The M&M Game: From Morsels to Modern Mathematics

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The M&M Game began as a simple question asked by the sixth author's curious four-year-old son, Cam: If two people are born on the same day, do they die on the same day? Of course, needing a way to explain randomness to children (two-year-old daughter Kayla was there as well), the three took the most logical next step and used M&M'S and created the M&M Game (see Figure 1 for an illustration).

You and some friends start with some number of M&M'S. Everyone flips a fair coin at the same time; if you get a head, you eat an M&M; if you get a tail, you don't. You continue tossing coins together until no one has any M&M'S left, and whoever is the last person with an M&M lives longest and "wins"; if all run out simultaneously, the game is a tie.

Figure 1 The first M&M Game; for young players, there is an additional complication in that it matters which colors you have and the order you place them down.

We can reformulate Cam's question on randomness as follows. If everyone starts with the same number of M&M'S, what is the chance everyone eats their last M&M at the same time? We'll concentrate on two people playing with c (for Cam) and k (for Kayla) M&M'S, though we encourage you to extend to the case of more people

Math. Mag. **90** (2017) 197–207. doi:10.4169/math.mag.90.3.197.
MSC: Primary 65Q30, Secondary 60C05; 33C05

playing, possibly with a biased coin. In the course of our investigations, we'll see some nice results in combinatorics and see applications of memoryless processes, statistical inference, and hypergeometric functions.

Recalling that the binomial coefficient $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ denotes the number of ways to choose r objects from n when order doesn't matter, we can compute the probability $P(k, k)$ of a tie when two people start with k M&M'S. If we let $P_n(k, k)$ denote the probability that the game ends in a tie with both people starting with k M&M'S after *exactly* n moves, then

$$P(k,k) \;=\; \sum_{n=k}^{\infty} P_n(k,k).$$

Note that we are starting the sum at k as it is impossible all the M&M'S are eaten in fewer than k moves.

We claim that

$$P_n(k,k) \;=\; \binom{n-1}{k-1}\left(\frac{1}{2}\right)^n \binom{n-1}{k-1}\left(\frac{1}{2}\right)^n.$$

This formula follows from the following observation: If the game ends in a tie after n tosses, then each person has *exactly* $k-1$ heads in their first $n-1$ tosses. As we have a fair coin, each string of heads and tails of length n for a player has probability $(1/2)^n$. The number of strings for each person where the first $n-1$ tosses have *exactly* $k-1$ heads, and the n^{th} toss is a head (we need this as otherwise we do not have each person eating their final M&M on the n^{th} move) is $\binom{n-1}{k-1}\binom{1}{1}$. The $\binom{1}{1}$ reflects the fact that the last toss must be a head. As there are two players, the probability that each has their k^{th} head after the n^{th} toss is the product, proving the formula. We have thus shown the following.

Theorem 1. *The probability the M&M Game ends in a tie with two people using fair coins and starting with k M&M'S is*

$$P(k,k) \;=\; \sum_{n=k}^{\infty}\binom{n-1}{k-1}\left(\frac{1}{2}\right)^n \binom{n-1}{k-1}\left(\frac{1}{2}\right)^n \;=\; \sum_{n=k}^{\infty}\binom{n-1}{k-1}^2 \frac{1}{2^{2n}}. \tag{1}$$

While the above solves the problem, it is unenlightening and difficult to work with. The first difficulty is that it involves an infinite sum over n. (In general, we need to be careful and make sure any infinite sum converges; while we are safe here as we are summing probabilities, we can elementarily prove convergence. Note $\binom{n-1}{k-1} \le n^{k-1}/(k-1)!$, and thus the sum is bounded by $(k-1)!^{-2}\sum_{n\ge k} n^{2k-2}/2^{2n}$; as the polynomial n^{2k-2} grows significantly slower than the exponential factor 2^{2n}, the sum rapidly converges.) Second, it is very hard to sniff out the k-dependence: If we double k, what does that do to the probability of a tie? It is desirable to have exact, closed-form solutions so we can not only quickly compute the answer for given values of the parameter but also get a sense of how the answer changes as we vary those inputs. In the sections below, we'll look at many different approaches to this problem, most of them trying to convert the infinite sum to a more tractable finite problem.

