INFINITE FAMILIES OF PARTITIONS INTO MSTD SUBSETS

HÙNG VIỆT CHU, NOAH LUNTZLARA, STEVEN J. MILLER, AND LILY SHAO

ABSTRACT. A set A is MSTD (more-sum-than-difference) if $|A+A| > |A-A|$. Though MSTD sets are rare, Martin and O'Bryant proved that there exists a positive constant lower bound for the proportion of MSTD subsets of $\{1, 2, \dots, r\}$ as $r \to \infty$. Asada et al. [AMMS] showed that there exists a positive constant lower bound for the proportion of decompositions of $\{1, 2, \dots, r\}$ into two MSTD subsets as $r \to \infty$, which implies the result of Martin and O'Bryant. However, the method is probabilistic and does not give explicit decompositions.

Continuing this work, we provide an efficient method to partition $\{1, 2, \dots, r\}$ (for r sufficiently large) into $k \geq 2$ MSTD subsets, positively answering a question raised in [AMMS] as to whether or not this is possible for all such k. Next, let R be the smallest integer such that for all $r \geq R$, $\{1, 2, \dots, r\}$ can be k-decomposed into MSTD subsets, while $\{1, 2, \dots, R-1\}$ cannot be k-decomposed into MSTD subsets. We establish rough lower and upper bounds for R and the gap between the two bounds grows linearly with k. Lastly, we provide a sufficient condition on when there exists a constant lower bound for the proportion of decompositions of $\{1, 2, \dots, r\}$ into k MSTD subsets as $r \to \infty$. This condition offers an alternative proof of Theorem 1.4 in [AMMS] and can be a promising approach to generalize the theorem.

CONTENTS

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1. INTRODUCTION

1.1. **Background.** Given a set A of natural numbers, define $A+A = \{a_i + a_j | a_i, a_j \in A\}$ and $A - A = \{a_i - a_j | a_i, a_j \in A\}$. Then A is said to be sum-dominant or MSTD (more-sums-than-differences) if $|A+A| > |A-A|$, balanced if $|A+A| = |A-A|$ and difference-dominated if $|A+A| < |A-A|$; see [He, Ma, Na1, Na2, Ru1, Ru2, Ru3] for some history and early results in the subject. Research on MSTD sets has made great progress in the last twenty years. In particular, Martin and O'Bryant [MO] showed that with the uniform model, where each element is chosen with probability $1/2$, the proportion of MSTD subsets of $\{1, 2, \dots, r\}$ is uniformly bounded below by a positive constant for large enough r. Zhao [Zh2] showed that the proportion converges as $r \to \infty$ and improved the lower bound to $4.28 \cdot 10^{-4}$. On the other hand, Hegarty and Miller [HM] proved that with a sparse model, where each element is chosen with probability $p(r)$ such that $r^{-1} = o(p(r))$ and $p(r) \to 0$ as $r \to \infty$, almost all sets are difference-dominated. These two results do not contradict each other since the probability of being MSTD subsets depends on which model we are using. In proving a lower bound for the proportion of MSTD subsets, Martin and O'Bryant used the probabilistic method and did not give explicit constructions of MSTD sets. Later works gave explicit construction of large families of MSTD sets: Miller et al. [MOS] gave a family of MSTD subsets of $\{1, 2, \dots, r\}$ with density $\Theta(1/r^4)$[1], while Zhao [Zh1] gave a denser family with density $\Theta(1/r)$, the current record.

In [AMMS], the authors used a technique introduced by Zhao [Zh2] to show that the proportion of 2-decompositions (i.e., parititons into two sets) of $\{1, 2, \dots, r\}$ that gives two MSTD subsets is bounded below by a positive constant. This result is surprising in view of the conventional method of constructing MSTD sets, which is to fix a fringe pair (L, R) of two sets containing elements to be used in the fringe of the interval and argue that all the middle elements appear with some positive probability. (The fringe pair ensures that some of the largest and smallest differences are missed and that our set is MSTD.) However, the result in [AMMS] seems to suggest that we can find two (or more) disjoint fringe pairs (L_1, R_1) and (L_2, R_2) such that $L_1 \cup L_2$ and $R_1 \cup R_2$ cover a full set of left and right elements of $\{1, 2, \dots, r\}$ and $(L_1, R_1), (L_2, R_2)$ are two fringe pairs for two disjoint MSTD sets. Previous research has focused on each fringe pair independently, so it is interesting to see that two (or more) fringe pairs can complement each other nicely on both sides of $\{1, 2, \dots, r\}$. Motivated by that, we provide a method to construct these fringe pairs and study partitions of $\{1, 2, \dots, r\}$ into MSTD subsets more thoroughly.

[1][ILMZ2] showed that with slightly more work, the density is improved to $\Theta(1/r^2)$.

1.2. **Notation and Main Results.** Let $[a,b]$ denote $\{\ell \in \mathbb{Z} | a \leq \ell \leq b\}$ and I_r denote $[1,r]$. We use the idea of P_n sets described in [MOS]. A set A is said to be P_n if the following conditions are met. Let $a = \min A$ and $b = \max A$. Then,

$$A + A \supseteq [2a+n, 2b-n] \tag{1.1}$$

$$A - A \supseteq [(a-b)+n, (b-a)-n]. \tag{1.2}$$

A set A is P_n with respect to sums (SP_n) if condition (1.1) is satisfied, and P_n with respect to differences (DP_n) if condition (1.2) is satisfied. Next, let $[a,b]_2$ denote $\{\ell \in \mathbb{Z} | a \leq \ell \leq b$ and $\ell - a$ is even$\}$. Finally, a $2-$decomposition of a set S is $A_1 \cup A_2 = S$, where $A_1 \cap A_2 = \emptyset$. We use the word *decomposition* and *partition* interchangeably. Our main result is:

Theorem 1.1. *Let A_1 and A_2 be chosen such that both are MSTD and P_n for some $n \in \mathbb{N}$, and also*

(1) *(A_1, A_2) partition $[1, 2n]$,*

(2) *$A_i = L_i \cup R_i$ with $L_i \subseteq [1,n]$ and $R_i \subseteq [n+1, 2n]$ for $i = 1, 2$,*

(3) *$[1,4] \cup \{n\} \subseteq L_1$ and $\{n+1\} \cup [2n-3, 2n] \subseteq R_1$, and*

(4) *$[5,7] \subseteq L_2$, $[n+2, n+5] \subseteq L_2 + L_2$, $[2n-6, 2n-4] \subseteq R_2$ and $[3n-3, 3n] \subseteq R_2 + R_2$.*

(See Remark 1.2 for an example of such sets A_1 and A_2).

