ONE-LEVEL DENSITY FOR HOLOMORPHIC CUSP FORMS OF ARBITRARY LEVEL

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Abstract. In 2000 Iwaniec, Luo, and Sarnak proved for certain families of L-functions associated to holomorphic newforms of square-free level that, under the Generalized Riemann Hypothesis, as the conductors tend to infinity the one-level density of their zeros matches the one-level density of eigenvalues of large random matrices from certain classical compact groups in the appropriate scaling limit. We remove the square-free restriction by obtaining a trace formula for arbitrary level by using a basis developed by Blomer and Milićević, which is of use for other problems as well.

1. Introduction

Montgomery [Mo] conjectured that the pair correlation of critical zeros up to height T of the Riemann zeta function $\zeta(s)$ coincides with the pair correlation of eigenvalues of random unitary matrices of dimension N in the appropriate limit as $T, N \to \infty$. This remarkable connection initiated a new branch of number theory concerned with relating the statistics of zeros of $\zeta(s)$, and of L-functions more generally, to those of eigenvalues of random matrices. While additional support for this agreement was obtained by the work of Hejhal [Hej] on the triple correlation of $\zeta(s)$, Rudnick and Sarnak [RS] on the n-level correlation for cuspidal automorphic forms, and Odlyzko [Od1, Od2] on the spacings between adjacent zeros of $\zeta(s)$, the story cannot end here as these statistics are insensitive to the behavior of finitely many zeros. As the zeros at and near the central point play an important role in a variety of problems, this led Katz and Sarnak [KS1, KS2] to develop a new statistic which captures this behavior.

Definition 1.1. Let $L(s,f)$ be an L-function with zeros in the critical strip $\rho_f = 1/2 + i\gamma_f$ (note $\gamma_f \in \mathbb{R}$ if and only if the Generalized Riemann Hypothesis holds for f), and let ϕ be an even Schwartz function whose Fourier transform has compact support. The **one-level**

Date: April 1, 2016.

2010 *Mathematics Subject Classification.* 11M26 (primary), 11M41, 15A52 (secondary).

Key words and phrases. Low lying zeroes, one level density, cuspidal newforms, Petersson formula.

The first four named authors were supported by NSF grant DMS1347804 and Williams College; the fifth-named author was partially supported by NSF grant DMS0850577. We thank Jim Cogdell for discussions on the signs of functional equation, and Djorde Milićević for conversations on his work with Blomer.

density is

$$D_1(f;\phi) \ := \ \sum_{\rho_f} \phi\left(\frac{\gamma_f}{2\pi}\log c_f\right), \tag{1.1}$$

where c_f is the analytic conductor.

Their density conjecture [KS1, KS2] states that the scaling limits of eigenvalues of classical compact groups near 1 correctly model the behavior of these zeros in families of L-functions as the conductors tend to infinity. Specifically, if the symmetry group is $\mathcal{G}$, then we expect

$$\mathcal{D}_1(\mathcal{F};\phi) \ = \ \int_{-\infty}^{\infty}\phi(x)W_1(\mathcal{G})(x)dx \ = \ \int_{-\infty}^{\infty}\widehat{\phi}(t)\widehat{W}_1(\mathcal{G})(t)dt, \tag{1.2}$$

where $K(y) = \frac{\sin \pi y}{\pi y}$, $K_\epsilon(x,y) = K(x-y)+\epsilon K(x+y)$ for $\epsilon = 0, \pm 1$, and

$$\begin{aligned}
W_1(\mathrm{SO(even)})(x) &= K_1(x,x)\\
W_1(\mathrm{SO(odd)})(x) &= K_{-1}(x,x)+\delta_0(x)\\
W_1(\mathrm{O})(x) &= \frac{1}{2}W_1(\mathrm{SO(even)})(x)+\frac{1}{2}W_1(\mathrm{SO(odd)})(x)\\
W_1(\mathrm{U})(x) &= K_0(x,x)\\
W_1(\mathrm{Sp})(x) &= K_{-1}(x,x).
\end{aligned} \tag{1.3}$$

While the Fourier transforms of the densities of the orthogonal groups all equal $\delta_0(y)+1/2$ in $(-1,1)$, they are mutually distinguishable for larger support (and are distinguishable from the unitary and symplectic cases for any support). There is now an enormous body of work showing the 1-level densities of many families (such as Dirichlet L-functions, elliptic curves, cuspidal newforms, Maass forms, number field L-functions, and symmetric powers of GL_2 automorphic representations) agree with the scaling limits of a random matrix ensemble; see [AAILMZ, AM, DM1, FiMi, FI, Gao, GK, Gü, HM, HR, ILS, KS1, KS2, Mil, MilPe, OS1, OS2, RR, Ro, Rub1, Rub2, ShTe, Ya, Yo] for some examples, and [DM1, DM2, ShTe] for discussions on how to determine the underlying symmetry. For additional readings on connections between random matrix theory, nuclear physics and number theory see [BFMT-B, Con, CFKRS, FM, For, KeSn1, KeSn2, KeSn3, Meh]

We concentrate on extending the results of Iwaniec, Luo, and Sarnak in [ILS]. One of their key results is a formula for unweighted sums of Fourier coefficients of holomorphic newforms over all newforms of a given weight and level. This formula writes the unweighted sums in terms of weighted sums to which one can apply the Petersson trace formula; it is instrumental in performing any averaging over holomorphic newforms, since one can interchange summation and replace the average of Fourier coefficients with Kloosterman sums and Bessel functions, which are amenable to analysis.

A drawback of their formula is that it may only be applied to averages of newforms of square-free level. One reason is that the development of such a formula depends essentially on the construction of an explicit orthonormal basis for the space of cusp forms of a given weight and level, which they only computed in the case of square-free level. In 2011, Rouymi [R] complemented the square-free calculations of Iwaniec, Luo, and Sarnak, finding an orthonormal basis for the space of cusp forms of prime power level, and applying this explicit basis towards the development of a similar sum of Fourier coefficients over all newforms with level equal to a fixed prime power.

In 2015, Blomer and Milićević [BM] extended the results of Iwaniec, Luo, and Sarnak and Rouymi by writing down an explicit orthonormal basis for the space of cusp forms (holomorphic or Maass) of a fixed weight and, novelly, arbitrary level.

The purpose of this article is, first, to leverage the basis of Blomer and Milićević to prove an exact formula for sums of Fourier coefficients of holomorphic newforms over all newforms of a given weight and level, where now the level is permitted to be arbitrary (see below, as well as Proposition 5.2 for a detailed expansion). The basis of Blomer and Milićević requires one to split over the square-free and square-full parts of the level; this splitting combined with the loss of several simplifying assumptions for Hecke eigenvalues and arithmetic functions makes the case where the level is not square-free is much more complex. As an application, we use this formula to show the 1-level density agrees only with orthogonal symmetry.

1.1. **Harmonic averaging.** Throughout we assume that $k, N \geqslant 1$ with k even. To state our formula for sums of Fourier coefficients, we let $H_k^\star(N)$ denote the set of holomorphic cusp forms of weight k and level N which are new of level N in the sense of Atkin and Lehner, and let $\lambda_f(n)$ denote its n^{th} Fourier coefficient (see the next section for more details).

For any cusp form f, we introduce the normalized Fourier coefficients

$$\Psi_f(n) \ := \ \left(\frac{\Gamma(k-1)}{(4\pi)^{k-1}}\right)^{1/2} ||f||^{-1}\lambda_f(n), \tag{1.4}$$

where $\|f\|^2 = \langle f, f\rangle$. We then define

$$\Delta_{k,N}(m,n) \ := \ \sum_{g\in\mathcal{B}_k(N)} \overline{\Psi_g(m)}\Psi_g(n), \tag{1.5}$$

where $\mathcal{B}_k(N)$ is an orthonormal basis for the space of cusp forms of weight k and level N. The importance of $\Delta_{k,N}(m,n)$ is clarified by the introduction of the Petersson formula in the next section.

Using the orthonormal basis $\mathcal{B}_k(N)$ of Milićević and Blomer, we then prove the following (unconditional) formula.

Theorem 1.2. *Suppose that* $(n, N) = 1$. *Then*

$$\sum_{f\in H_k^\star(N)} \lambda_f(n) \ = \ \frac{k-1}{12} \sum_{LM=N} \mu(L)M \prod_{p^2|M} \left(\frac{p^2}{p^2-1}\right)^{-1} \sum_{(m,M)=1} m^{-1}\Delta_{k,M}(m^2,n). \tag{1.6}$$

This formula may be immediately applied to variety of applications involving holomorphic cusp form Fourier coefficients and L-functions. Rouymi uses his basis and formula to study the non-vanishing at the central point of L-functions attached to primitive cusp forms; we elect to apply our formula to generalize [ILS, Theorem 1.1] on the one-level density of families of holomorphic newform L-functions by removing the condition that N must pass to infinity through the square-free integers.

1.2. **The Density Conjecture.** Before stating our results, we introduce the L-function $L(s,f)$ associated to a $f \in H_k^\star(N)$ as the Dirichlet series

$$L(s,f) \;=\; \sum_1^\infty \lambda_f(n)n^{-s}. \tag{1.7}$$

See Section 3 of [ILS] for the Euler product, analytic continuation, and functional equation of $L(s,f)$ (or its symmetric square); $L(s,f)$ may be analytically continued to an entire function on $\mathbf{C}$ with a functional equation relating s to $1-s$. We now assume the Generalized Riemann Hypothesis for $L(s,f)$, and, for technical reasons, $L(s,\mathrm{sym}^2 f)$ as well as for all Dirichlet L-functions (see Remark 1.5). Then we may write all nontrivial zeros of $L(s,f)$ as

$$\varrho_f \;=\; \frac{1}{2}+i\gamma_f. \tag{1.8}$$

For any $f \in H_k^\star(N)$, we denote by c_f its analytic conductor; for our family

$$c_f \;=\; k^2N. \tag{1.9}$$

Towards the definition of the one-level density, we first define for a fixed form f

$$D_1(f;\phi) \;:=\; \sum_{\gamma_f} \phi\left(\frac{\gamma_f}{2\pi}\log c_f\right) \tag{1.10}$$

where the ordinates γ_f are counted with their corresponding multiplicities, and $\phi(x)$ is an even function of Schwartz class such that its Fourier transform

$$\widehat{\phi}(y) \;=\; \int_{-\infty}^{\infty} \phi(x)e^{-2\pi ixy}dx \tag{1.11}$$

has compact support so that $\phi(x)$ extends to an entire function.