The basketball problem, memoryless games, and the geometric series formula

A basketball game We can convert the infinite M&M Game sum, Equation (1), into a finite sum as we have a memoryless game: The behavior of the system only depends on its state at a given moment in time and not on how we got there. There are many examples where all that matters is the configuration, not the path taken to reach it. For example, imagine a baseball game. If the lead-off hitter singles or walks, the net effect is to have a runner on first, and the two results are (essentially) the same. For another example, consider a game of Tic-Tac-Toe; what matters is where the X's and O's are on the board, not the order they are placed.

We first look at a related problem that's simpler but illustrates the same point and yields the famous geometric series formula. Imagine two of the greatest basketball players of all time, Larry Bird of the Boston Celtics and Michael Jordan of the Chicago Bulls, are playing a basketball game. The rules below are a slight modification of a famous Superbowl ad, "The Showdown," between the two where the first one to miss gives up his claim to a bag of McDonald's food to the other. In that commercial, the two keep taking harder and harder shots; for our version, we'll have them do the same shot each time. Explicitly, the rules of their one-on-one game of hoops are as follows.

> *Bird and Jordan alternate shooting free throws, with Bird going first, and the first player to make a basket wins. Assume Bird always makes a basket with probability p_B, while Jordan always gets a basket with probability p_J. If the probability Bird wins is x_B, what is x_B?*

Note that this is almost a simplified M&M Game: There is only one M&M, but the players take turns flipping their coins. We'll see, however, that it is straightforward to modify the solution and solve our original problem.

Solution from the geometric series formula The standard way to solve our basketball problem uses a geometric series. The probability that Bird wins is the sum of the probabilities that he wins on his n^{th} shot. We'll see in the analysis below that it's algebraically convenient to define $r := (1 - p_B)(1 - p_J)$, which is the probability they both miss. Let's go through the cases. We assume that p_B and p_J are not both zero; if they were, then neither can hit a basket. Not only would this mean that our ranking of them as two of the all-time greats is wrong, but the game will never end, and thus there's no need to do any analysis!

1. Bird wins on his 1^{st} shot with probability p_B.
2. Bird wins on his 2^{nd} shot with probability $(1 - p_B)(1 - p_J)p_B = rp_B$.
3. Bird wins on his n^{th} shot with probability $(1 - p_B)(1 - p_J) \cdot (1 - p_B)(1 - p_J) \cdots (1 - p_B)(1 - p_J)p_B = r^{n-1}p_B$.

To see this, if we want Bird to win on shot n, then we need to have him and Jordan miss their first $n - 1$ shots, which happens with probability $((1 - p_B)(1 - p_J))^{n-1} = r^{n-1}$, and then Bird hits his n^{th} shot, which happens with probability p_B. Thus,

$$\text{Prob(Bird wins)} \;=\; x_B \;=\; p_B + rp_B + r^2p_B + r^3p_B + \cdots \;=\; p_B \sum_{n=0}^{\infty} r^n,$$

which is a geometric series. As we assumed p_B and p_J are not both zero, $r = (1 - p_B)(1 - p_J)$ satisfies $|r| < 1$, and we can use the geometric series formula to deduce

$$x_B \;=\; \frac{p_B}{1-r} \;=\; \frac{p_B}{1-(1-p_B)(1-p_J)}.$$

We have made enormous progress. We converted our infinite series into a ***closed-form expression***, and we can easily see how the probability of Bird winning changes as we change p_B and p_J.

Solution through memoryless game and the geometric series formula We now give a second solution to the basketball game; instead of requiring the geometric series formula as an input, we obtain it as a consequence of our arguments.

Recall the assumptions we made. The probability Bird makes a shot is p_B, the probability Jordan hits a basket is p_J, and the probability they both miss is $r := (1 - p_B)(1 - p_J)$. We can use this to compute x_B, the probability Bird wins, in another way. Before, we wrote x_B as a sum over the probabilities that Bird won in n games. Now, we claim that

$$\text{Prob(Bird wins)} \; = \; x_B \; = \; p_B + r x_B.$$

To see this, note either Bird makes his first basket and wins (which happens with probability p_B) or he misses (with probability $1 - p_B$). If Bird is going to win, then Jordan must miss his first shot, and this happens with probability $1 - p_J$. Something interesting happens, however, if both Bird and Jordan miss: *We have reset our game to its initial state!* Since both have missed, it's as if we just started playing the game right now. Since both miss and Bird has the ball again, by definition, the probability Bird wins from this configuration is x_B, and thus the probability he wins is $p_B + (1 - p_B)(1 - p_J)x_B$.

