LEADING DIGIT LAWS ON LINEAR LIE GROUPS

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ABSTRACT. We study the leading digit laws for the matrix entries of a linear Lie group G. For non-compact G, these laws generalize the following observations: (1) the normalized Haar measure of the Lie group $\mathbb{R}^+$ is dx/x and (2) the scale invariance of dx/x implies the distribution of the digits follow Benford's law. Viewing this scale invariance as left invariance of Haar measure, we see either Benford or power law behavior in the significands from one matrix entry of various such G. When G is compact, the leading digit laws we obtain come as a consequence of digit laws for a fixed number of components of a unit sphere. The sequence of digit laws for the unit sphere exhibits periodic behavior as the dimension tends to infinity.

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1. INTRODUCTION

1.1. **Background.** Given a positive number x and a base $B > 1$, we write $x = S_B(x)B^{k(x)}$, where $S_B(x) \in [1, B)$ is the significand and $k(x) \in \mathbb{Z}$. The distribution of $S_B(x)$ has interested

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researchers in a variety of fields for over a hundred years, as frequently it is not uniformly distributed over $[1, B)$ but exhibits a profound bias. If $\mathrm{Prob}(S_B(x) \leq s) = \log_B(s)$ we say the system[1] follows Benford's law, which implies the probability of a first digit of $d \leq B-1$ is $\log_B(d+1) - \log_B(d) = \log_B(1+1/d)$; in particular, a sequence of numbers base 10 has a first digit of 1 about 30% of the time, and 9 for only around 4.5% of the values. This bias was first observed by Newcomb [New] in the 1880s, and then rediscovered by Benford [Ben] nearly 50 years later.

Many systems follow Benford's law; on the pure math side these include the Fibonacci numbers (and most solutions to linear recurrence relations) [BrDu], iterates of the $3x+1$ map [KonMi, LagSo], and values of L-functions on the critical strip among many others; on the applied side examples range from voter and financial data [Meb, Nig] to the average error in floating point calculations [Knu]. See [BH3, Mil] for two recent books on the subject, the latter describing many of the applications from detecting fraud in taxes, images, voting and scientific research, [BH2, Dia, Hi1, Hi2, Pin, Rai] for some classic papers espousing the theory, and [BH1, Hu] for online collections of articles on the subject.

Our purpose is to explore the distribution of leading digits of components chosen from some random process. We concentrate on two related types of systems. The first type consists of various $n \times n$ matrix ensembles, which of course can be viewed as vectors living in $\mathbb{R}^{n^2}$. The second are components of a point uniformly chosen on a unit sphere, which turn out to imply results for some of our matrix ensembles.

Following the work of Montgomery [Mon], Odlyzko [Od1, Od2], Katz-Sarnak [KaSa1, KaSa2], Keating-Snaith [KeSn1, KeSn2, KeSn3], Conrey-Farmer-Keating-Rubinstein-Snaith [CFKRS], Tracy-Widom [TW] and many others, random matrix ensembles in general, and the classical compact groups in particular, have been shown to successfully model a variety of number-theoretic objects, from special values to distribution of zeros to moments.[2] In some of these systems Benford's law has already been observed (such as values of L-functions and characteristic polynomials of random matrix ensembles in [KonMi], or values of Fourier coefficients in [ARS]); thus our work can be interpreted as providing another explanation for the prevalence of Benford's law. We first quickly review some needed background material on Haar measures (§1.2) and definitions (§1.3), and then state our results for compact groups in §1.4 and non-compact groups in §1.5.

1.2. **Haar Measure Review.** Random matrix theory has enjoyed numerous successes over the past few decades, successfully modeling a variety of systems from energy levels of heavy nuclei to zeros of L-functions [BFMT-B, FiMil, Ha]. Early work in the subject considered ensembles where the matrix elements were drawn independently from a fixed probability distribution p. This of course led to questions and conjectures on how various statistics (such as spacings between normalized eigenvalues) depends on p. For example, while the density of normalized eigenvalues in matrix ensembles (Wigner's semi-circle law) was known for all ensembles where the entries were chosen independently from nice distributions, the universality of the spacings between adjacent normalized eigenvalues resisted proof until this century (see, among others, [ERSY, ESY, TV1, TV2]).

[1]By a system x we mean either a sequence of numbers or a measurable function for which $\mathrm{Prob}(S_B(x) \leq s)$ exists. This will be made precise in the next section

[2]For example, the Tracy-Widom distributions describe the behavior of the largest eigenvalues of many ensembles, and frequently these control the behavior of the system; see [MNS] for an example from random graphs and network theory.

Instead of choosing the matrix elements independently and having to choose a probability distribution p, we can consider matrix groups where the Haar measure gives us a canonical choice for randomly choosing a matrix element.[3] On an n-dimensional Lie group G there exists a unique, non-trivial, countably additive measure μ which is left translation invariant (so $\mu(gE) = \mu(E)$ over all $g \in G$ and Borel sets E); μ is called the Haar measure. If our space is compact we may normalize μ so that it assigns a measure of 1 to G and thus may be interpreted as a probability. See §15 of [HR] for more details on Haar measures and Lie groups.

We are especially interested in the distribution of the leading digits in the (i, j)-th entry of a connected linear Lie subgroup $G \subset \mathrm{GL}(V)$. For many G the resulting behavior is easily determined and follows immediately from the observation that a system whose density is $\frac{1}{\log B}\frac{1}{x}$ on $[1, B)$ follows Benford's law (see Definition 1.1 and Lemma 3.1). After introducing some terminology we state five cases which are immediately analyzed from the Haar density; Theorem 1.12 is the main result which interprets Haar measure from a matrix decomposition of $SL_n(\mathbb{R})$. Care must be taken to separate the notion of the digit law for the compact and non-compact cases since many non-compact G do not posses a G-invariant probability measure. Thus we have two definitions of leading digit law: for non-compact G, we average the measure of significands over a neighborhood of a specific one-parameter subgroup (see Definition 1.3 for a precise statement). If G is compact, the Haar measure affords a global average over all matrix elements. In this light, one may think of the non-compact digit law as a local law and the compact digit law as global law (see Definition 1.6). A review of standard conventions and a list of the linear Lie groups we study are given in Appendix A. Throughout the paper, we write μ to mean the Haar measure on G and dg for the Haar density on G.

