n-k-1} - f_{k-1}F_{n-k-2}}{F_{n-k}} \\
&= (f_{k+1} + f_{k-1}) + \frac{f_{k-2}F_{n-k-1} - f_{k-1}F_{n-k-2}}{F_{n-k}}. \tag{2.7}
\end{aligned}
$$

As $F_{n-k-1} = f_k F_{n-k-k} + f_{k-1}F_{n-k-k-1}$ and $F_{n-k-2} = f_{k-1}F_{n-k-k-1} + f_{k-2}F_{n-k-k-2}$, (2.7) becomes

$$
\begin{aligned}
\frac{F_n}{F_{n-k}} &= (f_{k+1} + f_{k-1}) \\
&\quad + \frac{f_{k-2}(f_k F_{n-2k} + f_{k-1}F_{n-2k-1}) - f_{k-1}(f_{k-1}F_{n-2k} + f_{k-2}F_{n-2k-1})}{F_{n-k}} \\
&= (f_{k+1} + f_{k-1}) + \frac{f_{k-2}(f_k F_{n-2k}) - f_{k-1}(f_{k-1}F_{n-2k})}{F_{n-k}} \\
&= (f_{k+1} + f_{k-1}) + \frac{F_{n-2k}[f_{k-2}f_k - (f_{k-1})^2]}{F_{n-k}}. \tag{2.8}
\end{aligned}
$$

From Lemma 2.1(4) we know $f_{k-2}f_k - (f_{k-1})^2 = (-1)^{k-1}$. Using this and (2.8), (2.7) becomes

$$\frac{F_n}{F_{n-k}} = (f_{k+1} + f_{k-1}) + \frac{F_{n-2k}(-1)^{k-1}}{F_{n-k}}$$

$$= (f_{k+1} + f_{k-1}) + \frac{(-1)^{k-1}}{F_{n-k}/F_{n-2k}}. \tag{2.9}$$

If k is odd then $(-1)^{k-1} = 1$ and so (2.9) becomes

$$\frac{F_n}{F_{n-k}} = (f_{k+1} + f_{k-1}) + \frac{1}{F_{n-k}/F_{n-2k}}, \tag{2.10}$$

while if k is even then $(-1)^{k-1} = -1$ and so (2.9) becomes

$$\frac{F_n}{F_{n-k}} = (f_{k+1} + f_{k-1}) + \frac{-1}{F_{n-k}/F_{n-2k}}. \tag{2.11}$$

If k is even we must manipulate further, and (2.11) becomes

$$\frac{F_n}{F_{n-k}} = (f_{k+1} + f_{k-1} - 1) + 1 - \frac{1}{F_{n-k}/F_{n-2k}}. \tag{2.12}$$

Now

$$1 - \frac{1}{F_{n-k}/F_{n-2k}} = \frac{F_{n-k} - F_{n-2k}}{F_{n-k}}$$

$$= \frac{1}{\dfrac{F_{n-k}}{F_{n-k} - F_{n-2k}}}$$

$$= \frac{1}{1 + \dfrac{F_{n-2k}}{F_{n-k} - F_{n-2k}}}$$

$$= \frac{1}{1 + \dfrac{1}{\dfrac{F_{n-k} - F_{n-2k}}{F_{n-2k}}}}$$

$$= \frac{1}{1 + \dfrac{1}{\dfrac{F_{n-k}}{F_{n-2k}} - 1}}. \tag{2.13}$$

Taking the limit as $n \to \infty$ yields

$$\varphi^k = f_{k+1} + f_{k-1} - 1 + \frac{1}{1 + \dfrac{1}{\varphi^k - 1}}, \tag{2.14}$$

from which the continued fraction of φ^k easily follows. □

3. THE GENERAL DIFFERENCE EQUATION

We now consider the more general difference equation

$$G_n = mG_{n-1} + \ell G_{n-2} \tag{3.1}$$

with the initial conditions $G_0 = 0$, $G_1 = 1$ (we need to choose initial conditions to explicitly write down a generalized Binet's formula). The results from Lemma 2.1 immediately generalize; we omit the proofs (which are available in [Fi]) which are straightforward algebra, and just collect the results below. As before, we use both G_n and g_k for terms of the sequence to make the arguments below easier to visually parse.

Lemma 3.1. *Our sequence $\{G_n\}$ satisfies the following properties:*

(1) $G_n = mG_{n-1} + \ell G_{n-2}$ *with* $G_0 = 0$, $G_1 = 1$.

(2) $\varphi_{m,\ell} = \lim_{n\to\infty} \frac{G_n}{G_{n-1}} = \frac{m \pm \sqrt{m^2+4\ell}}{2}$.

(3) *Generalized Binet's Formula:*

$$G_n = \frac{1}{\sqrt{m^2+4\ell}} \left[\left(\frac{m+\sqrt{m^2+4\ell}}{2} \right)^n - \left(\frac{m-\sqrt{m^2+4\ell}}{2} \right)^n \right].$$

(4) $G_{n-2}G_n - (G_{n-1})^2 = (-1)^{n-1} \cdot \ell^{k-2}$.