Solving for x_B, the probability Bird beats Jordan is

$$x_B \; = \; \frac{p_B}{1 - r}.$$

As this must equal the infinite series expansion from the previous subsection, we deduce the geometric series formula:

$$\frac{p_B}{1 - r} \; = \; p_B \sum_{n=0}^{\infty} r^n \quad \text{therefore} \quad \sum_{n=0}^{\infty} r^n \; = \; \frac{1}{1 - r}.$$

Remark. We have to be a bit careful. It's important to keep track of assumptions. In our analysis, $r = (1 - p_B)(1 - p_J)$ with $0 \le p_B, p_J \le 1$, and at least one of p_B and p_J is positive (if both were zero the game would never end). Thus, we have only proved the geometric series formula if $0 \le r < 1$; we encourage you to find a way to pass to all $|r| < 1$.

Let's look closely at what we've done in this subsection. The key observation was to notice that we have a memoryless game. We now show how to similarly convert the solution to the M&M Game, Equation (1), into an equivalent finite sum.

Memoryless M&M Games

Setup. Remember (Equation (1)) that we have an infinite sum for the probability of a tie with both people starting with k M&M'S:

$$P(k, k) \; = \; \sum_{n=k}^{\infty} \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2} \cdot \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2}.$$

It's hard to evaluate this series as we have an infinite sum and a squared binomial coefficient whose top is changing. We want to convert it to something where we have more familiarity. From the hoops game, we should be thinking about how to obtain a *finite* calculation. The trick there was to notice we had a memoryless game, and all that mattered was the game state, not how we reached it. For our problem, we'll have many tosses of the coins, but in the end, what matters is where we are, not the string of heads and tails that got us there.

Let's figure out some way to do this by letting $k = 1$. In this case, we can do the same thing we did in the hoops game and boil the problem down into cases. There are four equally likely scenarios each time we toss coins, so the probability of each event occurring is 1/4 or 25%.

1. Both players eat.
2. Cam eats an M&M, but Kayla does not.
3. Kayla eats an M&M, but Cam does not.
4. Neither eat.

These four possibilities lead to the infinite series in Equation (1), as we calculate the probability the game ends in n tosses. It turns out one of the four events is not needed, and if we remove it, we can convert to a finite game.

Similar to the hoops game, we have another memoryless game. If Cam and Kayla both get tails and therefore don't eat their M&Ms, then it's as if the coin toss never happened. We can therefore ignore the fourth possibility. If you want, another way to look at this is that if we toss two tails, then there is no change in the number of M&M'S for either kid, and thus we may pretend such a toss never happened. This allows us to remove all the tosses of double tails, and now after each toss at least one player, possibly both, have fewer M&M'S. As we start with a finite number of M&M'S, the game terminates in a finite number of moves. Thus, instead of viewing our game as having four alternatives with each toss, there are only three, and they all happen with probability 1/3:

1. both players eat;
2. Cam eats an M&M, but Kayla does not;
3. Kayla eats an M&M, but Cam does not.

Notice that after each toss the number of M&M'S is decreased by either 1 or 2, so the game ends after at most $2k - 1$ tosses.

Solution We now replace the infinite sum of Equation (1) with a finite sum. Each of our three possibilities happens with probability 1/3. Since the game ends in a tie, the final toss must be double heads with both eating, and each must eat exactly $k - 1$ M&M'S in the earlier tosses. Let n denote the number of times both eat before the final toss (which again we know must be double heads); clearly, $n \in \{0, 1, \dots, k - 1\}$. We thus have $n + 1$ double heads, and thus Cam and Kayla must each eat $k - (n + 1) = k - n - 1$ times when the other doesn't eat.

We see that, in the case where there are $n + 1$ double heads (with the last toss being double heads), the total number of tosses is

$$(n + 1) + (k - n - 1) + (k - n - 1) \ = \ 2k - n - 1.$$

In the first $2k - n - 2$ tosses, we must choose n to be double heads, then of the remaining $(2k - n - 2) - n = 2k - 2n - 2)$ tosses before the final toss we must choose $k - n - 1$ to be just heads for Cam, and then the remaining $k - n - 1$ tosses before the final toss must all be just heads for Kayla. These choices explain the presence of the two binomial factors. As each toss happens with probability 1/3, this explains those factors; note we could have just written $(1/3)^{2k-n-1}$, but we prefer to highlight the sources. We have thus shown the following result.