1.3. **Definitions.**

Definition 1.1. *Given a base $B \in \mathbb{N}, B > 1$, a **digit law** is a probability density function $\psi : [1, B) \to [0, 1]$. A digit law ψ satisfies a (B, k) power law (for positive $k \neq 1$) if*

$$\psi(x) \ = \ \psi_k(x) \ := \ \frac{B^{k-1} - 1}{(k-1)B^{k-1}}\frac{1}{x^k}, \tag{1.1}$$

and ψ is B-Benford if

$$\psi(x) \ = \ \psi_1(x) \ := \ \frac{1}{\log B}\frac{1}{x}. \tag{1.2}$$

Definition 1.2. *Given a connected, non-compact, locally compact Lie group G with Lie algebra $L(G)$, and a subset $S \subset L(G)$ define the **tubular neighborhood around** S to be the set*

$$U_\epsilon(S) \ = \ \{X + Y \mid X \in S, Y \in L(G), Y \perp X, |Y| < \epsilon\}. \tag{1.3}$$

The definition of the local version of digit law, stated next, is technical but captures the essence of a leading digit law by averaging μ in the direction of X according to the significands base B. This definition has the advantage of producing digit laws for non-compact matrix groups which are not amenable (e.g. $\mathrm{SL}_2(\mathbb{R})$).

Definition 1.3 (Local formulation of digit law). *Given a connected, non-compact, locally compact Lie group G with Lie algebra $L(G)$, a unit direction $X \in L(G)$ which generates a one-parameter subgroup $x = x(t) = \exp(tX)$ of G, a base $B > 1$, a positive measure μ on G and*

[3]These are the ensembles that turn out to be most useful in number theory, not the ones arising from a fixed distribution.

probability density function $\psi : [1, B) \to [0, 1]$ *we say that* (G, μ, x) *satisfies the digit law* ψ *if*

$$\text{Prob}(S_B(x) \le s) \ = \ \lim_{k\to\infty}\lim_{\epsilon\to 0} \frac{\sum_{l=0}^{k-1} \mu(\exp(U_\epsilon([\log B^l, \log B^l s)X))}{\mu(\exp(U_\epsilon([0, \log B^k)X)))} \ = \ \int_0^s \psi(t)\, dt \tag{1.4}$$

where k *is a positive integer.*

Remark 1.4. *By the Baker-Campbell-Hausdorff formula* [VS], *the averaging condition* (1.4) *is equivalent to*

$$\text{Prob}(S_B(x) \le s) \ = \ \lim_{k\to\infty}\lim_{\epsilon\to 0} \frac{\sum_{l=0}^{k-1} \mu(\exp(\log B^l X)\exp(U_\epsilon([0, \log s)X)))}{\mu(\exp(U_\epsilon([0, k\log B)X)))}. \tag{1.5}$$

We typically take μ to be the left-invariant or right-invariant Haar measure on G. If μ is left-invariant or bi-invariant, (1.5) becomes

$$\text{Prob}(S_B(x) \le s) \ = \ \lim_{k\to\infty}\lim_{\epsilon\to 0} \frac{k\mu(\exp(U_\epsilon([0, \log s)X)))}{\mu(\exp(U_\epsilon([0, k\log B)X)))}. \tag{1.6}$$

Remark 1.5. *When a group* G *decomposes as the product of simultaneously commuting one-parameter subgroups (Theorems 1.11 and 1.10) The Haar measure* dg *on* G *decomposes as a product of measures along each one-parameter subgroup. In these instances we refer to the* joint leading digit law *on* G*.*

When G is compact, the Haar measure μ may be normalized to be an invariant probability measure on G, affording a global definition of a digit law, stated next.

Definition 1.6 (Global formulation of digit law). *Fix a base* $B > 1$*. Let* G *be a compact connected Lie group,* μ *a positive countably additive probability measure on* G *and* $f : G \to \mathbb{R}$ *measurable. We say that* (G, μ, f) *satisfies the digit law* ψ *if*

$$\text{Prob}(S_B(f(g)) < s) \ = \ \int_1^s \psi(x)\, dx. \tag{1.7}$$

Recapitulating, the definition of the digit law for compact G (Definition 1.6) averages significands from a measurable function $f : G \to \mathbb{R}$ over the entire group G, whereas the non-compact definition of the digit law (Definition 1.3) averages the significands over neighborhoods of a one parameter subgroup of G, since a G-invariant probability measure may not exist (i.e., G not amenable).

1.4. **Results (Compact Groups).** Our results about the distribution of entries of compact groups are a consequence of the following theorem about the coordinates of points chosen uniformly on spheres, which is also of independent interest. Write

$$S^n(r) \ := \ \{x \in \mathbb{R}^{n+1} : |x| = r\} \tag{1.8}$$

for the n-sphere of radius $r > 0$, and $S^n := S^n(1)$ for the unit sphere. Below erf is the standard error function:

$$\text{erf}(x) \ := \ \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \tag{1.9}$$

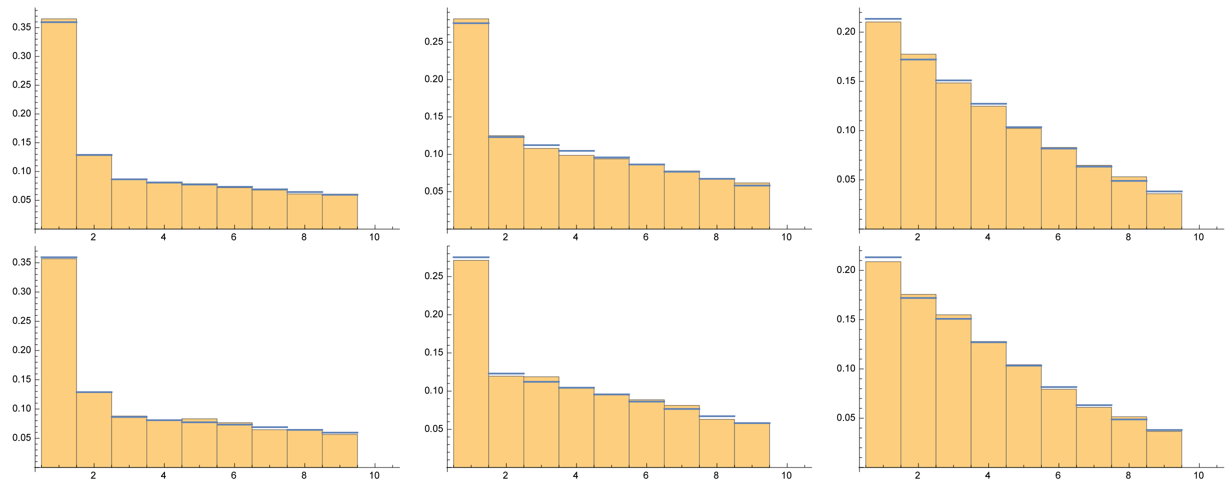

FIGURE 1. The distribution of the first digits base $B = 10$ (theory versus simulation) of the first component of points uniformly chosen on a sphere with n components. Top row: $n \in \{100, 200, 500\}$. Bottom row: $n \in \{10000, 20000, 50000\}$. Notice the periodicity when n increases by a factor of $B^2 = 100$.