Our family $\mathcal{F}(N)$ is $H_k^\star(N)$, where the level N is our asymptotic parameter (and $\mathcal{F} = \cup_{N\geqslant 1}\mathcal{F}(N)$). It is worth mentioning that $|H_k^\star(N)| \xrightarrow{N\to\infty} \infty$; precise asymptotics are given in Appendix C. Then the one-level density is the expectation of $D_1(f;\phi)$ averaged over our family:

$$D_1(H_k^*(N);\phi) \;:=\; \frac{1}{|H_k^*(N)|}\sum_{f\in H_k^*(N)} D_1(f;\phi). \tag{1.12}$$

Iwaniec, Luo, and Sarnak [ILS] prove prove the Density Conjecture with the support of $\widehat{\phi}$ in $(-2,2)$ and as N runs over square-free numbers. We prove the following theorem with no conditions on how $N\to\infty$.

Theorem 1.3. *Fix any $\phi \in \mathcal{S}(\mathbf{R})$ with $\operatorname{supp}\widehat{\phi} \subset (-2,2)$. Then, assuming the Generalized Riemann Hypothesis for $L(s,f)$ and $L(s,\operatorname{sym}^2 f)$ for $f \in H_k^*(N)$ and for all Dirichlet L-functions,*

$$\lim_{N\to\infty} \frac{1}{|H_k^\star(N)|} \sum_{f\in H_k^\star(N)} D_1(f;\phi) \;=\; \int_{-\infty}^{\infty} \phi(x) W_1(\mathrm{O})(x)\, dx \tag{1.13}$$

where $W_1(\mathrm{O})(x) = 1 + \frac{1}{2}\delta_0(x)$; thus the 1-level density agrees only with orthogonal symmetry.

Remark 1.4. While they are also able to split the family by the sign of the functional equation, we are unable to do so. The reason is that for square-free level N the sign of the functional equation, ϵ_f, is given by

$$\epsilon_f \;=\; i^k \mu(N)\lambda_f(N)N^{1/2} \tag{1.14}$$

(see equation (3.5) of [ILS]). By multiplying by $\frac{1}{2}(1 \pm \epsilon_f)$ we can restrict to just the even or odd forms, at the cost of having an additional $\lambda_f(N)$ factor in the Petersson formula. This leads to involved calculations of Bessel-Kloosterman terms, but these sums can be evaluated well enough to obtain support in $(-2,2)$. Unfortunately there is no analogue of their equation (3.5) for general level (once the level has a p^2 or larger factor, then the level does not determine the local representation and so doesn't determine the root number).

Remark 1.5. We briefly comment on the use of the various Generalized Riemann Hypotheses. First, assuming GRH for $L(s,f)$ yields a nice spectral interpretation of the 1-level density, as the zeros now lie on a line and it makes sense to order them; note, however, that this statistic is well-defined even if GRH fails. Second, GRH for $L(s,\operatorname{sym}^2 f)$ is used to bound certain sums which arise as lower order terms; in [ILS] (page 80 and especially page 88) the authors remark how this may be replaced by additional applications of the Petersson formula (assuming GRH allows us to trivially estimate contributions from each form, but a bound on average suffices). Finally, GRH for Dirichlet L-functions is needed when we follow [ILS] and expand the Kloosterman sums in the Petersson formula with Dirichlet characters; if we do not assume GRH here we are still able to prove the 1-level density agrees with orthogonal symmetry, but in a more restricted range than $(-2,2)$.

The structure of the paper is as follows. Our main goal is to prove the formula for sums of Hecke eigenvalues and then use this to compute the one-level density. We begin in §2 with a short introduction of the theory of primitive holomorphic cusp forms, as well as the Petersson trace formula and the basis of Blomer and Milićević. In §3 we find a formula for $\Delta_{k,N}(m,n)$, which we leverage in §4 to find a formula for $\Delta^\star_{k,N}(n)$ (Theorem 1.2). Using our formula, we find bounds for $\Delta^\star_{k,N}(n)$ in §5, culminating in the computation of the one-level density in §6 (Theorem 1.3).

2. Preliminaries

In this section we introduce some notation and results to be used throughout, much of which can be found in [IK].

2.1. **Hecke eigenvalues and the Petersson inner product.** Our setup is classical. Throughout k, N are positive integers, and k is even. We consider the linear space $S_k(N)$ of cusp forms of weight k and trivial nebentypus for the Hecke congruence group $\Gamma_0(N)$. Each $f \in S_k(N)$ admits a Fourier development

$$f(z) \;=\; \sum_{n\geqslant 1} a_f(n)e(nz), \tag{2.1}$$

where $e(z) := e^{2\pi iz}$ and the $a_f(n)$ are in general complex numbers, though as we only consider forms with trivial nebentypus, our Fourier coefficients are real.

It is well known that $S_k(N)$ is a finite-dimensional Hilbert space with respect to the Petersson inner product

$$\langle f, g\rangle \;=\; \int_{\Gamma_0(N)\backslash\mathfrak{H}} f(z)\overline{g(z)}y^{k-2}dxdy, \tag{2.2}$$

where $\mathfrak{H}$ denotes the upper-half plane $\mathfrak{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$. Given a form on $\Gamma_0(M)$, it is possible to induce a form on $\Gamma_0(N)$ for $M \mid N$. We call such forms for which $M < N$ "old forms"; the ones not induced from a form with $M < N$ are called the "new forms" or "primitive forms." Occasionally we will write the inner product with a subscript such as $\langle f, g\rangle_N$ to indicate we are considering f and g as forms on $\Gamma_0(N)$, when perhaps $\langle f, g\rangle_M$ might make sense as well.

Atkin and Lehner [AL] showed that the space $S_k(N)$ has a canonical orthogonal decomposition in terms of newforms. Let $H_k^\star(M)$ be the set of newforms of weight k and level M (typically we choose M to be a divisor of N). Then

$$S_k(N) \;=\; \bigoplus_{LM=N}\ \bigoplus_{f\in H_k^\star(M)} S_k(L; f) \tag{2.3}$$

where $S_k(L; f)$ is the linear space spanned by the forms

$$f_{|\ell}(z) \;=\; \ell^{\frac{k}{2}} f(\ell z) \quad \text{with } \ell \mid L. \tag{2.4}$$

Though the forms $f_{|\ell}(z)$ are linearly independent, they are not orthogonal.

If $f \in H_k^\star(M)$ then f is an eigenfunction of all Hecke operators $T_M(n)$, where

$$(T_M(n)f)(z) \;=\; \frac{1}{\sqrt{n}} \sum_{\substack{ad=n\\(a,M)=1}} \left(\frac{a}{d}\right)^{k/2} \sum_{b \pmod d} f\left(\frac{az+b}{d}\right). \tag{2.5}$$

For a fixed $f \in H_k^\star(M)$, let $\lambda_f(n)$ denote the eigenvalue of $T_M(n)$; i.e.,

$$T_M(n)f \;=\; \lambda_f(n)f \tag{2.6}$$

for all $n \geqslant 1$. The Hecke eigenvalues are multiplicative; more precisely, they satisfy the following identity for any $m, n \geqslant 1$:

$$\lambda_f(m)\lambda_f(n) \;=\; \sum_{\substack{d\mid(m,n)\\(d,M)=1}} \lambda_f(mn/d^2). \tag{2.7}$$

We normalize so that

$$a_f(1) \ = \ 1. \tag{2.8}$$

Then $a_f(n)$ and $\lambda_f(n)$ are related by

$$a_f(n) \ = \ \lambda_f(n) n^{(k-1)/2}. \tag{2.9}$$

Deligne showed that the Weil conjectures imply the Ramanujan-Petersson conjecture for holomorphic cusp forms, and then proved them. As a consequence, for $f \in S_k(N)$ we have the bound

$$|\lambda_f(n)| \ \leqslant \ \tau(n), \tag{2.10}$$

where $\tau(n)$ is the divisor function, and if $f \in H_k^\star(M)$ and $p \mid M$, then

$$\lambda_f(p)^2 \ = \ \begin{cases} \frac{1}{p} & \text{if } p \mid\mid N \\ 0 & \text{if } p^2 \mid N. \end{cases} \tag{2.11}$$

We recall the definition (1.4) of the normalized Fourier coefficients $\Psi_f(n)$ attached to any cusp form f:

$$\Psi_f(n) \ = \ \left(\frac{\Gamma(k-1)}{(4\pi n)^{k-1}} \right)^{1/2} ||f||^{-1} a_f(n) \ \ll_f \ \tau(n). \tag{2.12}$$

Let $\mathcal{B}_k(N)$ be an orthogonal basis of $S_k(N)$. Then

$$|\mathcal{B}_k(N)| \ = \ \dim S_k(N) \ \asymp \ \nu(N) k \tag{2.13}$$

where

$$\nu(N) \ := \ [\Gamma_0(1) : \Gamma_0(N)] \ = \ N \prod_{p|N} (1 + \tfrac{1}{p}). \tag{2.14}$$

From the Atkin-Lehner decomposition, we also deduce

$$\dim S_k(N) \ = \ \sum_{LM=N} \tau(L) \left| H_k^\star(M) \right|. \tag{2.15}$$

Recall Definition (1.5) of $\Delta_{k,N}(m,n)$:

$$\Delta_{k,N}(m,n) \ := \ \sum_{g \in \mathcal{B}_k(N)} \overline{\Psi_g(m)} \Psi_g(n). \tag{2.16}$$

The importance of $\Delta_{k,N}(m,n)$ is established by the Petersson trace formula.