Theorem 2. *The probability the M&M Game ends in a tie with two people using fair coins and starting with k M&M'S is*

$$P(k,k) \;=\; \sum_{n=0}^{k-1} \binom{2k-n-2}{n} \left(\frac{1}{3}\right)^{n} \binom{2k-2n-2}{k-n-1} \left(\frac{1}{3}\right)^{k-n-1} \left(\frac{1}{3}\right)^{k-n-1} \frac{1}{3}. \quad (2)$$

Viewing data

Plotting Before turning to additional ways to solve the problem, it's worthwhile to pause for a bit and discuss how to view data and use results for small expressions involving k; this finite sum is certainly easier to use than the infinite sum in Equation (1), and we plot it in Figure 2 (left).

While Equation (2) gives us a nice formula for finite computations, it's hard to see the k dependence. An important skill to learn is how to view data. Frequently, rather than plotting the data as given, it's better to do a log–log plot. What this means is that, instead of plotting the probability of a tie as a function of k, we plot the logarithm of the probability of a tie against the logarithm of k. We do this in Figure 2 (right).

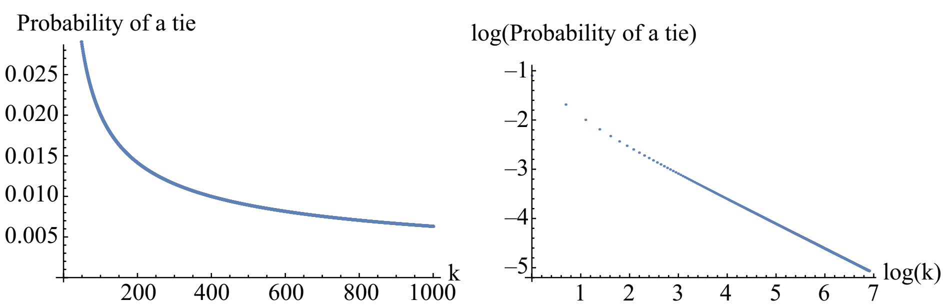

Figure 2 Left: the probability of a tie for $k \leq 1000$. Right: log–log plot.

Notice that the plot here looks *very* linear. Lines are probably the easiest functions to extrapolate, and if this linear relationship holds, we should be able to come up with a very good prediction for the logarithm of the probability (and hence, by exponentiating, obtain the probability). We do this in the next subsection.

Statistical inference Let's predict the answer for large values of k from smaller ones. The sixth named author gave a talk on this at the 110^{th} meeting of the Association of Teachers of Mathematics in Massachusetts in March 2013, which explains the prevalence of 110 and 220 below.

Figure 3 (left) gives the log-log plot for $k \leq 110$, while the right is for $50 \leq k \leq 110$. Using the Method of Least Squares with $P(k)$ the probability of a tie when we start with k M&M'S, we find a predicted best fit line of

$$\log\left(P(k)\right)\; \approx\; -1.42022 - 0.545568 \log k, \quad \text{or} \quad P(k) \;\approx\; 0.2412/k^{.5456}.$$

This predicts a probability of a tie when $k = 220$ of about 0.01274, but the answer is approximately 0.0137. While we are close, we are off by a significant amount. (In situations like this, it is better to look at not the difference in probabilities, which is small, but the percentage we are off; here, we differ by about 10%.)

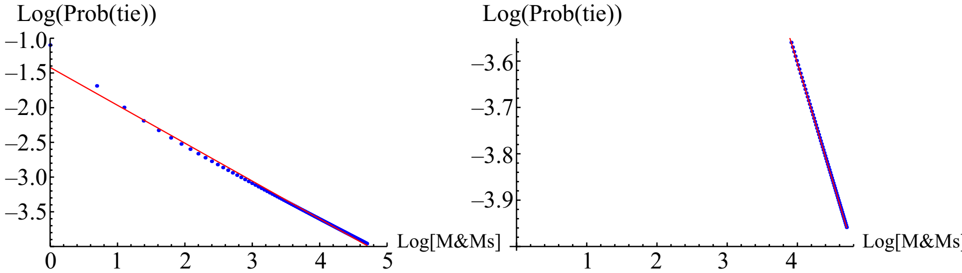

Figure 3 The probability of a tie. Left: $k \leq 110$. Right: $50 \leq k \leq 110$.