Theorem 1.7. *Consider the sequence of spheres $S^{nB^{2\ell}}$, $\ell \in \mathbb{N}$. For $\sqrt{\frac{n}{2}\frac{1}{B}} > 4$,*

$$\lim_{\ell\to\infty} \mathrm{Prob}(a < S_B(x_1) < b, x \in S^{nB^{2\ell}}) \;=\; \sum_{i=-\infty}^{\infty}\left[\mathrm{erf}\left(\sqrt{\frac{n}{2}}\frac{b}{B^i}\right) - \mathrm{erf}\left(\sqrt{\frac{n}{2}}\frac{a}{B^i}\right)\right]. \tag{1.10}$$

For fixed $n \in \mathbb{N}$, it follows that the leading digit law of x_1 in $S^{nB^{2\ell}}$, as $\ell \to \infty$, tends to the digit law $F_n : [1, B) \to [0, 1)$ whose cumulative distribution function is given by

$$F_n(x) \;:=\; \sum_{i=-\infty}^{\infty}\left[\mathrm{erf}\left(\sqrt{\frac{n}{2}}\frac{x}{B^i}\right) - \mathrm{erf}\left(\sqrt{\frac{n}{2}}\frac{1}{B^i}\right)\right]. \tag{1.11}$$

As $F_n(x) = F_{nB^2}(x)$ for any $n \in \mathbb{N}$, the leading digit law of x_1 in S^k, $k \to \infty$, falls into the periodic cycle of $B^2 - 1$ limiting digit laws F_n, $1 \le n < B^2$ as defined in (1.11).

We plot the behavior for a representative set of n from Theorem 1.7 in Figure 1.

We use Theorem 1.7 to analyze the digits of entries of compact groups G. We shall see in the case when $G = \mathrm{O}_n(\mathbb{R})$ or $\mathrm{U}_n(\mathbb{C})$, $p_{i,j}$ is a projection of G onto the (i, j)-th component and μ is Haar, the digit law of $(G, \mu, p_{i,j})$ is a consequence of digit laws from a point drawn at random from a unit sphere.

Theorem 1.8. *The leading digit law in the (i, j) component of $\mathrm{O}_n(\mathbb{R})$ (or the real or imaginary part of entries in $\mathrm{U}_n(\mathbb{C})$) with respect to Haar measure equals the leading digit law of x_1 in S^{n-1} with respect to the uniform measure.*

In particular, the asymptotic periodicity phenomenon for spheres (Theorem 1.7) is also observed in an entry of $\mathrm{O}_n(\mathbb{R})$; numerical simulations in this case yield identical behavior as in Figure 1.

1.5. **Results (Non-Compact Case).** The following theorems are a representative sample of what can be proved using the local definition (1.3) of digit law.

Theorem 1.9. *Let $G = U$ be the group of real-valued upper triangular matrices:*

$$U \;=\; \left\{ \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ 0 & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}, a_{i,i} \in \mathbb{R}/\{0\} \right\}. \tag{1.12}$$

The leading digit law of $a_{i,i}$ for the left-invariant Haar density dg_L is

- *B-Benford for all bases $B > 1$ when $i = j = 1$,*
- *a (B, k) power law when $i = j = k$ and $2 \leq k \leq n$,*
- *uniform for $1 \leq i < j \leq n$.*

The leading digit law of $a_{i,i}$ for the right-invariant Haar density dg_R is

- *B-Benford for all bases $B > 1$ when $i = j = n$,*
- *a $(B, n - k)$ power law when $i = j = k$ and $2 \leq k \leq n$,*
- *uniform for $1 < i < j \leq n$.*

Theorem 1.10. *Let D be the group of real-valued diagonal matrices:*

$$D \;=\; \left\{ \begin{bmatrix} a_{1,1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_{n,n} \end{bmatrix}, a_{i,i} \in \mathbb{R}/\{0\} \right\}. \tag{1.13}$$

For each i between 1 and n, the leading digit law of $a_{i,i}$ with respect to the bi-invariant Haar density dg is B-Benford for all bases $B > 1$.

Theorem 1.11. *Let D_1 be the group of real-valued, determinant 1 diagonal matrices:*

$$D_1 \;=\; \left\{ \begin{bmatrix} a_{1,1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_{n,n} \end{bmatrix}, \prod a_{i,i} \;=\; 1 \right\}; \tag{1.14}$$

For each i between 1 and n, the leading digit law of $a_{i,i}$ with respect to the bi-invariant Haar density dg is B-Benford for all bases $B > 1$.

Our next result concerns the distribution of digits in a single entry of $\mathrm{SL}_n(\mathbb{R})$. We first set some notation. Denote by $L, U, D_1 \subset G$ the subgroups of unipotent lower triangular matrices, unipotent upper triangular matrices, and the determinant 1 diagonal matrices of $\mathrm{SL}_n(\mathbb{R})$ respectively. Each $g \in G$ can be uniquely expressed as the product $g = lud$ where $l \in L, u \in U, d \in D_1$. Notice that each subgroup L, U, D_1 is topologically closed in $\mathrm{SL}_n(\mathbb{R})$ and hence is a Lie subgroup of G. If $\mathfrak{l}, \mathfrak{u}, \mathfrak{d}_1$ are the Lie algebras of L, U, D_1 respectively, then $\mathfrak{l}, \mathfrak{u}, \mathfrak{d}_1$ have the vector space bases (which we review in Appendix A)

$$\mathfrak{l} \;=\; \mathrm{sp}_{\mathbb{R}}(\{E_{i,j}\}_{i>j}), \quad \mathfrak{u} \;=\; \mathrm{sp}_{\mathbb{R}}(\{E_{i,j}\}_{i<j}), \quad \mathfrak{d}_1 \;=\; \mathrm{sp}_{\mathbb{R}}(E_{i,i} - E_{i+1,i+1})_{1 \leq i \leq n-1}, \tag{1.15}$$

where $E_{i,j}$ is the $n \times n$ matrix with 1 in the (i, j) position and zeroes elsewhere.