Pick $k \geq n/2 + 2$ and $m \in \mathbb{N}_0$. Set

$$\begin{aligned}
R_1' &= R_1 + m + 4k + 4, \\
R_2' &= R_2 + m + 4k + 4, \\
O_{11} &= \{n+4\} \cup [n+5, n+2k+1]_2 \cup \{n+2k+2\}, \\
O_{12} &= \{n+m+2k+3\} \cup [n+m+2k+4, n+m+4k]_2 \cup \{n+m+4k+1\}, \\
O_{21} &= [n+1, n+3] \cup [n+6, n+2k]_2 \cup [n+2k+3, n+2k+5], \\
O_{22} &= [n+m+2k, n+m+2k+2] \cup [n+m+2k+5, n+m+4k-1]_2 \\
&\qquad \cup [n+m+4k+2, n+m+4k+4].
\end{aligned}$$

Let $M_1 \subseteq [n+2k+6, n+m+2k-1]$ such that within M_1, there exists a sequence of pairs of consecutive elements, where consecutive pairs in the sequence are not more than $2k-1$ apart and the sequence starts with a pair in $[n+2k+6, n+4k+1]$ and ends with a pair in $[n+m+4, n+m+2k-1]$. Let $M_2 \subseteq [n+2k+6, n+m+2k-1]$ such that within M_2, there exists a sequence of triplets of consecutive elements, where consecutive triplets in the sequence are not more than $2k$ apart and the sequence starts with a triplet in $[n+2k+6, n+4k+5]$ and ends with a triplet in $[n+m, n+m+2k-1]$. Also, $M_1 \cap M_2 = \emptyset$ and $M_1 \cup M_2 = [n+2k+6, n+m+2k-1]$. Set $R_i' = R_i + m + 4k + 4$ for $i \in \{1,2\}$. Then

$$\begin{aligned}
A_1' &= L_1 \cup O_{11} \cup M_1 \cup O_{12} \cup R_1' \\
A_2' &= L_2 \cup O_{21} \cup M_2 \cup O_{22} \cup R_2'
\end{aligned}$$

are both MSTD and partition $[1, 2n + m + 4k + 4]$.

Remark 1.2. *To show that our family is not empty, we need to show the existence of at least one pair of* A_1 *and* A_2*. Note that our technique is similar to many other papers:* [He, MO, MOS, MPR, PW] *in the sense that we need a good fringe to start with. A random search yielded*

$$\begin{aligned} A_1 &= \{1, 2, 3, 4, 8, 9, 11, 13, 14, 15, 20, 21, 26, 27, 28, 31, 33, 37, 38, 39, 40\}, \\ A_2 &= \{5, 6, 7, 10, 12, 16, 17, 18, 19, 22, 23, 24, 25, 29, 30, 32, 34, 35, 36\}. \end{aligned}$$

We have

$$\begin{aligned} A_1 + A_1 &= [2, 80] \\ A_1 - A_1 &= [-39, 39]\backslash\{\pm 21\} \end{aligned} \quad \textit{and} \quad \begin{aligned} A_2 + A_2 &= [10, 72] \\ A_2 - A_2 &= [-31, 31]\backslash\{\pm 21\}. \end{aligned}$$

Clearly, both A_1 *and* A_2 *are* P_{20}*. It can be easily checked that all conditions mentioned in Theorem 1.1 are satisfied. These pairs of sets* A_1 *and* A_2 *are not hard for computers to find: for* $n = 20$*, computer search shows that there are about 48 such pairs.*

Remark 1.3. *Our method of decomposing* $[1, r]$ *into two MSTD sets allows a lot of freedom in choosing the middle elements. This is because once the fringe elements are chosen, the conditions placed on* M_1 *and* M_2 *are relatively weak.*

Next, we answer positively question (3) in [AMMS], where the authors ask: Can we decompose $\{1, 2, \ldots, r\}$ into three sets which are MSTD? For any finite number k, is there a sufficiently large r for which there is a k-decomposition into MSTD sets?

Theorem 1.4. *Let* $k \in \mathbb{N}_{\geq 2}$ *be chosen. There exists the smallest* $R \in \mathbb{N}$ *such that for all* $r \geq R$*,* $[1, r]$ *can be* k*-decomposed into MSTD subsets, while* I_{R-1} *cannot be* k*-decomposed into MSTD subsets.*

In particular, we find some rough bounds[2]*:*

(1) *when* k *is even,* $8k \leq R \leq 10k$*,*
(2) *when* $k \geq 5$ *odd,* $8k \leq R \leq 20k - 14$*, and*
(3) *when* $k = 3$*,* $24 \leq R \leq 24T + 24$*,*

where $T = \min\{\max A : |A + A| - |A - A| \geq 10|A|.\}$.

We prove Theorem 1.4 using sets constructed by the base expansion method[3] that helps generate an infinite family of MSTD sets from a given MSTD sets. The method is a very powerful tool and has been used extensively in literature including [He],[ILMZ1] and [ILMZ2]. However, the base expansion method turns out to be inefficient in terms of our MSTD sets' cardinality. Hence, we present a second, more efficient approach by using a particular family of MSTD sets. We present both proofs since they are of independent interest: the first proof is less technical but less efficient. Also, the second proof cannot resolve the case $k = 3$ while the first can.

[2]We make no attempt to optimize these bounds. Finer analysis may give us better bounds.

[3]We can generate an infinite family of MSTD sets from a given MSTD set through the base expansion method. Let A be an MSTD set, and let $A_{k,m} = \{\sum_{i=1}^{k} a_i m^{i-1} : a_i \in A\}$. If m is sufficiently large, then $|A_{k,m} \pm A_{k,m}| = |A \pm A|^k$ and $|A_{k,m}| = |A|^k$.

Lastly, we give a sufficient condition on when there exists a positive constant lower bound for the proportion of k-decompositions of $[1, r]$ into MSTD subsets. The condition offers an alternative proof of Theorem 1.4 in [AMMS] ($k = 2$). Due to the condition, we make the following conjecture.

Conjecture 1.5. *For any finite $k \geq 2$, the proportion of k-decompositions into MSTD subsets is bounded below by a positive constant.*

The outline of the paper is as follows. In section 2, we provide an efficient method to decompose $[1, r]$ into two MSTD subsets; section 3 presents two method to decompose $[1, r]$ into $k \geq 3$ MSTD subsets; Appendix B is devoted to establishing the bounds mentioned in Theorem 1.4 and the sufficient condition for a positive constant lower bound of the proportion of $k-$decompositions into MSTD subsets in Appendix A. Appendix D contains many examples illustrating our lemmas and theorems.

2. Explicit 2-decomposition into MSTD Subsets

In this section, we show how we can decompose $[1, r]$ into two MSTD subsets. We believe that the method can be applied to the case of k-decompositions, but the proof will be much more technical. However, for $k \geq 4$, we have a way to decompose $[1, r]$ into k MSTD subsets by simply using $2-$decompositions, which will be discussed later.

2.1. Explicit Construction of Infinite Families of MSTD sets.

The following lemma is useful in proving many of our results.

Lemma 2.1. *Let $A = L \cup R$ be an MSTD, P_n set where $L \subseteq [1, n]$ and $R \subseteq [n+1, 2n]$. Form $A' = L \cup M \cup R'$, where $M \subseteq [n+1, n+m]$ and $R' = R + m$ for some $m \in \mathbb{N}_0$. If A' is a SP_n set, then A' is MSTD.*

Proof. We prove that A' is MSTD by showing that the increase in the number of differences is at most the increase in the number of sums. As shown in the proof of Lemma 2.1 in [MOS], the number of new added sums is $2m$. Because $R' = R + m$, all differences in $[-(2n + m - 1), -(n + m)]$ can be paired up with differences in $[1 - 2n, -n]$ from $L - R$ and differences in $[n + m, 2n + m - 1]$ can be paired up with differences in $[n, 2n - 1]$ from $R - L$. Because the set A is P_n, A contains all differences in $[-n + 1, n - 1]$. In the worst scenario (in terms of the increase in the number of differences), $A' - A'$ contains all differences in $[-(n + m) + 1, (n + m) - 1]$. So, at most $|A' - A'| - |A - A| = |[-(n + m) + 1, (n + m) - 1]| - |[-n + 1, n - 1]| = 2m$. This completes our proof. □

Lemma 2.2. *Let an MSTD, P_n set A be chosen, where $A = L \cup R$ for $L \subseteq [1, n]$ and $R \subseteq [n + 1, 2n]$. Additionally, L and R must satisfy the following conditions: $[1, 4] \cup \{n\} \subseteq L$ and $\{n+1\} \cup [2n-3, 2n] \subseteq R$. Pick $k \geq n/2+2$ and $m \in \mathbb{N}_0$. Form*

$$O_1 = \{n + 4\} \cup [n + 5, n + 2k + 1]_2 \cup \{n + 2k + 2\}$$
$$O_2 = \{n + m + 2k + 3\} \cup [n + m + 2k + 4, n + m + 4k]_2 \cup \{n + m + 4k + 1\}.$$