Proposition 2.1 (Petersson [P])**.** *For any $m, n \geqslant 1$ we have*

$$\Delta_{k,N}(m,n) \ = \ \delta(m,n) + 2\pi i^k \sum_{c \equiv 0 \pmod N} c^{-1} S(m,n;c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c} \right). \tag{2.17}$$

Though the quantity $\Delta_{k,N}(m,n)$ is basis independent, we would like to compute with the Petersson trace formula using an explicit basis $\mathcal{B}_k(N)$ to average over newforms. However, as remarked, the spaces $S_k(L;f)$ do not have a distinguished orthogonal basis. Therefore, to produce a basis $\mathcal{B}_k(N)$, we need a basis for the spaces $S_k(L;f)$. Iwaniec, Luo, and Sarnak [ILS] write down an explicit basis when N is square-free. As we will see in the next section, Blomer and Milićević [BM] have recently obtained a basis for arbitrary level N. Our first key idea, a kind of trace formula for sums of Hecke eigenvalues over newforms in the case N is arbitrary, is an explicit computation with this new basis. Our second key idea on the one-level density of the L-functions $L(s,f)$ for $f \in H_k^\star(N)$ uses our first key idea in an essential way to reduce the problem to the one already treated by Iwaniec, Luo, and Sarnak.

To $H_k^\star(M)$ we often associate $\chi_{0;M}$, the trivial character mod M:

$$\chi_{0;M}(n) \;=\; \begin{cases} \chi_{0;M}(n) = 1 & \text{if } (n,M) = 1 \\ \chi_{0;M}(n) = 0 & \text{otherwise.} \end{cases} \tag{2.18}$$

2.2. An orthonormal basis for $S_k(N)$.

For $f \in H_k^\star(M)$ consider the following arithmetic functions, which coincide with the ones defined in [BM] up to a few corrections [BM2].

$$r_f(c) \;:=\; \sum_{b|c} \frac{\mu(b)\lambda_f(b)^2}{b\sigma_{-1}(b)^2}, \;\; \alpha(c) \;:=\; \sum_{b|c} \frac{\chi_{0;M}(b)\mu(b)}{b^2}, \;\; \beta(c) \;:=\; \sum_{b|c} \frac{\chi_{0;M}(b)\mu^2(b)}{b}, \tag{2.19}$$

where $\mu_f(c)$ is the multiplicative function given implicitly by

$$L(f,s)^{-1} \;=\; \sum_c \frac{\mu_f(c)}{c^s}, \tag{2.20}$$

or explicitly on prime powers by

$$\mu_f(p^j) \;=\; \begin{cases} -\lambda_f(p) & j = 1 \\ \chi_{0;M}(p) & j = 2 \\ 0 & j > 2 \end{cases} \tag{2.21}$$

and

$$\sigma_{-1}(b) \;=\; \sum_{r|b} \frac{\chi_{0;M}(r)}{r}. \tag{2.22}$$

For $\ell \mid d$ define

$$\xi'_d(\ell) \;:=\; \frac{\mu(d/\ell)\lambda_f(d/\ell)}{r_f(d)^{1/2}(d/\ell)^{1/2}\beta(d/\ell)}, \quad \xi''_d(\ell) \;:=\; \frac{\mu_f(d/\ell)}{(d/\ell)^{1/2}(r_f(d)\alpha(d))^{1/2}}. \tag{2.23}$$

Write $d = d_1 d_2$ where d_1 is square-free, d_2 is square-full, and $(d_1,d_2) = 1$. Thus $p \,||\, d$ implies $p \mid d_1$ and $p^2 \mid d$ implies $p^2 \mid d_2$. Then for $\ell \mid d$ define

$$\xi_d(\ell) \;:=\; \xi'_{d_1}((d_1,\ell))\xi''_{d_2}((d_2,\ell)), \tag{2.24}$$

and let $f\mid_\ell(z)$ be defined by $f\mid_\ell(z) = \ell^{k/2} f(\ell z)$. Blomer and Milićević prove the following.

Proposition 2.2 (Blomer and Milićević [BM, Lemma 9]). *Let*

$$f_d(z) \;:=\; \sum_{\ell|d} \xi_d(\ell) f\mid_\ell(z), \tag{2.25}$$

where $N = LM$ and $f \in H_k^\star(M)$ is Petersson-normalized with respect to Petersson norm on level N. Then $\{f_d : d \mid L\}$ is an orthonormal basis of $S_k(L; f)$.

We record the following identities, which are useful to later prove Proposition 3.1. For prime powers, $\xi_d(\ell)$ simplifies as

$$\begin{aligned} \xi_1(1) &= 1, & \xi_{p^\nu}(p^\nu) &= \left(r_f(p)\left(1-\chi_{0;M}(p)/p^2\right)\right)^{-1/2} \\ \xi_p(p) &= r_f(p)^{-1/2}, & \xi_{p^\nu}(p^{\nu-1}) &= \tfrac{-\lambda_f(p)}{\sqrt{p}}\xi_{p^\nu}(p^\nu) \\ \xi_p(1) &= \tfrac{-\lambda_f(p)}{\sqrt{p}\left(1+\chi_{0;M}(p)/p\right)}\xi_p(p), & \xi_{p^\nu}(p^{\nu-2}) &= \tfrac{\chi_{0;M}(p)}{p}\xi_{p^\nu}(p^\nu). \end{aligned} \tag{2.26}$$

In addition, we will make use of the following lemma. Originally stated in the context of square-free level, the same proof holds in general.

Lemma 2.3 (Iwaniec, Luo, Sarnak [ILS, Lemma 2.5])**.** *If f is a newform of weight k and level $M \mid N$, then*

$$\langle f, f\rangle_N = (4\pi)^{1-k}\Gamma(k)\frac{\nu(N)\varphi(M)}{12M}Z(1,f), \tag{2.27}$$

where φ is the Euler totient function, and

$$Z(s,f) := \sum_1^\infty \lambda_f(n^2)n^{-s}. \tag{2.28}$$

It is often convenient to work with the local zeta function

$$Z_N(s,f) := \sum_{\ell \mid N^\infty} \lambda_f(\ell^2)\ell^{-s}. \tag{2.29}$$

If $f \in H_k^\star(N)$, then one deduces from (2.11) that the local Euler factors of $Z(1,f)$ are given by

$$Z_p(1,f) = \begin{cases} \left(1+\frac{1}{p}\right)^{-1}\rho_f(p)^{-1} & \text{if } p \nmid N \\ \left(1+\frac{1}{p}\right)^{-1}\left(1-\frac{1}{p}\right)^{-1} & \text{if } p \mid\mid N \\ 1 & \text{if } p^2 \mid N, \end{cases} \tag{2.30}$$

where $\rho_f(c)$ is the multiplicative function

$$\rho_f(c) = \sum_{b \mid c}\mu(b)b\left(\frac{\lambda_f(b)}{\nu(b)}\right)^2 = \prod_{p \mid c}\left(1-p\left(\frac{\lambda_f(p)}{p+1}\right)^2\right). \tag{2.31}$$

Assume now that $N = LM$ and $f \in H_k^\star(M)$, and, writing

$$\mathfrak{p}(L,M) := \prod_{\substack{p^\beta \mid\mid L \\ p \mid M}} p^\beta, \tag{2.32}$$

note that

$$Z_{LM/\mathfrak{p}(L,M)}(s,f) = Z_N(s,f). \tag{2.33}$$

Specializing to $s = 1$, we find

$$\frac{MN}{\varphi(M)\nu(N)} \prod_{\substack{p|L \\ p\nmid M=1}} \rho_f(p)^{-1} \;=\; Z_N(1,f) \prod_{p^2|M} \left(\frac{p^2}{p^2-1}\right). \tag{2.34}$$

We also note that if $p \nmid M$, then $r_f(p) = \rho_f(p)$, and if $p \mid M$, then $r_f(p) = 1 - \lambda_f(p)^2/p$.

3. A Formula for $\Delta_{k,N}(m,n)$

In this section we provide an explicit formula for $\Delta_{k,N}(m,n)$ in terms of Hecke eigenvalues. We begin with a result about the coefficients inherited from the orthonormal basis defined in Proposition 2.2. Note that if $f(z) \in H_k^\star(M)$ has Fourier expansion

$$f(z) \;=\; \sum_{n\geqslant 1} a_f(n)e(nz), \tag{3.1}$$

then

$$f_d(z) \;:=\; \sum_{\ell|d} \xi_d(\ell) f\mid_\ell (z) \;=\; \sum_{\ell|d} \xi_d(\ell)\ell^{k/2} f(\ell z), \tag{3.2}$$

so the coefficients of the Fourier expansion of $f_d(z)$ are given by

$$a_{f_d}(n) \;=\; \sum_{\ell|(d,n)} \xi_d(\ell)\ell^{k/2} a_f(\tfrac{n}{\ell}). \tag{3.3}$$

Let $N = LM$ and let f be a newform of weight k and level M. Let $f' = f/||f||_N$ so that f' is Petersson-normalized with respect to level N. Then by Proposition 2.2, the set $\{f_d' : d \mid L\}$ is an orthonormal basis of $S_k(L; f)$. Let

$$\mathcal{B}_k(N) \;=\; \bigcup_{LM=N} \; \bigcup_{f\in H_k^\star(M)} \; \bigcup_{d|L} f_d' \tag{3.4}$$

be our orthonormal basis for $S_k(N)$, and note that

$$a_{f'}(n) \;=\; \frac{\lambda_f(n)n^{(k-1)/2}}{||f||}. \tag{3.5}$$

We have

$$\Delta_{k,N}(m,n) \;=\; \frac{12}{(k-1)\nu(N)} \sum_{LM=N} \frac{M}{\varphi(M)} \sum_{f\in H_k^\star(M)} \frac{1}{Z(1,f)}$$
$$\times \sum_{d|L} \left(\sum_{\ell|(d,m)} \xi_d(\ell)\ell^{1/2}\lambda_f(\tfrac{m}{\ell}) \right) \left(\sum_{\ell|(d,n)} \xi_d(\ell)\ell^{1/2}\lambda_f(\tfrac{n}{\ell}) \right). \tag{3.6}$$