Theorem 1.12. *Let $C_c(G)$ be the set of compactly supported continuous functions on G, dg be the normalized Haar density on $\mathrm{SL}_n(\mathbb{R})$, $\phi \in C_c(G)$. Then*

$$\int_G \phi(g)\, dg \;=\; \int_{\mathfrak{l}} \int_{\mathfrak{u}} \int_{D_1} \phi(\exp(X) \exp(Y) a)\, da\, dX\, dY, \tag{1.16}$$

where dX, dY are the Lebesgue measures on $\mathfrak{l}, \mathfrak{u}$, and

$$da \;=\; \prod_{i=1}^{n-1} \frac{da_{i,i}}{a_{i,i}} \tag{1.17}$$

is the Haar density on D_1. Consequently, the joint distribution of diagonal components is a product of B-Benford measures.

The next corollary follows immediately from the bi-invariance of dg on $\mathrm{SL}_n(\mathbb{R})$:

Corollary 1.13. *Let $P, Q \in \mathrm{SL}_n(\mathbb{R})$ be even order permutation matrices. For $A \in \mathrm{SL}_n(\mathbb{R})$, the joint distribution of the diagonal components of PAQ is a product of B-Benford measures.*

In other words, the joint distribution of n components is a product of B-benford measures if there is an even permutation of the rows and columns which sends the n components to the diagonal components. Lastly, we obtain results on the behavior of determinants of matrices from $\mathrm{GL}_n(\mathbb{R})^+$.

Theorem 1.14. *The leading digit law on the determinants of $\mathrm{GL}_n(\mathbb{R})$ is B-Benford.*

For other results related to Benford's law and matrices, see [B–], who prove that as the size of matrices with entries i.i.d.r.v. from a nice fixed distribution tends to infinity, the leading digits of the $n!$ terms in the determinant expansion converges to Benford's law. Also see [BH3] for results arising from powers of fixed matrices.

1.6. **Outline of Paper.** We prove Theorem 1.7 in §2, and give some additional consequences, including Theorem 1.8. We then turn to the non-compact cases in §3. After first proving our results for upper triangular and diagonal matrices, we derive Theorem 1.12 on components of $\mathrm{SL}_n(\mathbb{R})$ in §3.2 (see also Appendix B for a more geometric proof in two dimensions), which we immediately use to deduce the digit law on determinants, Theorem 1.14. We then end with some concluding remarks and thoughts on future research.

2. Proof of Compact Results

2.1. **Preliminaries.** A key ingredient in determining the limiting behavior is Stirling's formula (see [AS]): For z sufficiently large with $|\arg z| < \pi$,

$$\Gamma(z) \;\sim\; e^{-z} z^{z-1/2} (2\pi)^{1/2} (1 + O(1/z)). \tag{2.1}$$

For $r > 0$, recall $S^n(r)$ is the sphere of radius r in $\mathbb{R}^{n+1}$, with $S^n = S^n(1)$ the unit sphere. Denote by $V_n(r)$ and $S_n(r)$ the volume and surface area of the n-sphere.

Lemma 2.1. *Let x_1 be the first component of a point chosen at random from S^n. We have, for $1 \le a \le b \le B$, that*

$$\mathrm{Prob}(a < S_B(x_1) < b) \;=\; \frac{2}{\sqrt{\pi}} \frac{\Gamma(n/2 + 1/2)}{\Gamma(n/2)} \sum_{i=1}^{\infty} \int_{aB^{-i}}^{bB^{-i}} (1 - x_1^2)^{n/2-1}\, dx_1. \tag{2.2}$$

Proof. Pick a point $x \in S^n$ uniformly at random, and let x_1 be the first component of x. We are interested in the leading digit distribution of x_1. By symmetry, the distribution for other components is similar. Notice for $x \in \mathbb{R}^{n+1}$ with first component x_1 that for $0 < a < 1$

$$\{x : x_1 = a\} \cap S^n \;=\; S^{n-1}\left(\sqrt{1 - a^2}\right). \tag{2.3}$$

Approximating the surface area in the strip $\{a < x_1 < b, x \in S^n\}$ by a frustum, it follows for $n > 0$ that

$$\text{Prob}(a < x_1 < b, x \in S^n) \;=\; \frac{\int_a^b \frac{1}{\sqrt{1-x_1^2}} S_{n-1}\left(\sqrt{1-x_1^2}\right) dx_1}{S_n(1)}. \tag{2.4}$$

By the familiar relationship $S_n(r) = V_n'(r) = \frac{n+1}{r} V_n(r)$ and from the closed form solution

$$V_n(r) \;=\; \frac{\pi^{(n+1)/2} r^{n+1}}{\Gamma\left(\frac{n+1}{2}+1\right)}, \tag{2.5}$$

we find

$$\begin{aligned}
\text{Prob}(a < x_1 < b, x \in S^n) \;&=\; \frac{\int_a^b \frac{1}{\sqrt{1-x_1^2}} S_{n-1}\left(\sqrt{1-x_1^2}\right)\, dx_1}{S_n(1)} \\
&=\; \frac{n \int_a^b \frac{1}{1-x_1^2} V_{n-1}\left(\sqrt{1-x_1^2}\right)\, dx_1}{(n+1)V_n(1)} \\
&=\; \frac{1}{\sqrt{\pi}} \frac{n\Gamma(n/2+3/2)}{(n+1)\Gamma(n/2+1)} \int_a^b (1-x_1^2)^{n/2-1}\, dx_1 \\
&=\; \frac{1}{\sqrt{\pi}} \frac{\Gamma(n/2+1/2)}{\Gamma(n/2)} \int_a^b (1-x_1^2)^{n/2-1}\, dx_1.
\end{aligned} \tag{2.6}$$