Let $M \subseteq [n + 2k + 3, n + m + 2k + 2]$ be such that within M, there exists a sequence of pairs of consecutive elements, where consecutive pairs in the sequence are not more than $2k - 1$ apart and the sequence starts with a pair in $[n + 2k + 3, n + 4k + 1]$ and

ends with a pair in $[n+m+4, n+m+2k+2]$. *Denote* $A' = L \cup O_1 \cup M \cup O_2 \cup R'$, *where* $R' = R+m+4k+4$. *Then* A' *is MSTD.*

Proof. We know that $A' \subseteq [2, 4n+2m+8k+8]$. To prove that A' is MSTD, it suffices to prove that A' is $S-P_n$. In particular, we want to show that $[n+2, 3n+2m+8k+8] \subseteq A'+A'$. Due to symmetry[4], it suffices to show that $[n+2, 2n+m+4k+5] \subseteq A'+A'$. We have:

$$\begin{aligned} [n+2, n+4] &\subseteq A'+A' \quad (\text{because } 2,3,4,n \in A') \\ (1+O_1) \cup (2+O_1) &= [n+5, n+2k+4] \\ O_1 + O_1 &= [2n+8, 2n+4k+4]. \end{aligned}$$

Since $n+2k+4 \geq 2n+8$, $[n+2, 2n+4k+4] \subseteq A'+A'$. Consider $M+O_1$. In the worst scenario (in terms of getting necessary sums), the two smallest elements of M are $n+4k$ and $n+4k+1$, while the two largest elements of M are $n+m+4$ and $n+m+5$. So, $M+O_1 \supseteq [2n+4k+4, 2n+m+2k+7]$. We complete the proof by showing that $[2n+m+2k+8, 2n+m+4k+5] \subseteq A'+A'$. We have: $((n+4)+O_2) \cup ((n+5)+O_2) = [2n+m+2k+7, 2n+m+4k+6]$. So, A' is $S-P_n$ and thus, MSTD by Lemma 1.1. □

Lemma 2.3. *Let an MSTD,* P_n *set* A *be chosen, where* $A = L \cup R$ *for* $L \subseteq [1,n]$ *and* $R \subseteq [n+1, 2n]$. *Additionally,* L *and* R *must satisfy the following conditions:* $[5,7] \subseteq L, [n+2, n+5] \subseteq L+L, [2n-6, 2n-4] \subseteq R$ *and* $[3n-3, 3n] \subseteq R+R$. *Pick* $k \geq \lfloor n/2 \rfloor$ *and* $m \in \mathbb{N}_0$. *Form*

$$\begin{aligned} O_1 &= [n+1, n+3] \cup [n+6, n+2k]_2 \cup [n+2k+3, n+2k+5], \\ O_2 &= [n+m+2k, n+m+2k+2] \cup [n+m+2k+5, n+m+4k-1]_2 \\ &\quad \cup [n+m+4k+2, n+m+4k+4]. \end{aligned}$$

Let $M \subseteq [n+2k+6, n+m+2k-1]$ *such that within* M, *there exists a sequence of triplets of consecutive elements, where consecutive triplets in the sequence are not more than* $2k$ *apart and the sequence starts with a triplet in* $[n+2k+6, n+4k+5]$ *and ends with a triplet in* $[n+m, n+m+2k-1]$. *Denote* $A' = L \cup O_1 \cup M \cup O_2 \cup R'$, *where* $R' = R+m+4k+4$. *Then* A' *is MSTD.*

Proof. Similar as above, it suffices to prove that $[n+2, 2n+m+4k+5] \subseteq A'+A'$.[5] We have

$$\begin{aligned} [n+2, n+5] &\subseteq L+L \\ (5+O_1) \cup (6+O_1) \cup (7+O_1) &= [n+6, n+2k+12] \\ O_1 + O_1 &= [2n+2, 2n+4k+10]. \end{aligned}$$

[4]Due to symmetry, A' has the same structure as $2n+m+4k+5-A'$. If $[n+2, 2n+m+4k+5] \subseteq A'+A'$, then $[n+2, 2n+m+4k+5] \subseteq (2n+m+4k+5-A')+(2n+m+4k+5-A') = 4n+2m+8k+10-(A'+A')$ and so, $[2n+m+4k+5, 3n+2m+8k+8] \subseteq A'+A'$.

[5]Due to symmetry, A' has the same structure as $2n+m+4k+5-A'$. If $[n+2, 2n+m+4k+5] \subseteq A'+A'$, then $[n+2, 2n+m+4k+5] \subseteq (2n+m+4k+5-A')+(2n+m+4k+5-A') = 4n+2m+8k+10-(A'+A')$ and so, $[2n+m+4k+5, 3n+2m+8k+8] \subseteq A'+A'$

Because $n+2k+12 \geq 2n+2$, $A'+A'$ contains $[n+2, 2n+4k+10]$. Consider $M+O_1$. In the worst scenario (in terms of getting sums), the smallest elements in M are $n+4k+3, n+4k+4$ and $n+4k+5$, while the largest elements in M are $n+m, n+m+1$ and $n+m+2$. Then $M+O_1 \supseteq [2n+4k+4, 2n+m+2k+7]$. We only need to show that $[2n+m+2k+8, 2n+m+4k+5] \subseteq A'+A'$. We have: $((n+1)+O_2)\cup((n+2)+O_2)\cup((n+3)+O_2) = [2n+m+2k+1, 2n+m+4k+7]$. This completes our proof that A' is MSTD. □

Remark 2.4. *In order that our families of MSTD sets are nonempty, we must first find such a set A satisfying the conditions mentioned in each of the lemmas. For Lemma 2.2, an example of our set A is $A_1 = \{1,2,3,4,7,10,12,13,14,15,20,21,23,24\}$. We have*

$$\begin{aligned} A_1+A_1 &= [2,48], \\ A_1-A_1 &= [-23,23]\backslash\{\pm 15\}. \end{aligned}$$

Clearly, A_1 is P_{12} and A_1 satisfies all conditions required by Lemma 2.2. For Lemma 2.3, an example of our set A is $A_2 = \{5,6,7,9,11,16,17,18,19,22,23,24,25,29,31,33,34,35,36\}$. We have:

$$\begin{aligned} A_2+A_2 &= [10,72]\backslash\{19\}, \\ A_2-A_2 &= [-31,31]\backslash\{\pm 21\}. \end{aligned}$$

Clearly, A_2 is P_{20} and A_2 satisfies all conditions required by Lemma 2.3.

2.2. **Explicit Partitions into Two MSTD Sets.** We are now ready to prove Theorem 1.1. The proof follows from Lemmas 2.2 and 2.3.