Why are we so far off? The reason is that small values of k are affecting our prediction more than they should. If we have a main term in the log–log plot that is linear, it will eventually dominate lower order terms, *but* those lower order terms could have a sizable effect for low k. Thus, it's a good idea to ignore the smaller values when extrapolating our best fit line; in Figure 3 (right), we now go from $k = 50$ to 110. Our new best fit line is

$$\log\left(P(k)\right) \approx -1.58261 - 0.50553 \log k, \quad \text{or} \quad P(k) \approx 0.205437/k^{.50553}.$$

Using this formula, we predict 0.01344 for $k = 220$, which compares *very* favorably to the true answer of 0.01347.

Recurrences, hypergeometric functions, and the OEIS

The M&M recurrence Even though we have a finite sum for the probability of a tie (Equation 2), finding that formula required some knowledge of combinatorics and binomial coefficients. We give an alternate approach that avoids these ideas. Our first approach assumes we're still clever enough to notice that we have a memoryless game, and then we remark afterward how we would have found the same formula even if we didn't realize this.

We need to consider a more general problem. We always denote the number of M&M'S Cam has with c, and Kayla with k; we frequently denote this state by (c, k). Then we can rewrite the three equally likely scenarios, each with probability 1/3, as follows:

- $(c, k) \longrightarrow (c-1, k-1)$ (double heads and both eat),
- $(c, k) \longrightarrow (c-1, k)$ (Cam gets a head, and Kayla a tail),
- $(c, k) \longrightarrow (c, k-1)$ (Cam gets a tail, and Kayla a head).

If we let $x_{c,k}$ denote the probability the game ends in a tie when we start with Cam having c M&M'S and Kayla having k, we can use the above to set up a recurrence relation (see [**3**] for a brief introduction to recurrence relations). How so? Effectively, on each turn, we move from (c, k) in exactly one of the three ways enumerated above. Now, we can use simpler game states to figure out the probability of a tie when we start with more M&Ms, as in each of the three cases we have reduced the total number of M&M'S by at least one. We thus find that the recurrence relation satisfied by $\{x_{c,k}\}$ is

$$x_{c,k} = \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1} = \frac{x_{c-1,k-1} + x_{c-1,k} + x_{c,k-1}}{3}. \tag{3}$$

This cannot be the full story—we need to specify initial conditions. A little thought says $x_{0,0}$ must be 1 (if they both have no M&M'S then it must be a tie), while $x_{c,0} = 0$ if $c > 0$, and similarly, $x_{0,k} = 0$ if $k > 0$ (as in these cases, exactly one of them has an M&M, and thus the game cannot end in a tie).

We have made tremendous progress. We use these initial values and the recurrence relation (3) to determine $x_{c,k}$. Unfortunately, we cannot get a simple closed form expression, but we can easily compute the values by recursion. A good approach is to compute all $x_{c,k}$ where $c + k$ equals sum fixed sum s. We've already done the cases $s = 0$ and $s = 1$, finding $x_{0,0} = 1$, $x_{0,1} = x_{1,0} = 0$.

We now move to $s = 2$. We need only find $x_{1,1}$, as we know $x_{2,0} = x_{0,2} = 0$. Using the recurrence relation, we find

$$x_{1,1} \;=\; \frac{x_{0,0} + x_{0,1} + x_{1,0}}{3} \;=\; \frac{1 + 0 + 0}{3} \;=\; \frac{1}{3}.$$

Next is the case when the indices sum to 3. Of course, $x_{0,3} = x_{3,0} = 0$, so all we need are $x_{1,2}$ and $x_{2,1}$ (which by symmetry are the same). We find

$$x_{2,1} \;=\; x_{1,2} \;=\; \frac{x_{1,1} + x_{2,0} + x_{0,2}}{3} \;=\; \frac{1/3 + 0 + 0}{3} \;=\; \frac{1}{9}.$$

We can continue to $s = 4$, and after some algebra easily obtain

$$x_{2,2} \;=\; \frac{x_{1,1} + x_{2,1} + x_{1,2}}{3} \;=\; \frac{5}{27}.$$

If we continued on with these calculations, we would find that $x_{3,3} = \frac{11}{81}$, $x_{4,4} = \frac{245}{2187}$, $x_{5,5} = \frac{1921}{19863}$, $x_{6,6} = \frac{575}{6561}$, $x_{7,7} = \frac{42635}{531441}$, and $x_{8,8} = \frac{355975}{4782969}$. The beauty of this recursion process is that we have a sure-fire way to figure out the probability of a tie at different states of the M&M game. We leave it as an exercise to the interested reader to compare the computational difficulty of finding $x_{100,100}$ by the recurrence relation versus by the finite sum of Equation (2).