Next, fix a, b to satisfy $1 \leq a < b \leq B$. By symmetry, we may double the digit distribution in the positive half-space $x_1 > 0$ to get

$$\text{Prob}(a < S_B(x_1) < b, x \in S^n) \;=\; \frac{2}{\sqrt{\pi}} \frac{\Gamma(n/2+1/2)}{\Gamma(n/2)} \sum_{i=1}^{\infty} \int_{a \cdot B^{-i}}^{b \cdot B^{-i}} (1-x_1^2)^{n/2-1}\, dx_1, \tag{2.7}$$

which completes the proof. □

Example 2.2. *We write down the digit distribution of x_1 in S^n for small n. For the circle S^1, we have*

$$\begin{aligned}
\text{Prob}(a < x_1 < b, x \in S^1) \;&=\; \frac{\int_a^b \frac{1}{\sqrt{1-x_1^2}} S_0\left(\sqrt{1-x_1^2}\right) dx_1}{S_1(1)} \\
&=\; \frac{1}{\pi} \int_a^b \frac{1}{\sqrt{1-x_1^2}}\, dx_1 \\
&=\; \frac{\arcsin(b) - \arcsin(a)}{\pi},
\end{aligned} \tag{2.8}$$

so the digit law for S^1 is

$$\text{Prob}(a < S_B(x_1) < b, x \in S^1) \;=\; \frac{2}{\pi} \sum_{i=1}^{\infty} \left(\arcsin\left(\frac{b}{B^i}\right) - \arcsin\left(\frac{a}{B^i}\right) \right). \tag{2.9}$$

The leading digit distribution for S^2 is uniform with respect to any base, made evident from the calculation

$$\begin{aligned}\mathrm{Prob}(a < x_1 < b, x \in S^2) &= \frac{\int_a^b \frac{1}{\sqrt{1-x_1^2}} S_1\left(\sqrt{1-x_1^2}\right) dx_1}{S_2(1)} \\ &= \frac{\int_a^b 2\pi dx_1}{4\pi} \\ &= \frac{b-a}{2}, \qquad (2.10)\end{aligned}$$

which implies

$$\mathrm{Prob}(a < S_B(x_1) < b, x \in S^2) = \frac{b-a}{B-1}. \qquad (2.11)$$

Using Stirling's formula (2.1), we next prove an asymptotic result for the digit distribution on S^n.

Lemma 2.3. *Fix a base $B > 1$ and $1 \le a < b < B$. Let x_1 and x be as in Lemma 2.1. As $n \to \infty$, the difference between $\mathrm{Prob}(a < S_B(x_1) < b)$ and*

$$\sum_{i=1}^{\infty} \frac{2}{\sqrt{\pi}} \int_{\sqrt{\frac{n}{2}}\frac{a}{B^i}}^{\sqrt{\frac{n}{2}}\frac{b}{B^i}} e^{-x^2}\, dx = \sum_{i=1}^{\infty}\left[\mathrm{erf}\left(\sqrt{\frac{n}{2}}\frac{b}{B^i}\right) - \mathrm{erf}\left(\sqrt{\frac{n}{2}}\frac{a}{B^i}\right)\right] \qquad (2.12)$$

tends to zero.

Proof. Let $a, b \in \mathbb{R}$ satisfy $1 \le a < b < B$. From Lemma 2.1 we have

$$\mathrm{Prob}(a < S_B(x_1) < b, x \in S^n) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(n/2 + 1/2)}{\Gamma(n/2)} \sum_{i=1}^{\infty} \int_{a \cdot B^{-i}}^{b \cdot B^{-i}} (1 - x^2)^{n/2-1}\, dx. \qquad (2.13)$$

By Stirling's approximation (2.1)

$$\frac{\Gamma(n/2 + 1/2)}{\Gamma(n/2)} = \sqrt{\frac{n}{2}} + O(1) \qquad (2.14)$$

and the change of variables $x = y\sqrt{2/n}$ and $dx = dy\sqrt{2/n}$ we have

$$\begin{aligned}\mathrm{Prob}(a < S_B(x_1) < b) &= \left(\sqrt{\frac{n}{2}} + O(1)\right) \frac{2}{\sqrt{\pi}} \sum_{i=1}^{\infty} \int_{\sqrt{\frac{n}{2}}\frac{a}{B^i}}^{\sqrt{\frac{n}{2}}\frac{b}{B^i}} \left(1 - \frac{y^2}{n/2}\right)^{n/2-1} \sqrt{\frac{2}{n}}\, dy \\ &= \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right) \frac{2}{\sqrt{\pi}} \sum_{i=1}^{\infty} \int_{\sqrt{\frac{n}{2}}\frac{a}{B^i}}^{\sqrt{\frac{n}{2}}\frac{b}{B^i}} \left(1 - \frac{y^2}{n/2}\right)^{n/2-1} dy \qquad (2.15)\end{aligned}$$

for sufficiently large n. Further, the Dominated Convergence Theorem and the rapid decay of the sum over i shows that the difference between (2.15) and

$$\frac{2}{\sqrt{\pi}} \sum_{i=1}^{\infty} \int_{\sqrt{\frac{n}{2}}\frac{a}{B^i}}^{\sqrt{\frac{n}{2}}\frac{b}{B^i}} e^{-y^2}\, dy \qquad (2.16)$$

tends to zero as $n \to \infty$, completing the proof. □

2.2. **Proofs of Theorems 1.7 and 1.8.** Our first main result now immediately follows.

Proof of Theorem 1.7. Equation (1.10) follows from Lemma 2.3. Straightforward algebra yields (1.11). □

We need a few additional results before proving Theorem 1.8. Lemma 2.3 can be generalized to the first k components of a randomly selected point $x \in S^n \subset \mathbb{R}^{n+1}$. We consider the first k components $x_1, \ldots, x_k$ $(k < n+1)$. Similar to the analysis above, a point $(a_1, a_2, \ldots, a_k)$ which lies in the open unit disk D^k has the remaining $n-k+1$ components lying in a $n-k$ sphere of radius $\sqrt{1-a_1^2-\cdots-a_k^2}$. Rotational symmetry in the $n-k+1$ components affords a parameterization of the surface element dS_n of S^n by D^k as

$$\begin{aligned} dS_n(x_1,\ldots,x_k) &= S_{n-k}\left(\sqrt{1-x_1^2-\cdots-x_k^2}\right) dS_k(x_1,\ldots,x_k) \\ dS_n(x_1,\ldots,x_k) &= S_{n-k}\left(\sqrt{1-x_1^2-\cdots-x_k^2}\right) \frac{1}{\sqrt{1-x_1^2-\cdots-x_k^2}}\, dx \end{aligned} \tag{2.17}$$

where $dx = dx_1 dx_2 \cdots dx_k$.