Proof. As indicated in Remark 1.2, there exists pairs of sets A_1 and A_2 such that all conditions in Theorem 1.1 are satisfied. Pick $k \geq n/2+2$ and $m \in \mathbb{N}_0$. Set $R_i' = R_i + m + 4k + 4$ for $i = 1, 2$. Form

$$\begin{aligned} O_{11} &= \{n+4\} \cup [n+5, n+2k+1]_2 \cup \{n+2k+2\}, \\ O_{12} &= \{n+m+2k+3\} \cup [n+m+2k+4, n+m+4k]_2 \cup \{n+m+4k+1\}, \\ O_{21} &= [n+1, n+3] \cup [n+6, n+2k]_2 \cup [n+2k+3, n+2k+5], \\ O_{22} &= [n+m+2k, n+m+2k+2] \cup [n+m+2k+5, n+m+4k-1]_2 \\ &\qquad \cup [n+m+4k+2, n+m+4k+4]. \end{aligned}$$

We see that $O_{11}\cup O_{21} = [n+1, n+2k+5]$ and $O_{12}\cup O_{22} = [n+m+2k, n+m+4k+4]$. By Lemma 2.2 and Lemma 2.3, we know that $A_1' = L_1 \cup O_{11} \cup M_1 \cup O_{12} \cup R_1'$ and $A_2' = L_2 \cup O_{21} \cup M_2 \cup O_{22} \cup R_2'$ are MSTD sets and (A_1', A_2') partition $[1, 2n+m+4k+4]$, given that three following conditions are satisfied:

(1) $M_1 \subseteq [n+2k+6, n+m+2k-1]$ such that within M_1, there exists a sequence of pairs of consecutive elements, where consecutive pairs in the sequence are not more than $2k-1$ apart and the sequence starts with a pair in $[n+2k+6, n+4k+1]$ and ends with a pair in $[n+m+4, n+m+2k-1]$,
(2) $M_2 \subseteq [n+2k+6, n+m+2k-1]$ such that within M_2, there exists a sequence of triplets of consecutive elements, where consecutive triplets in the sequence are not more than $2k$ apart and the sequence starts with a triplet in $[n+2k+6, n+4k+5]$ and ends with a triplet in $[n+m, n+m+2k-1]$, and

(3) $M_1 \cup M_2 = [n+2k+6, n+m+2k-1]$ and $M_1 \cap M_2 = \emptyset$.

This completes the proof of the theorem. □

Remark 2.5. *Observe that our fringe pairs in this case are* $(L_1 \cup O_{11}, R_1' \cup O_{12})$ *and* $(L_2 \cup O_{21}, R_2' \cup O_{22})$. *Though disjoint, the union of the two fringe pairs gives us a full set of left and right elements of* $[1, 2n+m+4k+4]$ *and each is a fringe pair for a MSTD set.*

3. Explicit k-decomposition into MSTD Subsets

3.1. **Overview.** Theorem 1.1 gives us a way to partition $[1, r]$ into two MSTD subsets. Due to linear transformations, we can partition any (long enough) arithmetic progressions into two MSTD subsets. If we can find an MSTD subset S of $[1, r]$ such that $[1, r] \backslash S$ is a union of k arithmetic progressions $\cup_{j=1}^{k} I_j$, then we can partition $[1, r]$ into $1 + 2k$ MSTD subsets (because each I_j can be partitioned into two MSTD subsets). This is the central idea in both methods we use to k-decompose I_r into MSTD subsets presented later.

3.2. **Base Expansion Method ($k \geq 3$).** We explicitly provide a way to k-decompose $[1, r]$ into MSTD subsets. First, we need to define a "strong MSTD" set. We call a set S a $10-$strong MSTD set if $|S+S| - |S-S| \geq 10|S|$.[6] This type of MSTD set does exist. For example, using the base expansion method, we can construct them. Using $\tilde{S} = \{0, 2, 3, 4, 7, 11, 12, 14\}$, by the method, we can construct S such that $|S| = |\tilde{S}|^2 = 8^4 = 4096$, $|S+S| = 26^4 = 456976$ and $|S-S| = 25^4 = 390625$; then, $|S+S| - |S-S| > 10|S|$.

Lemma 3.1. *If* S *is a* $10-$*strong MSTD set, then* $S \cup \{a_1, a_2, a_3, a_4\}$, *where* $a_4 > a_3 > a_2 > a_1 > \max S$, *is a MSTD set. Similarly,* $S \cup \{b_1, b_2, b_3, b_4\}$, *where* $b_1 < b_2 < b_3 < b_4 < \min S$, *is also a MSTD set.*

Proof. We want to show that $S \cup \{a_1, a_2, a_3, a_4\}$ is MSTD. Adding one more element to a set S produces at least 0 new sums and at most $2|S|$ new differences. So, $|(S \cup \{a_1\}) + (S \cup \{a_1\})| - |(S \cup \{a_1\}) - (S \cup \{a_1\})| \geq 10|S| + 0 - 2|S| = 8|S|$. So, $S \cup \{a_1\}$ is MSTD. Define $S_1 = S \cup \{a_1\}$ with $|S_1| = |S| + 1$. Similarly, $|(S_1 \cup \{a_2\}) + (S_1 \cup \{a_2\})| - |(S_1 \cup \{a_2\}) - (S_1 \cup \{a_2\})| \geq 8|S| + 0 - 2|S_1| = 8|S| - 2(|S| + 1) = 6|S| - 2$. Again, $S_1 \cup \{a_2\}$ is MSTD. Repeating this argument, we can show that $S_4 = S \cup \{a_1, a_2, a_3, a_4\}$ is a MSTD set. The proof is similar for $S \cup \{b_1, b_2, b_3, b_4\}$. □

Remark 3.2. *The following fact will be useful later: An arithmetic progression of integers (assumed long enough) can contain an arbitrarily large number of disjoint* $10-$*strong MSTD sets.*

With the above remark, we are ready to prove the following.

Lemma 3.3. *There exists* $N \in \mathbb{N}$ *such that for all* $r \geq N$, $I_r = [1, r]$ *can be partitioned into exactly three MSTD subsets.*

[6] We pick the number 10 just to be safe for our later arguments. We make no attempt to provide an efficient way to decompose I_r into k MSTD subsets.

Proof. We use a pair of fringe elements described in [MO]: $L = \{1, 3, 4, 8, 9, 10, 11\}$ and $R = \{r-10, r-9, r-8, r-7, r-5, r-2, r-1, r\}$. We see that $I_r \backslash (L \cup R) = \{2, 5, 6, 7\} \cup [12, r-11] \cup \{r-3, r-4, r-6\}$. We have

$$\begin{aligned} L + L &= [2, 22] \setminus \{3\} \\ L + R &= [r-9, r+11] \\ R + R &= [2r-20, 2r]. \end{aligned}$$

Consider $K = \{\ell \,|\, 12 \leq \ell \leq r-11, \ell \text{ is even}\} \cup \{r-11\}$. We have $(\{11\} \cup K \cup \{r-10\}) + (\{11\} \cup K \cup \{r-10\}) = [22, 2r-20]$. So, $(L \cup K \cup R) + (L \cup K \cup R) = [2, 2r] \backslash \{3\}$. Because $\pm(R-L)$ lacks $\pm(r-7)$, $L \cup K \cup R$ is a MSTD set. It is not hard to see that adding numbers in $[12, r-11] \backslash K$ to $L \cup K \cup R$ still gives a MSTD set.