We now give an explicit formula for $\Delta_{k,N}(m,n)$ in terms of Hecke eigenvalues. Assume $(m,n,N)=1$ and let

$$\Xi_d(m,n;f) \;:=\; \Bigg(\sum_{\ell|(d,m)} \xi_d(\ell)\ell^{1/2}\lambda_f(\tfrac{m}{\ell})\Bigg)\Bigg(\sum_{\ell|(d,n)} \xi_d(\ell)\ell^{1/2}\lambda_f(\tfrac{n}{\ell})\Bigg) \tag{3.7}$$

for $d \mid L \mid N$, and note that

$$\Xi_d(m,n;f) \;=\; \sum_{\substack{\ell_1|(d,m)\\ \ell_2|(d,n)}} \xi_d(\ell_1)\xi_d(\ell_2)(\ell_1\ell_2)^{1/2}\lambda_f(\tfrac{m}{\ell_1})\lambda_f(\tfrac{n}{\ell_2}). \tag{3.8}$$

We can rewrite the formula (3.6) as

$$\Delta_{k,N}(m,n) \;=\; \frac{12}{(k-1)\nu(N)}\sum_{LM=N}\frac{M}{\varphi(M)}\sum_{f\in H_k^\star(M)}\frac{1}{Z(1,f)}\sum_{d|L}\Xi_d(m,n;f). \tag{3.9}$$

Using the multiplicativity of Fourier coefficients, one readily obtains that if $(\ell_1,\ell_2)=1$, and $\ell_1\ell_2 \mid m$, then

$$\lambda_f(\tfrac{m}{\ell_1})\lambda_f(\tfrac{m}{\ell_2}) \;=\; \lambda_f(m)\lambda_f(\tfrac{m}{\ell_1\ell_2}). \tag{3.10}$$

Using the multiplicativity of $\xi_d(\ell)$ and this identity, one finds that if $(g_1,g_2)=1$,

$$\Xi_{g_1}(m,n;f)\cdot\Xi_{g_2}(m,n;f) \;=\; \lambda_f(m)\lambda_f(n)\Xi_{g_1g_2}(m,n;f). \tag{3.11}$$

This allows us to reduce to the case of studying $\Xi_{p^\alpha}(m,n;f)$. Applying (3.11) to (3.6) yields

$$\begin{aligned}\Delta_{k,N}(m,n) \;=\; &\frac{12}{(k-1)\nu(N)}\\ &\times\sum_{LM=N}\frac{M}{\varphi(M)}\sum_{f\in H_k^\star(M)}\frac{1}{Z(1,f)}\left(\lambda_f(m)\lambda_f(n)\right)^{1-\omega(L)}\prod_{p^\alpha||L}\Big(\sum_{d|p^\alpha}\Xi_d(m,n;f)\Big),\end{aligned} \tag{3.12}$$

with $\omega(n)$ the number of distinct prime factors of n.

The task clearly becomes to understand the quantity

$$V_{p^\alpha}(m,n;f) \;:=\; \sum_{d|p^\alpha}\Xi_d(m,n;f). \tag{3.13}$$

The following proposition achieves this. To simplify notation, we introduce the following symbols. Put

$$\clubsuit_q(a) \;:=\; \begin{cases} q & q \mid a\\ 1 & \text{otherwise.}\end{cases} \tag{3.14}$$

$$\spadesuit_q(a_1,a_2,\dots,a_n) \;:=\; \begin{cases} 1 & q \mid a_i \text{ for some } i\\ 0 & \text{otherwise.}\end{cases} \tag{3.15}$$

Proposition 3.1. *Suppose* $(m,n,N)=1$. *Then*

$$\Delta_{k,N}(m,n) = \frac{12}{(k-1)N}\sum_{LM=N}\prod_{p^2|M}\left(\frac{p^2}{p^2-1}\right)\sum_{f\in H_k^\star(M)}\frac{Z_N(1,f)}{Z(1,f)}\left(\lambda_f(m)\lambda_f(n)\right)^{-\omega(L)+1} \times\prod_{\substack{p|L\\ p\nmid M}} r_f(p)\prod_{p^\alpha||L}V_{p^\alpha}(m,n;f) \tag{3.16}$$

where

$$\begin{aligned} V_{p^\alpha}(m,n;f) &= \lambda_f(m)\lambda_f(n)\left(1+\frac{\lambda_f(p)^2}{p\left(1+\chi_{0;M}(p)/p\right)^2\cdot r_f(p)}+\frac{\chi_{0;M}(p)}{p^2\cdot r_f(p)\left(1-\chi_{0;M}(p)/p^2\right)}\right)\\ &\quad+\spadesuit_p(nm)\lambda_f(m/\clubsuit_p(m))\lambda_f(n/\clubsuit_p(n))\\ &\qquad\times\left(\frac{-\lambda_f(p)}{r_f(p)\left(1+\chi_{0;M}(p)/p\right)}+\frac{-\lambda_f(p)\chi_{0;M}(p)}{p\cdot r_f(p)\left(1-\chi_{0;M}(p)/p^2\right)}\right)\\ &\quad+\spadesuit_{p^2}(n,m)\lambda_f(m/\clubsuit_{p^2}(m))\lambda_f(n/\clubsuit_{p^2}(n))\left(\frac{\chi_{0;M}(p)}{r_f(p)\left(1-\chi_{0;M}(p)/p^2\right)}\right) \end{aligned} \tag{3.17}$$

if $\alpha\geqslant 2$ *and*

$$\begin{aligned} V_{p^\alpha}(m,n;f) &= \lambda_f(m)\lambda_f(n)\left(1+\frac{\lambda_f(p)^2}{p\left(1+\chi_{0;M}(p)/p\right)^2\cdot r_f(p)}\right)\\ &\quad+\spadesuit_p(nm)\lambda_f(m/\clubsuit_p(m))\lambda_f(n/\clubsuit_p(n))\left(\frac{-\lambda_f(p)}{r_f(p)\left(1+\chi_{0;M}(p)/p\right)}\right) \end{aligned} \tag{3.18}$$

if $\alpha=1$.

(Note that the cases $p\mid m$ and $p\mid n$ are mutually exclusive. Thus the formulas given by (3.17) and (3.18) are well-defined.)

Proof of Proposition 3.1. Using (3.12), we write

$$\begin{aligned} &\Delta_{k,N}(m,n)\\ &= \frac{12}{(k-1)\nu(N)}\sum_{LM=N}\frac{M}{\varphi(M)}\sum_{f\in H_k^\star(M)}\frac{1}{Z(1,f)}\left(\lambda_f(m)\lambda_f(n)\right)^{1-\omega(L)}\prod_{p^\alpha||L}V_{p^\alpha}(m,n;f)\\ &= \frac{12}{(k-1)\nu(N)}\sum_{LM=N}\frac{M}{\varphi(M)}\sum_{f\in H_k^\star(M)}\frac{1}{Z(1,f)}\left(\lambda_f(m)\lambda_f(n)\right)^{1-\omega(L)}\\ &\qquad\times\prod_{\substack{p|L\\ p\nmid M}}\rho_f(p)^{-1}r_f(p)\prod_{p^\alpha||L}V_{p^\alpha}(m,n;f) \end{aligned}$$

$$= \frac{12}{(k-1)N} \sum_{LM=N} \prod_{p^2|M} \left(\frac{p^2}{p^2-1}\right) \sum_{f\in H_k^\star(M)} \frac{Z_N(1,f)}{Z(1,f)} \left(\lambda_f(m)\lambda_f(n)\right)^{1-\omega(L)} \times \prod_{\substack{p|L\\ p\nmid M}} r_f(p) \prod_{p^\alpha||L} V_{p^\alpha}(m,n;f).$$

The second line follows from the first as $p \nmid M$ implies that $r_f(p) = \rho_f(p)$. To conclude the proof, we need to obtain the correct expressions for $V_{p^\alpha}(m,n;f)$. These are proved in Appendix A. □

As the formula given by (3.1) is unwieldy, we assume $(m,N) = 1$ and $(n,N) = 1$ for the remainder of the paper. We have the following useful lemma:

Lemma 3.2. *Write $LM = N$. Fix $f \in H_k^\star(M)$. Then if $(n,N) = 1$ and $(m,N) = 1$ we have*

$$\left(\lambda_f(m)\lambda_f(n)\right)^{1-\omega(L)} \prod_{\substack{p|L\\ p\nmid M}} r_f(p) \prod_{p^\alpha||L} V_{p^\alpha}(m,n;f) = \lambda_f(m)\lambda_f(n) \prod_{\substack{p^2|L\\ p\nmid M}} \left(\frac{p^2}{p^2-1}\right) \prod_{\substack{p|L\\ p||M}} \left(\frac{p^2}{p^2-1}\right). \quad (3.19)$$

We give the proof in Appendix B. Proposition 3.1 and Lemma 3.2 imply

Lemma 3.3. *Suppose $(m,N) = 1$ and $(n,N) = 1$. Then*

$$\Delta_{k,N}(m,n) = \frac{12}{(k-1)N} \prod_{p^2|N} \left(\frac{p^2}{p^2-1}\right) \sum_{LM=N} \sum_{f\in H_k^\star(M)} \frac{Z_N(1,f)}{Z(1,f)} \lambda_f(m)\lambda_f(n). \quad (3.20)$$

4. Between weighted and unweighted sums

We now introduce the arithmetically weighted sums, as defined in [ILS, (2.53)],

$$\Delta^*_{k,N}(m,n) = \sum_{f\in H_k^\star(N)} \frac{\lambda_f(n)\lambda_f(m)Z_N(1,f)}{Z(1,f)}. \quad (4.1)$$

This allows us to state one of our main results, which generalizes Iwaniec, Luo, and Sarnak [ILS, Proposition 2.8] and Rouymi [R, Proposition 2.3].