We end with one final comment on this approach. We can recast this problem as one in counting weighted paths on a graph. We count the number of paths from (0, 0) to (n, n) where a path with m steps is weighted by $(1/3)^m$, and the permissible steps are (1, 0), (0, 1), and (1, 1). In Figure 4, we start with $(c, k) = (4, 4)$ and look at all the possible paths that end in (0, 0).

Remark. If we hadn't noticed it was a memoryless game, we would have found

$$x_{c,k} \;=\; \frac{1}{4}x_{c-1,k-1} + \frac{1}{4}x_{c-1,k} + \frac{1}{4}x_{c,k-1} + \frac{1}{4}x_{c,k}.$$

Straightforward algebra returns us to our old recurrence, equation (3):

$$x_{c,k} \;=\; \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

This means if we did not notice initially that there was a memoryless process, doing the algebra suggests there is one!

Hypergeometric functions We end our tour of solution approaches with a method that actually prefers the infinite sum to the finite one, hypergeometric functions (see, for example, [**1, 2**]). These functions arise as the solution of a particular linear second order differential equation:

$$x(1-x)y''(x) + [c - (1 - a + b)x]y'(x) - aby(x) \;=\; 0.$$

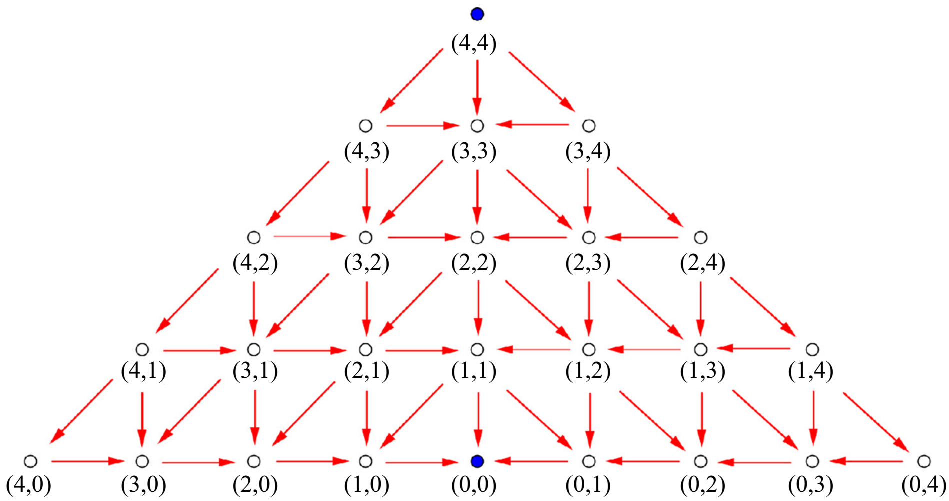

Figure 4 The M&M game when $k = 4$.

This equation is useful because every other linear second order differential equation with three singular points (in the case they are at 0, 1, and ∞) can be transformed into it. As this is a second order differential equation, there should be two solutions. One is

$$y(x) = 1 + \frac{abx}{c \cdot 1!} + \frac{a(a+1)b(b+1)x^2}{c(c+1) \cdot 2!} + \frac{a(a+1)(a+2)b(b+1)(b+2)x^3}{c(c+1)(c+2) \cdot 3!} + \cdots, \tag{4}$$

so long as c is not a nonpositive integer; we denote this solution by ${}_2F_1(a, b; c; z)$. By choosing appropriate values of a, b, and c, we recover many special functions. Wikipedia lists three nice examples:

$$\begin{aligned} \log(1+x) &= x\,{}_2F_1(1, 1; 2; -x) \\ (1-x)^{-a} &= {}_2F_1(a, 1; 1; x) \\ \arcsin(x) &= x\,{}_2F_1(1/2, 1/2; 3/2; x^2). \end{aligned} \tag{5}$$

By introducing some notation, we can write the series expansion more concisely. We define the Pochhammer symbol by