Now, $[12, r-11] \backslash K$ contains an arithmetic progression of consecutive odd integers. We can make this arithmetic progression arbitrarily large by increasing r. By Remark 3.2, this arithmetic progression can contain two disjoint $10-$strong MSTD sets, called S_1 and S_2. We write $[12, r-11] \backslash K = S_1 \cup S_2 \cup M$. By Lemma 3.1, $S_1^* = S_1 \cup \{2, 5, 6, 7\}$ and $S_2^* = S_2 \cup \{r-6, r-4, r-3\}$ are both MSTD. By what we say above, $K^* = M \cup L \cup K \cup R$ is also MSTD. Because $S_1^* \cup S_2^* \cup K^* = I_r$, we have completed the proof. □

Proof of Theorem 1.4. Let $k \geq 2$ be chosen. Write $k = 2m_1 + 3m_2$ for some m_1 and $m_2 \in \mathbb{N}_0$. We can find $N \in \mathbb{N}$ such that $I_N = [1, N] = \big([1, k_1] \cup [k_1+1, k_2] \cup \cdots \cup [k_{m_1-1}+1, k_{m_1}]\big) \cup \big([k_{m_1}+1, k_{m_1+1}] \cup [k_{m_1+1}+1, k_{m_1+2}] \cup \cdots \cup [k_{m_1+m_2-1}+1, N]\big)$. Each of the first m_1 intervals are large enough to be partitioned into two MSTD sets while the next m_2 intervals are large enough to be partitioned into three MSTD sets. So, I_N can be partitioned in exactly k MSTD sets. This completes our proof. □

3.3. **Efficient Methods** $(k \geq 4)$**.**

3.3.1. *Notations and Preliminary Results.* We introduce a notation to write a set; this notation was first used by Spohn [Sp]. Given a set $S = \{a_1, a_2, \ldots, a_n\}$, we arrange its elements in increasing order and find the differences between two consecutive numbers to form a sequence. Suppose that $a_1 < a_2 < \cdots < a_n$, then our sequence is $a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots, a_n - a_{n-1}$. Then we represent

$$S = (a_1 | a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots, a_n - a_{n-1}).$$

Take $S = \{3, 2, 5, 10, 9\}$, for example. We arrange the elements in the increasing order to have $2, 3, 5, 9, 10$ and form a sequence by looking at the difference between two consecutive numbers: $1, 2, 4, 1$. So, we write $S = (2|1, 2, 4, 1)$. All information about a set is preserved in this notation.

Lemma 3.4. *The following are also MSTD sets for a given $m \in \mathbb{N}$:*

$$(1|1,1,2,1,\underbrace{4\ldots,4}_{m\text{-times}},3,1,1,2),$$
$$(1|1,1,2,1,\underbrace{4\ldots,4}_{m\text{-times}},3,1,1,2,1),$$
$$(1|1,1,2,1,\underbrace{4\ldots,4}_{m\text{-times}},3,1,1).$$

We present the proof in Appendix C.

3.3.2. *Efficient Methods ($k \geq 4$).* In our decomposition of I_r into k MSTD subsets for $k \geq 3$, we use the base expansion method. However, the base expansion method is inefficient in terms of cardinalities of our sets. Is there a more efficient way to decompose? In answering this question, we present a method of decomposing I_r into k MSTD subsets ($k \geq 4$) that helps reduce the cardinalities of sets. We use the infinite family of MSTD sets in Lemma 3.4 to achieve this.

We want to decompose I_r for sufficiently large r into k ($k \geq 4$) MSTD subsets. If k is even, we can simply write I_r as the union of $k/2$ arithmetic progressions, each of which, by Theorem 1.1, can be decomposed into two MSTD subsets in an efficient way. If $k \geq 5$ is odd, then we consider $r \bmod 4$. If $r \equiv 1 \bmod 4$, write $r = 4m + 13$ for some $m \in \mathbb{N}$ and consider $(1|1,1,2,1,\underbrace{4\ldots,4}_{m\text{-times}},3,1,1,2)$. We have

$$I_r\backslash\{1,2,3,5,6,10,14,18,22,26,\ldots,6+4m,9+4m,10+4m,11+4m,13+4m\}$$
$$=\{4,8,12,16,20,\ldots,8+4m,12+4m\}\cup\{7,9,11,13,\ldots,7+4m\}.$$

Notice that both $\{4,8,12,16,20,\ldots,8+4m,12+4m\}$ and $\{7,9,11,13,\ldots,7+4m\}$ are arithmetic progressions and each of these sets can be decomposed into an even number of MSTD sets. So, our original sets $I_r = [1, 13 + 4m]$ can be decomposed into exactly k MSTD sets. If $r \equiv 2 \bmod 4$, write $r = 4m+14$ and consider $(1|1,1,2,1,\underbrace{4\ldots,4}_{m\text{-times}},3,1,1,2,1)$. If $r \equiv 3 \bmod 4$, write $r = 4m+11$ and consider $(1|1,1,2,1,\underbrace{4\ldots,4}_{m\text{-times}},3,1,1)$. If $r \equiv 0 \bmod 4$, write $r = 4m + 12$ and consider $(2|1,1,2,1,\underbrace{4\ldots,4}_{m\text{-times}},3,1,1)$. Using the same argument as above, we can show that I_r can be decomposed into exactly k MSTD sets. We prove the upper and lower bounds for R in Appendix B.

4. Future Work

We end with several additional questions to pursue.

(1) In [AMMS], the authors show that there is a positive constant lower bound for the percentage of decompositions into two MSTD sets? Is there a positive constant lower bound for the percentage of decompositions into k MSTD sets for $k \geq 3$? In other words, is Conjecture 1.5 true? A method is to find a family of sets $(A_i)_{i=1}^k$ that satisfies the condition in Theorem A.4.

(2) Is there a method of k-decomposition that is of high density, for example $\Theta(1/r^c)$ for small c?
(3) For the $3-$decomposition, we use the base expansion method, which is inefficient. Can we find an efficient way to decompose $[1, r]$ into three MSTD subsets.
(4) Can we find some better bounds for R in Theorem 1.4? There is a yawning gap between our upper and lower bounds.
(5) Suppose that I_r can be decomposed into k MSTD subsets. Can we conclude that I_{r+1} can be decomposed into k MSTD subsets?

Appendix A. Sufficient Condition on Constant Lower Bound

Lemma A.1. *Consider $S \subseteq \{0, 1, \dots, r-1\}$ and $S = L \cup M \cup R$. Fix $L \subseteq [0, \ell - 1]$ and $R \subseteq [r - \ell, r - 1]$ for some fixed ℓ. Let M be a uniformly randomly chosen subset of $[\ell, r - \ell - 1]$. Then for any $\varepsilon > 0$, there exists sufficiently large r such that*

$$\mathbb{P}([2\ell - 1, 2r - 2\ell - 1] \subseteq S + S) \ \geq \ 1 - 6(2^{-|L|} + 2^{-|R|}) - \varepsilon. \tag{A.1}$$

Proof. We write

$$\begin{aligned}
&\mathbb{P}([2\ell - 1, 2r - 2\ell - 1] \subseteq S + S) \ = \ 1 - \mathbb{P}([2\ell - 1, 2r - 2\ell - 1] \not\subseteq S + S) \\
&= \ 1 - \mathbb{P}([2\ell - 1, r - \ell - 1] \cup [r + \ell - 1, 2r - 2\ell - 1] \not\subseteq S + S \\
&\quad \text{or } \mathbb{P}([r - \ell, r + \ell - 2] \not\subseteq S + S) \\
&\geq \ 1 - \mathbb{P}([2\ell - 1, r - \ell - 1] \cup [r + \ell - 1, 2r - 2\ell - 1] \not\subseteq S + S \\
&\quad - \mathbb{P}([r - \ell, r + \ell - 2] \not\subseteq S + S).
\end{aligned} \tag{A.2}$$

By Proposition 8 in [MO],

$$\mathbb{P}([2\ell - 1, r - \ell - 1] \cup [r + \ell - 1, 2r - 2\ell - 1] \not\subseteq S + S) \ \leq \ 6(2^{-|L|} + 2^{-|R|}). \tag{A.3}$$