(5) $G_n = g_{k+1}G_{n-k} + \ell g_k G_{n-k-1}$.

We want to find closed form expressions for the continued fractions of the $\varphi_{m,\ell}^k$ for integral k. As the continued fraction of α is trivially related to that of $1/\alpha$, it suffices to study $k > 0$.

We argue as in §2. From Lemma 3.1(5) we have $G_n = g_{k+1}G_{n-k} + \ell g_k G_{n-k-1}$, and thus

$$\frac{G_n}{G_{n-k}} = \frac{g_{k+1}G_{n-k} + \ell g_k G_{n-k-1}}{G_{n-k}}$$

$$= g_{k+1} + \ell g_k \frac{G_{n-k-1}}{G_{n-k}}$$

$$= g_{k+1} + \ell \frac{mg_{k-1}G_{n-k-1} + \ell g_{k-2}G_{n-k-1}}{G_{n-k}}$$

$$= g_{k+1} + \ell \frac{(mg_{k-1}G_{n-k-1} + \ell g_{k-1}G_{n-k-2}) + \ell g_{k-2}G_{n-k-1} - \ell g_{k-1}G_{n-k-2}}{G_{n-k}}$$

$$= g_{k+1} + \ell \frac{g_{k-1}G_{n-k}}{G_{n-k}} + \ell \frac{\ell g_{k-2}G_{n-k-1} - \ell g_{k-1}G_{n-k-2}}{G_{n-k}}$$

$$= (g_{k+1} + \ell g_{k-1}) + \ell^2 \frac{g_{k-2}G_{n-k-1} - g_{k-1}G_{n-k-2}}{G_{n-k}}. \tag{3.2}$$

From Lemma 3.1(5) we have $G_{n-k-1} = g_k G_{n-k-k} + \ell g_{k-1}G_{n-k-k-1}$. Also $G_{n-k-2} = g_{k-1}G_{n-k-k-1} + \ell g_{k-2}G_{n-k-k-2}$. Thus (3.2) reduces to

$$\frac{G_n}{G_{n-k}} = (g_{k+1} + \ell g_{k-1})$$

$$+ \ell^2 \frac{g_{k-2}(g_k G_{n-2k} + \ell g_{k-1}G_{n-2k-1}) - g_{k-1}(g_{k-1}G_{n-2k} + \ell g_{k-2}G_{n-2k-1})}{G_{n-k}}$$

$$= (g_{k+1} + \ell g_{k-1}) + \ell^2 \frac{g_{k-2}g_k G_{n-2k} - g_{k-1}g_{k-1}G_{n-2k}}{G_{n-k}}$$

$$= (g_{k+1} + \ell g_{k-1}) + \ell^2 \frac{G_{n-2k}(g_{k-2}g_k - (g_{k-1})^2)}{G_{n-k}}. \tag{3.3}$$

From Lemma 3.1(4) we know

$$g_{k-2}g_k - (g_{k-1})^2 = (-1)^{k-1}\ell^{k-2}, \tag{3.4}$$

and thus

$$\frac{G_n}{G_{n-k}} = (g_{k+1} + \ell g_{k-1}) + \frac{G_{n-2k}\ell^k(-1)^{k-1}}{G_{n-k}}. \tag{3.5}$$

We now prove our main result.

Proof of Theorem 1.2.

Case 1: $\ell = 1$**:** In this case, the difference equation takes the form $G_n = mG_{n-1} + G_{n-2}$. The functional form of (3.5) is identical to that of (2.9); the only difference is that we have g's and G's instead of f's and F's. We can thus quote our results from that case, and obtain the desired result.

Case 2: $\ell = -1$**:** When $\ell = -1$ the difference equation is of the form $G_n = mG_{n-1} - G_{n-2}$; we choose the initial conditions so that no term is ever negative. Taking $\ell = -1$ in (3.5) gives

$$\frac{G_n}{G_{n-k}} = (g_{k+1} + g_{k-1}) + (-1)^{k-1}(-1)^k\frac{G_{n-2k}}{G_{n-k}}$$

$$= (g_{k+1} + g_{k-1}) - \frac{G_{n-2k}}{G_{n-k}}. \tag{3.6}$$

As this is of the same form as what we had in (2.11), we can argue as in §2. We've thus shown that all quadratic surds of the form $\varphi_{m,-1} = \frac{m+\sqrt{m^2-4}}{2}$, and their powers, have continued fraction expansions of the form

$$\varphi_{m,-1}^k = (g_{k+1} - g_{k-1} - 1) + \frac{1}{1 + \frac{1}{\varphi_{m,-1}^k - 1}}. \tag{3.7}$$

This clearly determines a continued fraction with repeating block of length 2 of the form:

$$\varphi_{m,-1}^k = [g_{k+1} - g_{k-1} - 1, \overline{1, (g_{k+1} - g_{k-1} - 2)}]. \tag{3.8}$$

completing the proof. □

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