Lemma 2.4. *For any base B,*

$$\begin{aligned} &\lim_{n\to\infty}\left|\mathrm{Prob}(|a_1| \le |x_1| < |b_1|, \ldots, |a_k| \le |x_k| < |b_k|, x \in S^n) - \prod_{i=1}^{k} \frac{2}{\sqrt{\pi}} \int_{a_i\sqrt{\frac{n}{2}}}^{b_i\sqrt{\frac{n}{2}}} e^{-x^2}\, dx\right| \\ &= 0. \end{aligned} \tag{2.18}$$

Proof. Similar to Lemma 2.3, we may reduce to the case when the bounds a_i, b_i on x_i are positive. By symmetry and substitution of (2.17),

$$\begin{aligned} &\mathrm{Prob}(a_1 < x_1 < b_1, \ldots, a_k < x_k < b_k, x \in S^n) \\ &= 2^k \int_{a_1<x_1<b_1,\ldots,a_k<x_k<b_k} dS(x_1,\ldots,x_k) \\ &= \frac{2^k}{S_n(1)} \int_{a_1}^{b_1} \cdots \int_{a_k}^{b_k} S_{n-k}\left(\sqrt{1-x_1^2-\cdots-x_k^2}\right) \frac{1}{\sqrt{1-x_1^2-\cdots-x_k^2}}\, dx_k \cdots dx_1 \\ &= \frac{2^k(n-k+1)}{(n+1)V_n(1)} \int_{a_1}^{b_1} \cdots \int_{a_k}^{b_k} V_{n-k}\left(\sqrt{1-x_1^2-\cdots-x_k^2}\right) \frac{1}{\sqrt{1-x_1^2-\cdots-x_k^2}}\, dx_k \cdots dx_1 \\ &= \left(\frac{2}{\sqrt{\pi}}\right)^k \frac{\Gamma(n/2+1/2)}{\Gamma(n/2-k/2+1/2)} \int_{a_1}^{b_1} \cdots \int_{a_k}^{b_k} (1-x_1^2-\cdots-x_k^2)^{(n-k-1)/2}\, dx_k \cdots dx_1. \end{aligned} \tag{2.19}$$

Stirling's approximation (2.1) gives

$$\frac{\Gamma(n/2+1/2)}{\Gamma(n/2-k/2+1/2)} = \left(\frac{n}{2}\right)^{k/2} + O(n^{k/2-\epsilon}), \tag{2.20}$$

and the change of variables $x_i = y_i/\sqrt{n/2},\ 1 \le i \le k$ complete the proof. □

Corollary 2.5. *For any base B*

$$\lim_{n\to\infty}\left|\text{Prob}(a_1 < S_B(x_1) < b_1, \dots, a_k \le S_B(x_k) < b_k, x \in S^n) - \prod_{j=1}^{k}\sum_{i=1}^{\infty}\frac{2}{\sqrt{\pi}}\int_{\sqrt{\frac{n}{2}}\frac{a_j}{B^i}}^{\sqrt{\frac{n}{2}}\frac{b_j}{B^i}} e^{-x^2}\,dx\right| = 0. \tag{2.21}$$

In particular, the joint leading digit distribution of the first k components is asymptotically periodic in n, with period B^2, tending to one of the $(B^2-1)^k$ limiting distributions

$$\prod_{j=1}^{k} F_n(x_j) \;=\; \prod_{j=1}^{k}\sum_{i=-\infty}^{\infty}\left[\text{erf}\left(\sqrt{\frac{n}{2}}\frac{x_j}{B^i}\right) - \text{erf}\left(\sqrt{\frac{n}{2}}\frac{1}{B^i}\right)\right] \tag{2.22}$$

with $1 \le n < B^2$.

Next we show that the leading digit distribution in the (i, j) entry of $\text{O}_n(\mathbb{R})$ (w.r.t. Haar measure) is equal to the first entry of the sphere S^{n-1} with respect to uniform. Thus, the asymptotic periodicity phenomenon for spheres (Lemma 2.3, 2.4) appears in the digit laws for a fixed number entries in $\text{O}_n(\mathbb{R})$, so long as all entries lie in the same row or column.

Proof of Theorem 1.8. As $\text{O}_n(\mathbb{R})$ contains every permutation matrix $P \in \text{GL}_n(\mathbb{R})$ there exist permutation matrices $P, Q \in \text{GL}_\text{n}(\mathbb{R})$ such that $PAQ \in \text{O}_n(\mathbb{R})$ sends the (i, j) entry to the $(1, 1)$ entry. By invariance of dg, it suffices to prove the claim for the $(1, 1)$ component of $\text{O}_n(\mathbb{R})$. Recall that any matrix $A \in \text{O}_n(\mathbb{R})$ satisfies $A^T A = I$, so that the columns of A form an orthonormal basis of $\mathbb{R}^n$. We may therefore embed $\text{O}_n(\mathbb{R})$ in the product of n copies of $(n-1)$-spheres $S^{n-1} \times \cdots \times S^{n-1}$. Consider the construction of a matrix in $\text{O}_n(\mathbb{R})$ one column at a time from left to right. The first column c_1 can be selected arbitrarily from S^{n-1}. The second column c_2 is a vector selected in the cross-section formed by the orthogonal plane to c_1 in S^{n-1} and this cross-section is isometric to $S^{n-2} \times \{0\}$. In general, the ith column is selected from the hyperplane orthogonal to vectors $c_1, \dots, c_{i-1}$ in S^{n-1}, a set that is isometric to $S^{n-i} \times \{(0, \dots, 0)\}$ (k times). Since the $\text{O}_n(\mathbb{R})$-action on a subset $A \subset \text{O}_n(\mathbb{R})$ preserves the Haar measure of A, there is a measure-preserving transformation between a basis of the Haar measurable subsets of $\text{O}_n(\mathbb{R})$ and the measurable subsets $A_1 \times A_2 \times \cdots \times A_n \subset S^{n-1} \times S^{n-2} \times \cdots \times S^0$ where each component S^i is equipped with the uniform measure. Therefore, the digit law in the $(1, 1)$ component of $\text{O}_n(\mathbb{R})$ equals the digit law of S^{n-1} with the uniform measure. The leading digit law follows.