Proposition 4.1. *Suppose $(m,N) = 1$ and $(n,N) = 1$. Then*

$$\Delta_{k,N}(m,n) = \frac{12}{(k-1)N} \prod_{p^2|N} \left(\frac{p^2}{p^2-1}\right) \sum_{LM=N} \sum_{\substack{\ell|L^\infty\\ (\ell,M)=1}} \ell^{-1} \Delta^\star_{k,M}(m\ell^2,n) \quad (4.2)$$

and

$$\Delta^{\star}_{k,N}(m,n) \ = \ \frac{k-1}{12}\sum_{LM=N}\mu(L)M\prod_{p^2|M}\left(\frac{p^2}{p^2-1}\right)^{-1}\sum_{\substack{\ell|L^\infty\\(\ell,M)=1}}\ell^{-1}\Delta_{k,M}(m\ell^2,n), \qquad (4.3)$$

where M' denotes the square-free part of M and N'' denotes the square-full part of N.

Proof of Proposition 4.1. We first prove (4.2). Note the following: $(m,N)=1$ and $(n,N)=1$ imply $(m,M)=1$, $(n,M)=1$, and $(\ell,m)=1$.

These observations together with Lemma 3.3 imply

$$\begin{aligned}\Delta_{k,N}(m,n) \ &= \ \frac{12}{(k-1)N}\prod_{p^2|N}\left(\frac{p^2}{p^2-1}\right)\sum_{LM=N}\sum_{f\in H^\star_k(M)}\frac{Z_{L/\mathfrak{p}(L,M)}(1,f)Z_M(1,f)}{Z(1,f)}\lambda_f(m)\lambda_f(n)\\ &= \ \frac{12}{(k-1)N}\prod_{p^2|N}\left(\frac{p^2}{p^2-1}\right)\sum_{LM=N}\sum_{f\in H^\star_k(M)}\Big(\sum_{\substack{\ell|L^\infty\\(\ell,M)=1}}\lambda_f(\ell^2)\ell^{-1}\Big)\frac{Z_M(1,f)}{Z(1,f)}\lambda_f(m)\lambda_f(n)\\ &= \ \frac{12}{(k-1)N}\prod_{p^2|N}\left(\frac{p^2}{p^2-1}\right)\sum_{LM=N}\sum_{\substack{\ell|L^\infty\\(\ell,M)=1}}\ell^{-1}\Delta^\star_{k,M}(m\ell^2,n). \qquad (4.4)\end{aligned}$$

We are now ready to prove (4.3) using Möbius inversion. We begin with

$$\begin{aligned}&\frac{k-1}{12}\sum_{LM=N}\mu(L)M\prod_{p^2|M}\left(\frac{p^2}{p^2-1}\right)^{-1}\sum_{\substack{\ell|L^\infty\\(\ell,M)=1}}\ell^{-1}\Delta_{k,M}(m\ell^2,n)\\ &= \ \frac{k-1}{12}\sum_{LM=N}\mu(L)M\prod_{p^2|M}\left(\frac{p^2}{p^2-1}\right)^{-1}\sum_{\substack{\ell|L^\infty\\(\ell,M)=1}}\ell^{-1}\frac{12}{(k-1)M}\prod_{p^2|M}\left(\frac{p^2}{p^2-1}\right)\\ &\qquad\times\sum_{QW=M}\sum_{f\in H^\star_k(W)}\frac{Z_M(1,f)}{Z(1,f)}\lambda_f(m\ell^2)\lambda_f(n)\\ &= \ \sum_{LM=N}\mu(L)\sum_{QW=M}\sum_{f\in H^\star_k(W)}\Big(\sum_{\substack{\ell|L^\infty\\(\ell,M)=1}}\lambda_f(\ell^2)\ell^{-1}\Big)\frac{Z_M(1,f)}{Z(1,f)}\lambda_f(m)\lambda_f(n)\\ &= \ \sum_{LM=N}\mu(L)\sum_{QW=M}\sum_{f\in H^\star_k(W)}\frac{Z_N(1,f)}{Z(1,f)}\lambda_f(m)\lambda_f(n). \qquad (4.5)\end{aligned}$$

Let

$$\heartsuit_N(W) \ := \ \sum_{f\in H^\star_k(W)}\frac{Z_N(1,f)}{Z(1,f)}\lambda_f(m)\lambda_f(n). \qquad (4.6)$$

Interchanging orders of summation yields

$$\sum_{LM=N}\mu(L)\sum_{QW=M}\heartsuit_N(W) \;=\; \sum_{W|N}\heartsuit_N(W)\sum_{L|\frac{N}{W}}\mu(L) \;=\; \Delta^\star_{k,N}(m,n), \tag{4.7}$$

as the Möbius sum vanishes unless $W = N$ and $\heartsuit_N(N) = \Delta^\star_{k,N}(m,n)$. □

One of our primary applications of Proposition 4.1 is to obtain a formula for pure sums of Hecke eigenvalues. We define the pure sum

$$\Delta^\star_{k,N}(n) \;:=\; \sum_{f\in H_k^\star(N)}\lambda_f(n) \tag{4.8}$$

and prove Theorem 1.2 from the introduction, which we restate here for convenience.

Theorem 1.2. *Suppose that* $(n,N)=1$. *Then*

$$\Delta^\star_{k,N}(n) \;=\; \frac{k-1}{12}\sum_{LM=N}\mu(L)M\prod_{p^2|M}\left(\frac{p^2}{p^2-1}\right)^{-1}\sum_{(m,M)=1}m^{-1}\Delta_{k,M}(m^2,n). \tag{4.9}$$

Proof. We remove the weights in (4.1) by summing $m^{-1}\Delta^\star_{k,N}(m^2,n)$ over all $(m,N)=1$. On one side we have

$$\begin{aligned}\sum_{(m,N)=1}m^{-1}\Delta^\star_{k,N}(m^2,n) \;&=\; \sum_{(m,N)=1}m^{-1}\sum_{f\in H_k^\star(N)}\frac{\lambda_f(m^2)\lambda_f(n)Z_N(1,f)}{Z(1,f)}\\ &=\; \sum_{f\in H_k^\star(N)}\frac{\lambda_f(n)}{Z(1,f)}\sum_{(m,N)=1}\sum_{\ell|N^\infty}(\ell m)^{-1}\lambda_f(\ell^2)\lambda_f(m^2)\\ &=\; \sum_{f\in H_k^\star(N)}\frac{\lambda_f(n)}{Z(1,f)}\sum_{r\geqslant 1}r^{-1}\lambda_f(r^2)\\ &=\; \sum_{f\in H_k^\star(N)}\lambda_f(n)\\ &=\; \Delta^\star_{k,N}(n).\end{aligned} \tag{4.10}$$

On the other hand we have, using (4.3), for $(n,N)=1$,

$$\begin{aligned}&\sum_{(m,N)=1}m^{-1}\Delta_{k,N}(m^2,n)\\ &=\; \sum_{(m,N)=1}m^{-1}\frac{k-1}{12}\sum_{LM=N}\mu(L)M\prod_{p^2|M}\left(\frac{p^2}{p^2-1}\right)^{-1}\sum_{\substack{\ell|L^\infty\\(\ell,M)=1}}\frac{1}{\ell}\Delta_{k,M}((m\ell)^2,n)\\ &=\; \frac{k-1}{12}\sum_{LM=N}\mu(L)M\prod_{p^2|M}\left(\frac{p^2}{p^2-1}\right)^{-1}\sum_{(m,N)=1}\sum_{\substack{\ell|L^\infty\\(\ell,M)=1}}\frac{1}{m\ell}\Delta_{k,M}((m\ell)^2,n)\end{aligned}$$

$$= \frac{k-1}{12} \sum_{LM=N} \mu(L) M \prod_{p^2|M} \left(\frac{p^2}{p^2-1}\right)^{-1} \sum_{(m,M)=1} \frac{1}{m} \Delta_{k,M}(m^2,n). \tag{4.11}$$

This completes the proof. □

5. Estimating tails of pure sums

One might inquire about the convergence of the innermost sum in Theorem (1.2). It is assured by the holomorphy of $L(s, \mathrm{sym}^2 f)$, but is not absolute (see [ILS, p. 79] for a full discussion). For this reason, following [ILS, §2], we begin our work towards the Density Conjecture by splitting

$$\Delta^\star_{k,N}(n) = \Delta'_{k,N}(n) + \Delta^\infty_{k,N}(n) \tag{5.1}$$

where

$$\Delta'_{k,N}(n) := \frac{k-1}{12} \sum_{\substack{LM=N \\ L\leqslant X}} \mu(L) M \prod_{p^2|M} \left(\frac{p^2}{p^2-1}\right)^{-1} \sum_{\substack{(m,M)=1 \\ m\leqslant Y}} m^{-1} \Delta_{k,M}(m^2,n), \tag{5.2}$$

and $\Delta^\infty_{k,N}(n)$ is the complementary sum. Here $X, Y \geqslant 1$ are free parameters.

We consider sequences $\mathcal{A} = \{a_q\}$ that satisfy

$$\sum_{(q,nN)=1} \lambda_f(q) a_q \ll (nkN)^\varepsilon \tag{5.3}$$

for all $f \in H^\star_k(M)$ with $M \mid N$ such that the implied constant depends only on ε. The sequence we need for our application, given by

$$a_q = p^{-1/2} \log p \qquad \text{if } q = p \leqslant Q, \tag{5.4}$$

and $a_q = 0$ elsewhere, satisfies this property provided $\log Q \ll \log kN$; see [ILS, p. 80] for more details.