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{(a+n-1)!}{(a-1)!}$$

(where the last equality holds for integer a; for real a, we need to interpret the factorial as its completion, the Gamma function). Our solution becomes

$${}_2F_1(a, b, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n x^n}{(c)_n n!}.$$

Note the factorials in the above expression suggest that there should be connections between hypergeometric functions and products of binomial coefficients. In this notation, the 2 represents the number of Pochhammer symbols in the numerator, the 1 the number of Pochhammer symbols in the denominator, and the a, b, and c are what we evaluate the symbols at (the first two are the ones in the numerator, the last the denominator). One could of course consider more general functions, such as

$$ {}_sF_t(\{a_i\}, \{b_j\}; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_s)_n x^n}{(b_1)_n \cdots (b_t)_n n!}. $$

The solution ${}_2F_1(a, b, c; x)$ is called a hypergeometric function, and if you look closely at it while recalling the infinite sum solution to the M&M Game you might see the connection. After some algebra where we convert the binomial coefficients in the infinite sum solution of Equation (1) to the falling factorials that are the Pochhammer symbols, we find the following closed form solution.

Theorem 3. *The probability the M&M Game ends in a tie with two people using fair coins and starting with k M&M'S is*

$$ P(k, k) = {}_2F_1(k, k, 1; 1/4)4^{-k}. \tag{6} $$

It is not immediately clear that this is progress; after all, it looks like we've just given a fancy name to our infinite sum. Fortunately, special values of hypergeometric functions are well studied (see, for example, [**1, 2**]), and a lot is known about their behavior as a function of their parameters. We encourage the interested reader to explore the literature and discover how "useful" the above is.

OEIS If we use our finite series expansion of Equation (2) or the recurrence relation of Equation (3), we can easily calculate the probability of a tie for some small k. We give the probabilities for k up to 8 in Table 1. In addition, we also give $3^{2k-1}P(k, k)$ as multiplying by 3^{2k-1} clears the denominators and allows us to use the On-Line Encyclopedia of Integer Sequences (OEIS, http://oeis.org/).

TABLE 1: Probability of a tie.

k	$P(k, k)$	$3^{2k-1}P(k, k)$
1	1/3	1
2	5/27	5
3	11/81	33
4	245/2187	245
5	1921/19683	1921
6	575/6561	15525
7	42635/531441	127905
8	355975/4782969	1067925

Thus, to the M&M Game with two players, we can associate the integer sequence 1, 5, 33, 245, 1921, 15,525, 127,905, 1,067,925, ... We plug that into the OEIS and find that it knows that sequence: A084771 (see http://oeis.org/A084771, and note that the first comment on this sequence is that it equals the number of paths in the graph we discussed).

Conclusion and further questions

We've seen many different ways of solving the M&M Game, each leading to a different important aspect of mathematics. We leave the reader with some additional questions to pursue using the techniques from this and related articles.

How long do we expect a game to take? What would happen to the M&M problem if we increased the number of players? What if all of the players started with different numbers of M&M'S? What if the participants used biased coins?

In one of the first games ever played, starting with five M&M'S Kayla tossed five consecutive heads, losing immediately; years later, she still talks about that memorable performance. There is a lot known about the longest run of heads or tails in tosses of a fair (or biased) coin (see, for example, [**4**]). We can ask related questions here. What is the expected longest run of heads or tails by any player in a game? What is the expected longest run of tosses where all players' coins have the same outcome?

Acknowledgment This work was partially supported by NSF Grants DMS1265673 and DMS1561945. We thank Stan Wagon for suggesting and sharing his solution to the probability the first shooter wins the hoops game when both probabilities are independent uniform random variables, Adam Lott for help with images, and Ann Ewalt, Michael Jones, Elizabeth Miller, Frank Morgan,and the referees for comments on an earlier draft. The sixth named author thanks Donald Cameron for the opportunity to talk on this problem at the 110^{th} meeting of the Association of Teachers of Mathematics in Massachusetts.

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Summary. To an adult, it's obvious that the day of someone's death is not precisely determined by the day of birth, but it's a very different story for a child. We invented what we call the *the M&M Game* to help explain randomness: Given k people, each simultaneously flips a fair coin, with each eating an M&M on a head and not eating on a tail. The process then continues until all M&M'S are consumed, and two people are deemed to die at the same time if they run out of M&M'S together. We analyze the game and highlight connections to the memoryless process, combinatorics, statistical inference, and hypergeometric functions.

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