Analogous digit laws for the real and imaginary parts in a fixed number of entries in $\text{U}_n(\mathbb{C})$ are immediate, since $\text{U}_n(\mathbb{C})$ contains every permutation matrix and the first column of $\text{U}_n(\mathbb{C})$ is a point on S^{2n-1}. $\square$

3. Proof of Non-Compact Results

3.1. **Proofs for Upper Triangular and Diagonal Matrices.** The starting point of our investigations is the following lemma on the multiplicative group of positive real numbers.

Lemma 3.1. *Let $\mathbb{R}^+$ be the multiplicative group of positive real numbers with Haar density dx/x. Then $(\mathbb{R}^+, dx/x, x)$ is B-Benford for all bases $B > 1$.*

Proof. As the Lie algebra $L(\mathbb{R}^+) = \mathbb{R}$ of $\mathbb{R}^+$ is one dimensional, the perpendicular subspace to $\mathbb{R}$ is $\{0\}$. Thus, for any $s \in [1, B)$, one has $U_\epsilon([0, \log s)X) = [0, \log s)X$, whence (1.6) becomes

$$\text{Prob}(S_B(X) \leq s) \;=\; \lim_{k\to\infty} \frac{k\int_1^s dx/x}{\int_1^{B^k} dx/x} \;=\; \lim_{k\to\infty} \frac{k\log s}{k\log B} \;=\; \log_B s. \tag{3.1}$$

□

Our first three theorems in the non-compact setting follow by applying Lemma 3.1 to non-compact G whose Haar density decompose as a product of densities on the matrix components; the digit laws are then easily determined from the local formulation of digit law (1.6).

Proof of Theorem 1.9. The left-invariant Haar measure on U has density

$$dg_L \;=\; \frac{1}{a_{11}a_{22}^2\cdots a_{nn}^n} \prod_{i<j} da_{ij} \tag{3.2}$$

and the right-invariant Haar measure on U has density

$$dg_R \;=\; \frac{1}{a_{11}^n a_{22}^{n-1}\cdots a_{nn}} \prod_{i<j} da_{ij}, \tag{3.3}$$

where da_{ij} is the Lebesgue density on $\mathbb{R}$ in both cases. All leading digit laws follow. □

Proof of Theorem 1.10. The bi-invariant Haar measure on D is

$$dg \;=\; \frac{1}{a_{1,1}a_{2,2}\cdots a_{nn}}\, da_{1,1}da_{2,2}\cdots da_{n,n}, \tag{3.4}$$

where $da_{i,i}$ is the Lebesgue measure on $\mathbb{R}$. The digit laws follow. □

Proof of Theorem 1.11. D_1 is diffeomorphic to the graph of

$$(a_{11},\ldots,a_{n-1,n-1}) \;\mapsto\; \frac{1}{a_{11}a_{22}\cdots a_{n-1,n-1}} \tag{3.5}$$

and hence is diffeomorphic to an open sub-manifold of $\mathbb{R}^{n-1}$. The bi-invariant Haar measure on D_1 is thus

$$dg \;=\; \frac{1}{a_{11}a_{22}\cdots a_{n-1,n-1}}\, da_{11}da_{22}\cdots da_{nn}, \tag{3.6}$$

where da_{ii} is the Lebesgue measure on $\mathbb{R}$. The digit laws follow. □

The explicit formulations of Haar densities in Theorems 1.9, 1.10, 1.11 can be found in [HR] §15.

3.2. **Proof of Theorems 1.12 and 1.14.** Recall L, U, D_1 are the subgroups of $\mathrm{SL}_n(\mathbb{R})$ of lower triangular, upper triangular, and diagonal determinant 1 matrices, with Lie algebras $\mathfrak{l}, \mathfrak{u}, \mathfrak{d}_1$ respectively (see Appendix A).

Proof of Theorem 1.12. We decompose the density of dg with respect to the matrix decomposition $\mathrm{SL}_n(\mathbb{R}) = LUD_1$. To accomplish this task, we pick $g_0 \in G$ and calculate the Jacobian at 0 under the change to exponential coordinates

$$g(X,Y,Z) \;= g_0 \exp X \exp Y \exp Z \;\; (X \in \mathfrak{l}, Y \in \mathfrak{u}, Z \in \mathfrak{d}_1) \tag{3.7}$$

As a function, g is a local isomorphism from a neighborhood of 0 in $\mathfrak{l}\times\mathfrak{u}\times\mathfrak{d}_1$ onto a neighborhood U of g_0. [VS] §2.10. To calculate the Jacobian we compute directional derivatives. To this end, if we let

$$g(t) \;=\; g(tX,Y,Z) \;= g_0 \exp tX \exp Y \exp Z \tag{3.8}$$

be a curve through g_0 in the direction of $X\in\mathfrak{l}$, then

$$g'(t) \;=\; g_0(\exp tX)X \exp Y \exp Z. \tag{3.9}$$

Therefore

$$\begin{aligned} g^{-1}(t)g'(t) \;&=\; (g_0 \exp tX \exp Y \exp Z)^{-1} g_0(\exp tX)X\exp Y\exp Z \\ &=\; \mathrm{Ad}((\exp Y \exp Z)^{-1})(X) \;=\; e^{-\operatorname{ad} Z}e^{-\operatorname{ad} Y}X \end{aligned} \tag{3.10}$$

Similarly, if

$$h(t) \;=\; h(X,tY,Z) \;=\; g_0 \exp X \exp tY \exp Z \;\; (X\in\mathfrak{l}, Y\in\mathfrak{u}, Z\in\mathfrak{d}_1) \tag{3.11}$$

is a curve through g_0 in the direction of $Y\in\mathfrak{u}$, then

$$\begin{aligned} h(t)^{-1}h'(t) \;&=\; (g_0\exp X \exp tY \exp Z)^{-1} g_0(\exp X)(\exp tY)Y\exp Z \\ &=\; \mathrm{Ad}((\exp Z)^{-1})(Y) \end{aligned} \tag{3.12}$$