Lemma 5.1. *Suppose* $(n,N) = 1$ *and that* $\mathcal{A}$ *satisfies* (5.3). *Then*

$$\sum_{(q,nN)=1} \Delta^\infty_{k,N}(nq) a_q \ll kN(X^{-1} + Y^{-1/2})(nkNXY)^\varepsilon. \tag{5.5}$$

Proof. Suppose $(q, nN) = 1$. By Lemma 3.3 we write

$$\Delta^\infty_{k,N}(nq) = \sum_{\substack{KLM=N \\ L>X}} \mu(L) \sum_{f\in H^\star_k(M)} \lambda_f(nq) + \sum_{\substack{KLM=N \\ L\leqslant X}} \mu(L) \sum_{f\in H^\star_k(M)} \lambda_f(nq) R_f(KM;Y) \tag{5.6}$$

where

$$R_f(KM;Y) := \frac{Z_{KM}(1,f)}{Z(1,f)} \sum_{\substack{(m,KM)=1 \\ m>Y}} m^{-1} \lambda_f(m^2). \tag{5.7}$$

By the Riemann Hypothesis for $L(s, \mathrm{sym}^2(f))$ we have

$$R_f(KM;Y) \ll Y^{-1/2}(kKMY)^{\varepsilon}. \tag{5.8}$$

Combining this fact with the Deligne bound for $|\lambda_f(n)|$, we have

$$\begin{aligned}
\sum_{(q,nN)=1} \Delta^{\infty}_{k,N}(nq)a_q &= \sum_{\substack{KLM=N\\ L>X}} \mu(L) \sum_{f\in H_k^\star(M)} \lambda_f(n) \sum_{(q,nN)=1} \lambda_f(q)a_q \\
&\quad + \sum_{\substack{KLM=N\\ L\leqslant X}} \mu(L) \sum_{f\in H_k^\star(M)} \lambda_f(n)R_f(KM;Y) \sum_{(q,nN)=1} \lambda_f(q)a_q \\
&\ll \sum_{\substack{KLM=N\\ L>X}} \mu(L)|H_k^\star(M)|\tau(n)(nkN)^{\varepsilon} \\
&\quad + \sum_{\substack{KLM=N\\ L\leqslant X}} \mu(L)|H_k^\star(M)|\tau(n)Y^{-1/2}(kKMY)^{\varepsilon}(nkN)^{\varepsilon} \\
&\ll \sum_{\substack{KLM=N\\ L>X}} \mu(L)\left(\frac{k-1}{12}\right)\varphi(M)\tau(n)(nkN)^{\varepsilon} \\
&\quad + \sum_{\substack{KLM=N\\ L\leqslant X}} \mu(L)\left(\frac{k-1}{12}\right)\varphi(M)\tau(n)Y^{-1/2}(kKMY)^{\varepsilon}(nkN)^{\varepsilon} \\
&\ll \sum_{\substack{KLM=N\\ L>X}} k\tfrac{N}{X}(nkN)^{\varepsilon} + \sum_{\substack{KLM=N\\ L\leqslant X}} kNY^{-1/2}(kKMY)^{\varepsilon}(nkN)^{\varepsilon} \\
&\ll kN(X^{-1}+Y^{-1/2})(nkNXY)^{\varepsilon}.
\end{aligned} \tag{5.9}$$

This establishes the lemma. □

We now substitute the Petersson formula (Proposition 2.1) for each instance of $\Delta_{k,M}(m^2,n)$ to obtain an exact formula for $\Delta'_{k,N}(n)$ in terms of Kloosterman sums.

Proposition 5.2. *Suppose* $(n,N)=1$. *Then*

$$\begin{aligned}
\Delta'_{k,N}(n) &= \delta_Y(m^2,n)\frac{k-1}{12}n^{-1/2} \sum_{\substack{LM=N\\ L\leqslant X}} \mu(L)M \prod_{p^2|M}\left(\frac{p^2}{p^2-1}\right)^{-1} \\
&\quad + \frac{k-1}{12} \sum_{\substack{LM=N\\ L\leqslant X}} \mu(L)M \prod_{p^2|M}\left(\frac{p^2}{p^2-1}\right)^{-1} \\
&\qquad \times \sum_{\substack{(m,M)=1\\ m\leqslant Y}} m^{-1}2\pi i^k \sum_{c\equiv 0 \pmod M} c^{-1}S(m^2,n;c)J_{k-1}\left(\frac{4\pi m\sqrt{n}}{c}\right),
\end{aligned}$$

where

$$\delta_Y(m^2,n) \;=\; \begin{cases} 1 & \text{if } n=m^2 \text{ and } m \leqslant Y, \\ 0 & \text{otherwise.} \end{cases} \tag{5.10}$$

We recover the bounds for $|H_k^\star(N)|$ given by Martin in [Ma, Theorem 6(c)] and use them to prove the following proposition in Appendix C.

Proposition 5.3. *We have that as $kN \to \infty$*

$$\frac{k-1}{12}\varphi(N)\prod_p\left(1-\frac{1}{p^2-p}\right)+O\left((kN)^{2/3}\right) \;\leqslant\; |H_k^\star(N)| \;\leqslant\; \frac{k-1}{12}\varphi(N)+O\left((kN)^{2/3}\right). \tag{5.11}$$

6. The Density Conjecture for $H_k^\star(N)$

Fix some $\phi \in \mathcal{S}(\mathbf{R})$ with $\widehat{\phi}$ supported in $(-u,u)$. We reprise some basic definitions from the introduction.

To a holomorphic newform f, we associate the L-function

$$L(s,f) \;=\; \sum_1^\infty \lambda_f(n)n^{-s}. \tag{6.1}$$

Assuming the Riemann Hypothesis for $L(s,f)$, we can write its non-trivial zeros as

$$\varrho_f \;=\; \frac{1}{2}+i\gamma_f. \tag{6.2}$$

We are interested in the one-level densities of low-lying zeroes. We recall the definition of $D_1(f;\phi)$ in (1.10):

$$D_1(f;\phi) \;=\; \sum_{\gamma_f}\phi\left(\frac{\gamma_f}{2\pi}\log c_f\right), \tag{6.3}$$

where c_f is the analytic conductor of f which in our case is k^2N. We also introduce a scaling parameter R which we take to satisfy $1 < R \asymp k^2N$.

Iwaniec, Luo, and Sarnak [ILS, §4] establish that for $f \in H_k^\star(N)$,

$$D_1(f;\phi) \;=\; E(\phi)-P(f;\phi)+O\left(\frac{\log\log kN}{\log R}\right) \tag{6.4}$$

where

$$E(\phi) \;=\; \widehat{\phi}(0)+\frac{1}{2}\phi(0) \tag{6.5}$$

and

$$P(f;\phi) \;=\; \sum_{p\nmid N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}. \tag{6.6}$$

Note that their argument does not depend on N being square-free. The Density Conjecture concerns the average over $H_k^\star(N)$, so we consider the sum

$$\mathcal{B}_k^\star(\phi) \;=\; \sum_{f\in H_k^\star(N)} D_1(f;\phi). \tag{6.7}$$

Substituting (6.4) into the above we find that

$$\mathcal{B}_k^\star(\phi) \;=\; |H_k^\star(N)|\,E(\phi) - \mathcal{P}_k^\star(\phi) + O\left(|H_k^\star(N)|\,\frac{\log\log kN}{\log R}\right) \tag{6.8}$$

where

$$\mathcal{P}_k^\star(\phi) \;=\; \sum_{p\nmid N} \Delta_{k,N}^\star(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}. \tag{6.9}$$

In order to establish that as $kN\to\infty$ that the main term of $\mathcal{B}_k^\star(\phi)/\,|H_k^\star(N)|$ is $E(\phi)$, we need to establish that $\mathcal{P}_k^\star(\phi)=o(k\varphi(N))$. This is sufficient because $|H_k^\star(N)|\asymp k\varphi(N)$, as we showed in Proposition 5.3.

We can now write

$$\mathcal{P}_k^\star(\phi) \;=\; \sum_{p\nmid N}\left(\Delta'_{k,N}(p)+\Delta^\infty_{k,N}(p)\right)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}. \tag{6.10}$$

We first bound

$$\sum_{p\nmid N}\Delta^\infty_{k,N}(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}. \tag{6.11}$$

Let a_q be as in (5.4) for $q\leqslant R^u$ and 0 for $q>R^u$ (the latter is due to the appearance of $\widehat{\phi}$, which is zero for $P>R^u$). We see that this sequence satisfies the condition on Q in the definition (5.4), and since ϕ is of Schwartz class, we may apply Lemma 5.1 with $X=Y=(kN)^\delta$ for small positive δ to find

$$\sum_{p\nmid N}\Delta^\infty_{k,N}(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R} \;\ll\; kN(X^{-1}+Y^{-1/2})(kNXY)^\varepsilon \;=\; o(k\varphi(N)). \tag{6.12}$$

Next we must estimate the other term from (6.10),

$$\mathcal{M}_k^\star(\phi) \;:=\; \sum_{p\nmid N}\Delta'_{k,N}(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}. \tag{6.13}$$

To begin, define

$$Q^\star_{k;N}(m;c) \;=\; 2\pi i^k\sum_{p\nmid N} S(m^2,p;c)J_{k-1}\left(\frac{4\pi m}{c}\sqrt{p}\right)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}. \tag{6.14}$$

Then we apply Lemma 5.2 to each instance of $\Delta'_{k,N}$. Note that the first term in Lemma 5.2 disappears because p is never a square. Then, moving the initial summation over $p\nmid N$

into the expression, we can rewrite in terms of $Q^{\star}_{k;N}(m;c)$:

$$\mathcal{M}^{\star}_{k}(\phi) \;=\; \frac{k-1}{12}\sum_{\substack{LM=N\\ L\leqslant X}}\mu(L)M\prod_{p^2|M}\left(\frac{p^2}{p^2-1}\right)^{-1}\sum_{\substack{(m,M)=1\\ m\leqslant Y}}m^{-1}\sum_{c\equiv 0(M)}c^{-1}Q^{\star}_{k;N}(m;c). \tag{6.15}$$