We find a upper bound for

$$\mathbb{P}([r - \ell, r + \ell - 2] \not\subseteq S + S) \ \leq \ \mathbb{P}([r - \ell, r + \ell - 2] \not\subseteq M + M). \tag{A.4}$$

because $[r-2\ell, r-2] \not\subseteq S+S$ implies $[r-\ell, r+\ell-2] \not\subseteq M+M$. By a linear shift of ℓ, we can consider M a subset of $[0, r-2\ell-1]$ and $\mathbb{P}([r-\ell, r+\ell-2] \not\subseteq M+M)$ turns into $\mathbb{P}([r - 2\ell, r - 2] \not\subseteq M + M)$. Use the change of variable $N = r - 2\ell$. We have: $M \subseteq [0, N-1]$ and we estimate: $\mathbb{P}([r - 2\ell, r - 2] \not\subseteq M + M) = \mathbb{P}([N, N + 2\ell - 2] \not\subseteq M + M) \leq \sum_{k=N}^{N+2\ell-2} \mathbb{P}(k \notin M + M)$. Lemma 7 in [MO] shows that the last quantity tend to 0 as N goes to infinity. So, for any $\varepsilon > 0$, there exists sufficiently large r such that $\mathbb{P}([r - \ell, r + \ell - 2] \not\subseteq S + S) < \varepsilon$. This completes our proof. □

Lemma A.2. *Consider $S \subseteq \{0, 1, \dots, r-1\}$ and $S = L \cup M \cup R$. Fix $L \subseteq [0, \ell - 1]$ and $R \subseteq [r - \ell, r - 1]$ for some fixed ℓ. Let M be a uniformly randomly chosen subset of $[\ell, r - \ell - 1]$. Let a denote the smallest integer such that both $[\ell, 2\ell - a] \subseteq L + L$ and $[2r - 2\ell + a - 2, 2r - \ell - 2] \subseteq R + R$. Then, for all $\varepsilon > 0$, there exists sufficiently large r such that*

$$\begin{aligned}
&\mathbb{P}([2\ell - a + 1, 2r - 2\ell + a - 3] \subseteq S + S) \\
&\quad \geq \ 1 - (a-2)(2^{-\tau(R)} + 2^{-\tau(L)}) - 6(2^{-|L|} + 2^{-|R|}) - \varepsilon,
\end{aligned} \tag{A.5}$$

where $\tau(L) = \{i \in L | i \leq \ell - a + 1\}$ and $\tau(R) = \{i \in R | i \geq r - \ell + a - 2\}$.

Proof. We have:

$$\begin{aligned}
&\mathbb{P}([2\ell-a+1,2r-2\ell+a-3]\subseteq S+S)\\
&=\ \mathbb{P}([2\ell-1,2r-2\ell-1]\subseteq S+S \text{ and } [2\ell-a+1,2\ell-2]\subseteq S+S\\
&\qquad\qquad \text{and } [2r-2\ell,2r-2\ell+a-3]\subseteq S+S)\\
&=\ 1-\mathbb{P}([2\ell-1,2r-2\ell-1]\not\subseteq S+S \text{ or } [2\ell-a+1,2\ell-2]\not\subseteq S+S\\
&\qquad\qquad \text{or } [2r-2\ell,2r-2\ell+a-3]\not\subseteq S+S)\\
&\geq\ 1-\mathbb{P}([2\ell-a+1,2\ell-2]\not\subseteq S+S)-\mathbb{P}([2r-2\ell,2r-2\ell+a-3]\not\subseteq S+S)\\
&\quad -\mathbb{P}([2\ell-1,2r-2\ell-1]\not\subseteq S+S).
\end{aligned}$$

By Lemma A.1, $\mathbb{P}([2\ell-1,2r-2\ell-1]\not\subseteq S+S)\leq 6(2^{-|L|}+2^{-|R|})+\varepsilon$. We have

$$\mathbb{P}([2\ell-a+1,2\ell-2]\not\subseteq S+S)\ \leq\ \sum_{k=2\ell-a+1}^{2\ell-2}\mathbb{P}(k\notin S+S). \tag{A.6}$$

Let $\tau(L)=\{i\in L|i\leq \ell-a+1\}$ and $\tau(R)=\{i\in R|i\geq r-\ell+a-2\}$. For each value of k in $[2\ell-a+1,2\ell-2]$, in order that $k\notin S+S$, all pairs of numbers that sum up to k must not be both in S. Take $k=2\ell-a+1$, for example. For a number $x\leq \ell-a+1$, the number y that when added to x gives $2\ell-a+1$ is at least ℓ and $y\notin S$. So,

$$\mathbb{P}([2\ell-a+1,2\ell-2]\not\subseteq S+S)\ \leq\ (a-2)2^{-\tau(L)} \tag{A.7}$$

$$\mathbb{P}([2r-2\ell,2r-2\ell+a-3]\not\subseteq S+S)\ \leq\ (a-2)2^{-\tau(R)}. \tag{A.8}$$

We have shown that

$$\begin{aligned}
&\mathbb{P}([2\ell-a+1,2r-2\ell+a-3]\subseteq S+S)\\
&\qquad\qquad\geq\ 1-(a-2)(2^{-\tau(R)}+2^{-\tau(L)})-6(2^{-|L|}+2^{-|R|}))-\varepsilon.
\end{aligned}$$

This completes our proof. □

Corollary A.3. *Let A be an MSTD, P_n set such that $A=L\cup R$, where $L\subseteq[0,n-1]$ and $R\subseteq[n,2n-1]$. Let $m\in\mathbb{N}$ be chosen. Set $R'=R+m\subseteq[n+m,2n+m-1]$. Let $M\subseteq[n,n+m-1]$ be randomly chosen. Let a be the smallest integer such that $[n,2n-a]\subseteq L+L$ and $[2n+a-2,3n-2]\subseteq R+R$. Then for all $\varepsilon>0$, there exists sufficiently large m such that*

$$\begin{aligned}
&\mathbb{P}([2n-a+1,2n+2m+a-3]\not\subseteq S+S)\\
&\quad\leq\ (a-2)(2^{-\tau(R)}+2^{-\tau(L)})+6(2^{-|L|}+2^{-|R|})+\varepsilon,
\end{aligned} \tag{A.9}$$

where $\tau(L)=\{i\in L|i\leq n-a+1\}$ and $\tau(R)=\{i\in R|i\geq n+a-2\}$.

Proof. The corollary follows immediately by setting $r=2n+m$ and $\ell=n$ in the theorem. Also, notice that R' is a linear shift of R. □

For conciseness, we denote

$$f(L,R)\ =\ (a-2)(2^{-\tau(R)}+2^{-\tau(L)})+6(2^{-|L|}+2^{-|R|})+\varepsilon.$$

Theorem A.4. *Suppose that there exist sets $(A_i)_{i=1}^k$, which are pairwise disjoint, MSTD, P_n and $\cup(A_i)_{i=1}^k = [0, 2n-1]$. In particular, each $A_i = L_i \cup R_i$, where $L_i \subseteq [0, n-1]$ and $R_i \subseteq [n, 2n-1]$. Let $m \in \mathbb{N}_0$ be chosen. Form $R_i' = R_i + m$ and $(M_i)_{i=1}^k \subseteq [n, n+m-1]$ such that $(M_i)_{i=1}^k$ are pairwise disjoint and $\cup_{i=1}^k M_i = [n, n+m-1]$. Then for a positive percentage of the time, all $S_i = L_i \cup M_i \cup R_i$ are MSTD given the following condition:*

$$1 - \sum_{i=1}^{k} f(L_i, R_i) \ > \ 0, \tag{A.10}$$

for m sufficiently large. In other words, there exists a positive constant lower bound for the proportion of k-decompositions into MSTD subsets as $m \to \infty$.