Iwaniec, Luo, and Sarnak [ILS, §6] prove the bound (which still holds for N not square-free)

$$Q^{\star}_{k;N}(m;c) \;\ll\; \widetilde{\gamma}(z)mP^{1/2}(kN)^{\varepsilon}(\log 2c)^{-2}, \tag{6.16}$$

where $z = 4\pi m\sqrt{P}/c$, $P = R^{u'}$ with some $u' < u$, and $\widetilde{\gamma}(z) = 2^{-k}$ if $3z \leqslant k$; this bound appears after their equation (6.17), and uses GRH for Dirichlet L-functions (they expand the Kloosterman sums with Dirichlet characters). In order to apply this bound we need to secure $12\pi m P^{1/2} \leqslant kc$ (so as to satisfy a condition on an estimate for the Bessel function). Noting that $m \leqslant Y$ and $c \geqslant M \geqslant N/X$, it suffices to have $12\pi XYP^{1/2} \leqslant kN$. Taking logarithms, this becomes a condition on u, namely

$$u \;\leqslant\; \frac{2(1-2\delta)\log(kN)}{\log(k^2N)}. \tag{6.17}$$

For u in this range we can apply the estimate (6.16) to find

$$\begin{aligned}\mathcal{M}^{\star}_{k}(\phi) \;&=\; \frac{k-1}{12}\sum_{\substack{LM=N\\ L\leqslant X}}\mu(L)M\\ &\quad\times\prod_{p^2|M}\left(\frac{p^2}{p^2-1}\right)^{-1}\sum_{\substack{(m,M)=1\\ m\leqslant Y}}m^{-1}\sum_{c\equiv 0(M)}c^{-1}2^{-k}mP^{1/2}(kN)^{\varepsilon}(\log 2c)^{-2}\\ &=\; \frac{k-1}{12}2^{-k}P^{1/2}(kN)^{\varepsilon}\sum_{\substack{LM=N\\ L\leqslant X}}\mu(L)M\prod_{p^2|M}\left(\frac{p^2}{p^2-1}\right)^{-1}\sum_{\substack{(m,M)=1\\ m\leqslant Y}}\sum_{c\equiv 0(M)}c^{-1}(\log 2c)^{-2}.\end{aligned} \tag{6.18}$$

Trivial estimation plus the bound

$$\sum_{c\equiv 0(M)}\frac{1}{c(\log c)^2} \;\ll\; \frac{1}{M} \tag{6.19}$$

yields

$$\sum_{p\nmid N}\Delta'_{k,N}(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R} \;\ll\; \frac{k-1}{12}2^{-k}P^{1/2}(kN)^{\varepsilon}XY, \tag{6.20}$$

which is $o(k\varphi(N))$ for $\varepsilon + 2\delta < 1/2$.

Thus by taking δ sufficiently small and applying the combined estimates for the completed sums, (6.12) and (6.20), we have established that $\mathcal{P}^{\star}_{k}(\phi) = o(k\varphi(N))$ where $\widehat{\phi}$ is supported in $(-u, u)$ and

$$u \ < \ \frac{2\log kN}{\log k^2N}, \tag{6.21}$$

which implies the following.

Theorem 6.1. *Assuming the Generalized Riemann Hypothesis for $L(s,f)$ and $L(s,\operatorname{sym}^2 f)$ and for all Dirichlet L-functions, the Density Conjecture holds for the family $H_k^\star(N)$ for any test function $\phi(x)$ whose Fourier transform is supported inside $(-u,u)$ with u given by* (6.21).

We immediately obtain the following.

Theorem 1.3. *Fix any $\phi \in \mathcal{S}(\mathbf{R})$ with $\operatorname{supp}\widehat{\phi} \subset (-2,2)$. Then, assuming the Generalized Riemann Hypothesis for $L(s,f)$ and $L(s,\operatorname{sym}^2 f)$ for $f \in H_k^*(N)$ and for all Dirichlet L-functions,*

$$\lim_{N\to\infty} \frac{1}{\left|H_k^\star(N)\right|} \sum_{f\in H_k^\star(N)} D_1(f;\phi) \ = \ \int_{-\infty}^{\infty} \phi(x) W_1(\mathrm{O})(x)\, dx \tag{6.22}$$

where $W_1(\mathrm{O})(x) = 1 + \frac{1}{2}\delta_0(x)$; thus the 1-level density agrees only with orthogonal symmetry.

Appendix A. Simplifying $V_{p^\alpha}(m,n;f)$

Definitions needed in the appendix can be found in Section 3. We have the following:

$$\begin{aligned}\sum_{d|p^\alpha} \Xi_d(m,n;f) \ &= \ \Xi_1(m,n;f) + \Xi_p(m,n;f) + \cdots + \Xi_{p^\alpha} \\ &= \ \xi_1(1)\xi_1(1)\lambda_f(m)\lambda_f(n) + \sum_{\substack{\ell_1|(p,m)\\ \ell_2|(p,n)}} \xi_p(\ell_1)\xi_p(\ell_2)(\ell_1\ell_2)^{1/2}\lambda_f(\tfrac{m}{\ell_1})\lambda_f(\tfrac{n}{\ell_2}) \\ &\quad + \sum_{\substack{\ell_1|(p^2,m)\\ \ell_2|(p^2,n)}} \xi_{p^2}(\ell_1)\xi_{p^2}(\ell_2)(\ell_1\ell_2)^{1/2}\lambda_f(\tfrac{m}{\ell_1})\lambda_f(\tfrac{n}{\ell_2}),\end{aligned} \tag{A.1}$$

where the remaining summands vanish for the following reason: Suppose $p \mid m$. Then $p \mid (m,L)$ so $(m,n,N) = 1$ implies that $p \nmid n$. Then if $\ell_2 \mid (p,n)$ we must have $\ell_2 = 1$, and similarly if $p \mid n$; hence $\xi_{p^\beta}(1)$ vanishes for $\beta \geqslant 3$. Thus we have

$$\begin{aligned}V_{p^\alpha}(m,n;f) \ &:= \ \sum_{d|p^\alpha} \Xi_d(m,n;f) \\ &= \ \lambda_f(m)\lambda_f(n) + \xi_p(1)^2\lambda_f(m)\lambda_f(n) \\ &\quad + \spadesuit_p(nm)\xi_p(p)\xi_p(1)p^{1/2}\lambda_f(m/\clubsuit_p(m))\lambda_f(n/\clubsuit_p(n)) \\ &\quad + \xi_{p^2}(1)^2\lambda_f(m)\lambda_f(n) + \spadesuit_p(nm)\xi_{p^2}(p)\xi_{p^2}(1)p^{1/2}\lambda_f(m/\clubsuit_p(m))\lambda_f(n/\clubsuit_p(n)) \\ &\quad + \spadesuit_{p^2}(n,m)\xi_{p^2}(p^2)\xi_{p^2}(1)p\lambda_f(m/\clubsuit_{p^2}(m))\lambda_f(n/\clubsuit_{p^2}(n)) \\ &= \ \lambda_f(m)\lambda_f(n)(1 + \xi_p(1)^2 + \xi_{p^2}(1)^2)\end{aligned}$$

$$\begin{aligned}
&+ \spadesuit_p(nm)\lambda_f(m/\clubsuit_p(m))\lambda_f(n/\clubsuit_p(n))\sqrt{p}(\xi_p(p)\xi_p(1)+\xi_{p^2}(p)\xi_{p^2}(1))\\
&+ \spadesuit_{p^2}(n,m)\xi_{p^2}(p^2)\xi_{p^2}(1)p\lambda_f(m/\clubsuit_{p^2}(m))\lambda_f(n/\clubsuit_{p^2}(n))\\
= \;& \lambda_f(m)\lambda_f(n)\left(1+\frac{\lambda_f(p)^2}{p\left(1+\chi_{0;M}(p)/p\right)^2 r_f(p)}+\frac{\chi_{0;M}(p)}{p^2 r_f(p)\left(1-\chi_{0;M}(p)/p^2\right)}\right)\\
&+ \spadesuit_p(nm)\lambda_f(m/\clubsuit_p(m))\lambda_f(n/\clubsuit_p(n))\\
&\quad\times\left(\frac{-\lambda_f(p)}{r_f(p)\left(1+\chi_{0;M}(p)/p\right)}+\frac{-\lambda_f(p)\chi_{0;M}(p)}{pr_f(p)\left(1-\chi_{0;M}(p)/p^2\right)}\right)\\
&+ \spadesuit_{p^2}(n,m)\lambda_f(m/\clubsuit_{p^2}(m))\lambda_f(n/\clubsuit_{p^2}(n))\left(\frac{\chi_{0;M}(p)}{r_f(p)\left(1-\chi_{0;M}(p)/p^2\right)}\right).
\end{aligned}\tag{A.2}$$

and now (3.18) follows by removing the contribution from the p^2 terms.

Appendix B. Verification of Lemma 3.2

Lemma B.1 (Lemma 3.2)**.** *Write $LM = N$. Fix $f \in H_k^\star(M)$. Then if $(n,N)=1$ and $(m,N)=1$ we have*

$$\begin{aligned}
&(\lambda_f(m)\lambda_f(n))^{1-\omega(L)}\prod_{\substack{p|L\\ p\nmid M}} r_f(p)\prod_{p^\alpha||L} V_{p^\alpha}(m,n;f)\\
&\qquad\qquad = \lambda_f(m)\lambda_f(n)\prod_{\substack{p^2|L\\ p\nmid M}}\left(\frac{p^2}{p^2-1}\right)\prod_{\substack{p|L\\ p||M}}\left(\frac{p^2}{p^2-1}\right).
\end{aligned}\tag{B.1}$$

Note that the conditions $(m,N)=1$ and $(n,N)=1$ imply that if $p \mid L$ then $p\nmid m$ and $p\nmid n$. Thus by Proposition 3.1, Appendix B, (2.11), and (2.31), we have

$$\begin{aligned}
&\left(\frac{1}{\lambda_f(m)\lambda_f(n)}\right)^{\omega(L)-1}\prod_{\substack{p|L\\ p\nmid M}} r_f(p)\prod_{p^\alpha||L}\left(\sum_{d|p^\alpha}\Xi_d(m,n;f)\right)\\
&= \lambda_f(m)\lambda_f(n)\prod_{\substack{p||L\\ p\nmid M}} r_f(p)\left(1+\frac{\lambda_f(p)^2}{p\left(1+\frac{\chi_{0;M}(p)}{p}\right)^2 r_f(p)}\right)\\
&\quad\times\prod_{\substack{p^2|L\\ p\nmid M}} r_f(p)\left(1+\frac{\lambda_f(p)^2}{p\left(1+\frac{\chi_{0;M}(p)}{p}\right)^2 r_f(p)}+\frac{\chi_{0;M}(p)}{p^2 r_f(p)\left(1-(\frac{\chi_{0;M}(p)}{p})^2\right)}\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \prod_{\substack{p||L\\ p|M}} \left(1 + \frac{\lambda_f(p)^2}{p\left(1+\frac{\chi_{0;M}(p)}{p}\right)^2 r_f(p)}\right)\\
&\times \prod_{\substack{p^2|L\\ p|M}} \left(1 + \frac{\lambda_f(p)^2}{p\left(1+\frac{\chi_{0;M}(p)}{p}\right)^2 r_f(p)} + \frac{\chi_{0;M}(p)}{p^2 r_f(p)\left(1-(\frac{\chi_{0;M}(p)}{p})^2\right)}\right)\\
= \ & \lambda_f(m)\lambda_f(n) \prod_{\substack{p||L\\ p\nmid M}} r_f(p) \left(1 + \frac{\lambda_f(p)^2}{p\left(1+\frac{1}{p}\right)^2 r_f(p)}\right)\\
&\times \prod_{\substack{p^2|L\\ p\nmid M}} r_f(p) \left(1 + \frac{\lambda_f(p)^2}{p\left(1+\frac{1}{p}\right)^2 r_f(p)} + \frac{1}{p^2 r_f(p)\left(1-(\frac{1}{p})^2\right)}\right) \prod_{p|(L,M)} \left(1+\frac{\lambda_f(p)^2}{p r_f(p)}\right)\\
= \ & \lambda_f(m)\lambda_f(n) \prod_{\substack{p||L\\ p\nmid M}} \left(\rho_f(p) + \frac{\lambda_f(p)^2}{p\left(1+\frac{1}{p}\right)^2}\right) \prod_{\substack{p^2|L\\ p\nmid M}} \left(\rho_f(p) + \frac{\lambda_f(p)^2}{p\left(1+\frac{1}{p}\right)^2} + \frac{1}{p^2-1}\right)\\
&\times \prod_{p|(L,M)} \left(1 + \frac{\lambda_f(p)^2}{p\left(1-\frac{\lambda_f(p)^2}{p}\right)}\right)\\
= \ & \lambda_f(m)\lambda_f(n) \prod_{\substack{p||L\\ p\nmid M}} \left(1 - p\left(\frac{\lambda_f(p)}{p+1}\right)^2 + \frac{\lambda_f(p)^2}{p\left(1+\frac{1}{p}\right)^2}\right)\\
&\times \prod_{\substack{p^2|L\\ p\nmid M}} \left(1 - p\left(\frac{\lambda_f(p)}{p+1}\right)^2 + \frac{\lambda_f(p)^2}{p\left(1+\frac{1}{p}\right)^2} + \frac{1}{p^2-1}\right) \prod_{\substack{p|L\\ p||M}} \left(1 + \frac{1}{p^2\left(1-\frac{1}{p^2}\right)}\right)\\
= \ & \lambda_f(m)\lambda_f(n) \prod_{\substack{p^2|L\\ p\nmid M}} \left(1+\frac{1}{p^2-1}\right) \prod_{\substack{p|L\\ p||M}} \left(1+\frac{1}{p^2-1}\right)\\
= \ & \lambda_f(m)\lambda_f(n) \prod_{\substack{p^2|L\\ p\nmid M}} \left(\frac{p^2}{p^2-1}\right) \prod_{\substack{p|L\\ p||M}} \left(\frac{p^2}{p^2-1}\right).
\end{aligned}
$$

Appendix C. Estimates for $|H_k^\star(N)|$

In this section we recover the main terms of some bounds of Martin [Ma] on the cardinality of the set $H_k^\star(N)$. Since $\lambda_f(1) = 1$, we see that $\Delta_{k,N}^\star(1) = |H_k^\star(N)|$. So, in order to determine the cardinality of $H_k^\star(N)$ it suffices to have an estimate of $\Delta_{k,N}^\star(1)$. Taking one term $q = 1$ from Lemma 5.1, we find that

$$\Delta_{k,N}^\infty(n) \ \ll\ kN\left(X^{-1} + Y^{-1/2}\right)(nkNXY)^\varepsilon. \tag{C.1}$$

Next we turn to evaluating $\Delta'_{k,N}(1)$.

$$\Delta'_{k,N}(n) \ =\ \frac{k-1}{12}\sum_{\substack{LM=N\\ L\leqslant X}}\mu(L)M\prod_{p^2|M}\left(\frac{p^2}{p^2-1}\right)^{-1}\sum_{\substack{(m,M)=1\\ m\leqslant Y}} m^{-1}\Delta_{k,M}(m^2,n). \tag{C.2}$$

An application of Weil's bound for Kloosterman sums and a crude bounding of the Bessel function in the Petersson formula (Proposition 2.1) yields an estimate such as [ILS, Corollary 2.2], which we reproduce now. For any $m, n \geqslant 1$,

$$\Delta_{k,M}(m,n) \ =\ \delta(m,n) + O\left(\frac{\tau(M)}{Mk^{5/6}}\frac{(m,n,M)\tau_3((m,n))}{((m,M)+(n,M))^{1/2}}\left(\frac{mn}{\sqrt{mn}+kM}\right)^{1/2}\log 2mn\right), \tag{C.3}$$

where the implied constant is absolute. When $n = 1$ and $m \leqslant Y$ we find

$$\Delta_{k,M}(m^2,1) \ =\ \delta(m,1) + O\left(\frac{M^\varepsilon m^{1+\varepsilon}}{M^{3/2}k^{4/3}}\right). \tag{C.4}$$

In our case this gives that

$$\Delta'_{k,N}(1) \ =\ \frac{k-1}{12}\sum_{\substack{LM=N\\ L\leqslant X}}\mu(L)M\prod_{p^2|M}\left(\frac{p^2}{p^2-1}\right)^{-1} + O\left(\frac{X^{1/2}Y(kNY)^\varepsilon}{N^{1/2}k^{1/3}}\right). \tag{C.5}$$

We now turn to the evaluation of

$$\eta(N) \ :=\ \sum_{LM=N}\mu(L)M\prod_{p^2|M}\left(1-\frac{1}{p^2}\right). \tag{C.6}$$

Let M'' denote the square-full part of M and let $g(M) = M/\zeta_{M''}(2)$, which is multiplicative. Then $\eta = \mu \star g$ is multiplicative as it is the Dirichlet convolution of two multiplicative functions and we can compute directly the value of η on prime powers:

$$\eta(p^v) \ =\ \begin{cases} p\left(1-\frac{1}{p}\right) & \text{if } v = 1\\ p^2\left(1-\frac{1}{p}-\frac{1}{p^2}\right) & \text{if } v = 2\\ p^v\left(1-\frac{1}{p^2}\right)\left(1-\frac{1}{p}\right) & \text{if } v > 2. \end{cases} \tag{C.7}$$

It is also useful to establish a bound relating $\eta(N)$ to $\varphi(N)$. By inspection we have that $\eta(N) \leqslant \varphi(N)$. Then as the ratio $\eta(p^v)/\varphi(p^v)$ is minimized when $v = 2$ and as

$$\eta(p^2)/\varphi(p^2) \ = \ \frac{p^2 - p - 1}{p^2 - p} \ = \ 1 - \frac{1}{p^2 - p}, \tag{C.8}$$

we find that

$$\varphi(N) \prod_p \left(1 - \frac{1}{p^2 - p}\right) \ \leqslant \ \eta(N) \ \leqslant \ \varphi(N). \tag{C.9}$$

Now combining (C.1) (C.5) and (C.6), we find that

$$\Delta^\star_{k,N}(1) \ = \ \frac{k-1}{12}\eta(N)\left(1 + O\left(\frac{\tau(N)N}{\eta(N)X}\right)\right) + O\left(\frac{X^{1/2}Y(kNY)^\varepsilon}{N^{1/2}k^{1/3}}\right) + O\left(kN\left(X^{-1} + Y^{-1/2}\right)(kNXY)^\varepsilon\right). \tag{C.10}$$

Taking $X = Y^{1/2} = k^{8/21}N^{3/7}$ then gives

$$\Delta^\star_{k,N}(1) \ = \ \frac{k-1}{12}\eta(N) + O\left((kN)^{2/3}\right), \tag{C.11}$$

which recovers the tight asymptotic bounds given on $|H^\star_k(N)|$ in [Ma, Theorem 6(c)]. Combining this with (C.9) we establish the following.

Proposition C.1 (Proposition 5.3)**.** *We have that as* $kN \to \infty$

$$\frac{k-1}{12}\varphi(N)\prod_p\left(1 - \frac{1}{p^2 - p}\right) + O\left((kN)^{2/3}\right) \ \leqslant \ |H^\star_k(N)| \ \leqslant \ \frac{k-1}{12}\varphi(N) + O\left((kN)^{2/3}\right). \tag{C.12}$$

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