Proof. Let a_i be the corresponding a value (defined in Lemma A.2) for L_i and R_i. By Corollary A.3, the probability

$$\begin{aligned} &\mathbb{P}(\forall i, S_i \text{ is MSTD}) \geq \mathbb{P}(\forall i, S_i + S_i \supseteq [2n - a_i + 1, 2n + 2m + a_i - 3]) \\ &\geq \ 1 - \mathbb{P}(\exists i, S_i + S_i \not\supseteq [2n - a_i + 1, 2n + 2m + a_i - 3]) \\ &\geq \ 1 - \sum_{i=1}^{k} f(L_i, R_i) > 0. \end{aligned} \tag{A.11}$$

The first inequality is because $[2n - a_i + 1, 2n + 2m + a_i - 3] \subseteq S_i + S_i$ guarantees that S_i is SP_n and thus, MSTD. By Lemma 2.1, S_i is MSTD. □

Now, we prove Theorem 1.4 in [AMMS] easily.

Corollary A.5. *There exists a constant $c > 0$ such that the percentage of 2-decompositions of $[0, r-1]$ into two MSTD subsets is at least c.*

Proof. Let

$$\begin{aligned} L_1 &= \{0, 1, 2, 3, 7, 8, 10, 12, 13, 14, 19\}, \\ R_1 &= \{20, 25, 26, 27, 30, 32, 36, 37, 38, 39\}, \\ L_2 &= \{4, 5, 6, 9, 11, 15, 16, 17, 18\}, \\ R_2 &= \{21, 22, 23, 24, 28, 29, 31, 33, 34, 35\}. \end{aligned}$$

Notice that $n = 20$. We find that $a_1 = 12$ and $a_2 = 4$. From that we calculate $\tau(L_1) = 6, \tau(R_1) = 6, \tau(L_2) = 8$ and $\tau(R_2) = 9$. So, $f(L_1, R_1)$ is less than 0.33 for m sufficiently large, while $f(L_2, R_2)$ is less than 0.03 for m sufficiently large. By Theorem A.4, we are done. □

Appendix B. Lower and Upper Bounds for R in Theorem 1.4

Given $k \geq 2$, the lower bound is obvious since by [He], the smallest cardinality of an MSTD sets is 8. In [AMMS], it is shown that for all $r \geq 20$, I_r can be partitioned into two MSTD subsets. To decompose $I_r = [1, r]$ into $k \geq 2$ (even) MSTD subsets, we write I_r to be the union of $k/2$ arithmetic progressions and require each to be of length at least 20. So, for all $r \geq 10k$, I_r can be decomposed into k MSTD subsets. Hence, $R \leq 10k$.

In our method to decompose $I_r = [1, r]$ into $k \geq 5$ (odd) MSTD subsets, we use particular MSTD sets, which are

$$\begin{aligned}
A_1 &= (1|1, 1, 2, 1, \underbrace{4 \ldots, 4}_{m\text{-times}}, 3, 1, 1, 2), \\
A_2 &= (1|1, 1, 2, 1, \underbrace{4 \ldots, 4}_{m\text{-times}}, 3, 1, 1, 2, 1), \\
A_3 &= (1|1, 1, 2, 1, \underbrace{4 \ldots, 4}_{m\text{-times}}, 3, 1, 1), \\
A_4 &= (2|1, 1, 2, 1, \underbrace{4 \ldots, 4}_{m\text{-times}}, 3, 1, 1).
\end{aligned}$$

These sets have the property that $I_{\max A_i} \backslash A_i, i \in [1, 4]$ is the union of two arithmetic progressions. Given m, $I_{\max A_3} \backslash S_3$ gives a pair of arithmetic progressions of shortest length, $m + 2$ and $2m + 1$, while $\max A_2 = 4m + 14 = \max\{\max A_i | i \in [1, 4]\}$. We consider two cases.

(1) $k = 4j + 1$ $(j \geq 1)$. We require that all arithmetic progressions of length at least $m + 2$ can be partitioned into $2j$ MSTD sets. Then $m + 2 \geq 20j$ and so, $m \geq 20j - 2$, which also guarantees that all arithmetic progressions of length at least $2m + 1$ can be partitioned into $2j$ MSTD sets. So, we find out that for $r \geq 4(20j - 2) + 14 = 20k - 14$, I_r can be partitioned into k MSTD subsets. Hence, $R \leq 20k - 14$.

(2) $k = 4j + 3$ $(j \geq 1)$. We require that all arithmetic progressions of length at least $m + 2$ can be partitioned into $2j$ MSTD sets. Then $m + 2 \geq 20j$ and so, $m \geq 20j - 2$, which also guarantees that all arithmetic progressions of length at least $2m + 1$ can be partitioned into $2j + 2$ MSTD sets. So, we find out that for $r \geq 4(20j - 2) + 14 = 20k - 14$, I_r can be partitioned into k MSTD subsets. Hence, $R \leq 20k - 14$.

Finally, for $3-$decomposition, we use the base expansion method, where we require a run of consecutive odd numbers (an arithmetic progression) to contain two disjoint $10-$strong MSTD sets. The length of the arithmetic progression is at least $\frac{r-26}{2} + 1$. Let T be $\min\{\max A : A \text{ is } 10 - \text{strong }\}$. Then we require $\frac{r-26}{2} + 1 \geq T$ or $r \geq 2T + 24$. Hence, $24 \leq R \leq 2T + 24$.

APPENDIX C. PROOF OF LEMMA 3.4

We prove that for a fixed $m \in \mathbb{N}$, $S = (0|1, 1, 2, 1, \underbrace{4 \ldots, 4}_{m\text{-times}}, 3, 1, 1, 2)$ is MSTD. The proof for other sets in the lemma follows similarly.

Note that $\max S = 12 + 4m$. We will prove that $|S + S| \geq 26 + 6m$. Since S contains 1 and 2, if the difference between two numbers, say $x < y$, in S is less than or equal to 3, then $S + S$ contains $[x, y]$. If $a, b \in S$ and $a - b = 4$, then in the worst case (in term of cardinality of the sum set), $S + S$ does not contain $a - 1$. So, for the interval $[0, 12 + 4m]$, $S + S$ misses at most $m - 1$ sums because there are m differences of 4 and $8 = 4 + 4 \in S + S$. Next, consider $[13 + 4m, 24 + 8m]$ and observe that

$S_1 = \{\ell | 1 \leq \ell \leq 9 + 4m \text{ and } \ell \equiv 1 \bmod 4\} \subseteq S + S$. We have

$$\begin{aligned}(12 + 4m) + S_1 &= \{\ell | 13 + 4m \leq \ell \leq 21 + 8m \text{ and } \ell \equiv 1 \bmod 4\},\\ (10 + 4m) + S_1 &= \{\ell | 11 + 4m \leq \ell \leq 19 + 8m \text{ and } \ell \equiv 3 \bmod 4\},\\ (9 + 4m) + S_1 &= \{\ell | 10 + 4m \leq \ell \leq 18 + 8m \text{ and } \ell \equiv 2 \bmod 4\}.\end{aligned}$$

Note that

$$\begin{aligned}16 + 4m &= (12 + 4m) + 4 \in S + S,\\ 16 + 8m &= (8 + 4m) + (8 + 4m) \in S + S,\\ 20 + 8m &= (10 + 4m) + (10 + 4m) \in S + S,\\ 22 + 8m &= (10 + 4m) + (12 + 4m) \in S + S,\\ 24 + 8m &= (12 + 4m) + (12 + 4m) \in S + S.\end{aligned}$$

On the interval $[13 + 4m, 24 + 8m]$, $S + S$ misses at most the whole set $\{\ell | 20 + 4m \leq \ell \leq 12 + 8m \text{ and } \ell \equiv 0 \bmod 4\} \cup \{23 + 8m\}$, which has m numbers. Therefore, in total, $S + S$ misses at most $2m - 1$ numbers.

Next, we show that the difference set $S - S$ misses at least $2m$ numbers by proving that $S - S$ contains none of the elements in $\{6 + 4\ell | 0 \leq \ell \leq m - 1\}$. We use proof by contradiction. Suppose that there exists $0 \leq \ell \leq m - 1$ such that $6 + 4\ell$ is in $S - S$. Then, there must exists a run within $1, 1, 2, 1, \underbrace{4 \ldots, 4}_{m\text{-times}}, 3, 1, 1, 2$ that sums up to $6 + 4\ell$. Because $6 + 4\ell \equiv 2 \bmod 4$, the run must either starts within $1, 1, 2, 1$ or ends within $3, 1, 1, 2$. Consider two following cases:

(1) **Case I**: the run starts within $1, 1, 2, 1$. Because $1 + 1 + 2 + 1 = 5 < 6$, the run must end within $3, 1, 1, 2$. Therefore, the run sums up to a number of the form $a + 4m + b$, where the value of a and b depend on where the run starts and where it ends, respectively. Since $a + 4m + b = 6 + 4\ell \leq 6 + 4(m - 1)$, $a + b \leq 2$. This is a contradiction because $b \geq 3$.

(2) **Case II**: the run ends within $3, 1, 1, 2$. Because there is no runs within $\underbrace{4 \ldots, 4}_{m\text{-times}}$, $3, 1, 1, 2$ that sum up to $6 + 4\ell$, the runs must start within $1, 1, 2, 1$. Repeating the argument used in **Case I** and we have a contradiction.

Therefore, $(S - S) \cap \{6 + 4\ell | 0 \leq \ell \leq m - 1\} = \emptyset$ and so, $S - S$ misses at least $2m$ elements. This completes our proof that S is MSTD. □

APPENDIX D. EXAMPLES

D.1. **Lemma 2.2.** We use the set A_1 mentioned in Remark 2.4. As $n = 12$, we have $k \geq 8$. Pick $k = 8$ and $m = 40$. Then

$$\begin{aligned}O_1 &= \{16\} \cup [17, 29]_2 \cup \{30\}\\ O_2 &= \{71\} \cup [72, 84]_2 \cup \{85\}.\end{aligned}$$

By Lemma 2.2, we must choose $M \subseteq [31, 70]$ such that within M, there exists a sequence of pairs of consecutive elements, where consecutive pairs in the sequence are

not more than 15 apart and the sequence starts with a pair in $[31, 45]$ and ends with a pair in $[56, 70]$. We pick M:

$$M = \{38, 39, 54, 55, 59, 60\}.$$

Our set A' is

$$\begin{aligned} A' = &\{1, 2, 3, 4, 7, 10, 12\} \cup \{16\} \cup [17, 29]_2 \cup \{30\} \cup \{38, 39, 54, 55, 59, 60\} \\ &\cup \{71\} \cup [72, 84]_2 \cup \{85\} \cup \{89, 90, 91, 96, 97, 99, 100\}. \end{aligned}$$

It can be verified that $|A' + A'| = 199 > |A' - A'| = 197$, so A' is MSTD.

D.2. **Lemma 2.3.** We use the set A_2 mentioned in Remark 2.4. As $n = 20$, we have $k \geq 10$. Pick $k = 12$ and $m = 35$. Then

$$\begin{aligned} O_1 &= [21, 23] \cup [26, 44]_2 \cup [47, 49] \\ O_2 &= [79, 81] \cup [84, 102]_2 \cup [105, 107]. \end{aligned}$$

By Lemma 2.2, we must choose $M \subseteq [50, 78]$ such that within M, there exists a sequence of triplets of consecutive elements, where consecutive triplets in the sequence are not more than 23 apart and the sequence starts with a triplet in $[50, 73]$ and ends with a triplet in $[55, 78]$. We pick M:

$$M = \{60, 61, 62\}.$$

Our set A' is

$$\begin{aligned} A' = &\{5, 6, 7, 9, 11, 16, 17, 18, 19\} \\ &\cup [21, 23] \cup [26, 44]_2 \cup [47, 49] \\ &\cup \{60, 61, 62\} \cup [79, 81] \cup [84, 102]_2 \cup [105, 107] \\ &\cup \{109, 110, 111, 112, 116, 118, 120, 121, 122, 123\}. \end{aligned}$$

It can be verified that $|A' + A'| = 236 > |A' - A'| = 235$, so A' is MSTD.

D.3. **Theorem 1.1.** We use A_1 and A_2 mentioned in Remark 1.2. Pick $k = 12$ and $m = 30$. Set

$$\begin{aligned} A'_{1e} = &\{1, 2, 3, 4, 8, 9, 11, 13, 14, 15, 20\} \\ &\cup \{24\} \cup [25, 45]_2 \cup \{46\} \cup \{50, 51\} \cup [54, 69] \cup \{72, 73\} \\ &\cup \{77\} \cup [78, 98]_2 \cup \{99\} \\ &\cup \{103, 108, 109, 110, 113, 115, 119, 120, 121, 122\}, \\ A'_{2e} = &\{5, 6, 7, 10, 12, 16, 17, 18, 19\} \\ &\cup [21, 23] \cup [26, 44]_2 \cup [47, 49] \cup \{52, 53, 70, 71\} \\ &\cup [74, 76] \cup [79, 97]_2 \cup [100, 102] \\ &\cup \{104, 105, 106, 107, 111, 112, 114, 116, 117, 118\}. \end{aligned}$$

We have $|A'_{1e} + A'_{1e}| - |A'_{1e} - A'_{1e}| = 243 - 241 = 2$, $|A'_{2e} + A'_{2e}| - |A'_{2e} - A'_{2e}| = 227 - 225 = 2$ and (A'_{1e}, A'_{2e}) partitions $[1, 122]$.

D.4. **5-decompositions.** We do not give an example of a 3-decomposition because our method is inefficient and involves a large set arising from the base expansion method. Neither do we give an example of a 4-decomposition because the method is straightforward. We use the efficient method to have a 5-decomposition into MSTD sets. Set

$$\begin{aligned} M_1 &= (1|1,1,2,1,\underbrace{4\ldots,4}_{119\text{-times}},3,1,1,2) \\ &= \{1,2,3,5\} \cup [6,482]_4 \cup \{485,486,487,489\}. \end{aligned}$$

Observe that

$$[1,489]\backslash M_1 = [4,488]_4 \cup [7,483]_2.$$

Because $[4,488]_4$ is an arithmetic progression of length 122, we have

$$\begin{aligned} M_2 &= 4A'_{1e} \text{ and} \\ M_3 &= 4A'_{2e} \end{aligned}$$

partition $[4,488]_4$. Notice that in the example mentioned in D.3, we can pick $m = 147$ and find A''_{1e} (containing 1) and A''_{2e} that partition $[1,239]$. We have

$$\begin{aligned} M_4 &= 2A''_{1e} + 5 \text{ and} \\ M_5 &= 2A''_{2e} + 5 \end{aligned}$$

partition $[7,483]_2$. We have found (M_1,M_2,M_3,M_4,M_5) that partition $[1,489]$.

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E-mail address: chuh19@mail.wlu.edu

DEPARTMENT OF MATHEMATICS, WASHINGTON AND LEE UNIVERSITY, LEXINGTON, VA 24450

E-mail address: nluntzla@umich.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109

E-mail address: sjm1@williams.edu, Steven.Miller.MC.96@aya.yale.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, WILLIAMS COLLEGE, WILLIAMSTOWN, MA 01267

E-mail address: ls12@williams.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, WILLIAMS COLLEGE, WILLIAMSTOWN, MA 01267