Lastly, if $k(t)$ is a curve through g_0 in the direction of Z, then $k(t)^{-1}k'(t)=Z$. By left-invariance of dg, the Jacobian at g_0 with respect to the coordinate bases of $\mathfrak{l},\mathfrak{u},\mathfrak{d}_1$ is given by the block matrix

$$\left[\begin{array}{c|c|c} [\mathrm{Ad}((\exp Y\exp Z)^{-1})(X)]_{\mathfrak{l}} & 0 & 0 \\ \hline * & [\mathrm{Ad}(\exp Z)^{-1}(Y)]_{\mathfrak{u}} & 0 \\ \hline * & * & id_{\mathfrak{d}_1} \end{array}\right], \tag{3.13}$$

where $[\mathrm{Ad}((\exp Y\exp Z)^{-1})(X)]_{\mathfrak{l}}$ are the terms of the vector $\mathrm{Ad}((\exp Z\exp Y)^{-1})(X)$ which lie in the subspace $\mathfrak{l}$ and $[\mathrm{Ad}((\exp Z)^{-1})(X)]_{\mathfrak{u}}$ are the terms of $\mathrm{Ad}((\exp Z)^{-1})(X)$ which lie in $\mathfrak{u}$. It follows that the volume element around g_0 decomposes as

$$dg = |\det \mathrm{Ad}(u^{-1})_{\mathfrak{l}}||\det\mathrm{Ad}(d^{-1})_{\mathfrak{l}}||\det\mathrm{Ad}((d)^{-1})_{\mathfrak{u}}|\; da\; dX\; dY \tag{3.14}$$

with $u=\exp(Y)\in U$, $d=\exp(Z)\in D_1$. Notice that (3.13) independent of g_0. We compute (3.14) explicity, by first observing that $\mathrm{Ad}(d^{-1})$ acts by scalar multiplication on $\mathfrak{sl}_n$. In particular, each matrix $E_{i,j}, i\neq j$ in $\mathfrak{sl}_n$ is an eigenvector of $\mathrm{Ad}(d^{-1})$ with eigenvalue $d_{j,j}/d_{i,i}$. Therefore, in the coordinate basis $\{E_{i,j}\}_{i<j}$ of $\mathfrak{u}$ we have

$$\det \mathrm{Ad}(d^{-1})_{\mathfrak{u}} \;=\; \prod_{1\le i<j\le n}\frac{d_{j,j}}{d_{i,i}}. \tag{3.15}$$

In the coordinate basis $\{E_{i,j}\}_{i>j}$ of $\mathfrak{l}$ we have

$$\det \mathrm{Ad}(d^{-1})_{\mathfrak{l}} \;=\; \prod_{1\le i<j\le n}\frac{d_{i,i}}{d_{j,j}} \tag{3.16}$$

and

$$\det \mathrm{Ad}(u^{-1})_{\mathfrak{l}} \;= id_{\mathfrak{l}}. \tag{3.17}$$

Thus, the decomposition of dg in (3.14) becomes

$$dg = \; da\; dX\; dY \tag{3.18}$$

and the density da was determined in Theorem 1.11, completing the proof of Theorem 1.12. □

Appendix B provides a geometric proof of Theorem 1.12 based on the area of hyperbolic sectors.

Proof of Theorem 1.14. Let $\mathrm{GL}_n(\mathbb{R})^+$ be the group of all invertible $n \times n$ matrices with positive determinant. The map

$$f\colon \mathrm{GL}_n(\mathbb{R})^+ \to \mathbb{R}^+ \times \mathrm{SL}_n(\mathbb{R}) \tag{3.19}$$

given by $f(g) = (\det(g), (\det(g))^{-1/n} g)$ is a Lie isomorphism. Commutativity between the subgroups $\mathbb{R}^+$ (embedded in $\mathrm{GL}_n(\mathbb{R})^+$ as scalar matrices) and $\mathrm{SL}_n(\mathbb{R})$ admits a decomposition of the Haar density (up to positive constant) as $dg = r^{-1} dr dh$ where dh is the Haar density on $\mathrm{SL}_n(\mathbb{R})$. □

4. Conclusions and Future Work

Our results can serve as a means for detecting underlying symmetries of a physical system. For example, imagine we are trying to construct matrices from one of the classical compact groups according to Haar measure (see [Mez] for a description of how to do this). We can use our digit laws as a test of whether or not we are simulating the matrices correctly. Our results should also generalize to other groups of matrices, including those over fields other than the reals. Theorems 1.9, 1.11, 1.10, and 1.12 found digit laws in matrix entries of noncompact Lie groups. A general treatment of digit laws via Haar decompositions should also be possible through the theory of modular functions, which we leave as future research.

Appendix A. Linear Lie groups

For a vector space V over a field F of characteristic 0, a Lie group $G \subset \mathrm{GL}(V)$ is a group equipped with a differentiable structure such that group multiplication and inversion are differentiable. The Lie algebra $L(G)$ may be naturally identified with the tangent space $T_e(G)$ to the identity. The exponential map $\exp : L(G) \to G$ maps a line $tX \in L(G)$, $t \in F, X \in L(G)$ through X to its unique one parameter subgroup $\exp(tX)$. Let E_{ij} be the $n \times n$ matrix with 1 in the (i, j) entry and zeroes elsewhere.

We study the following linear Lie groups.

- The general linear group $\mathrm{GL}_n(\mathbb{R})$ of matrices of nonzero determinant and its Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ of all $n \times n$ matrices.
- The special linear group $\mathrm{SL}_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(V) \mid \det A = 1\}$ and its Lie algebra $\mathfrak{sl}_n(\mathbb{R}) = \{X \in \mathfrak{gl}_n(\mathbb{R}) \mid \mathrm{tr} X = 0\}$ of traceless matrices. The proof of Theorem 1.12 uses the vector space decomposition $\mathfrak{sl}_n = \mathfrak{l} + \mathfrak{d}_1 + \mathfrak{u}$, where

$$\begin{aligned} \mathfrak{l} &= \mathrm{span}_{\mathbb{R}}(\{E_{i,j}\}_{i>j}) \\ \mathfrak{u} &= \mathrm{span}_{\mathbb{R}}(\{E_{i,j}\}_{i<j}) \\ \mathfrak{d}_1 &= \mathrm{span}_{\mathbb{R}}(E_{i,i} - E_{i+1,i+1})_{1 \le i \le n-1}; \end{aligned} \tag{A.1}$$

  here $E_{i,j}$ is the $n \times n$ matrix with 1 in the (i, j) position and zeroes elsewhere.
- The group $\mathrm{D} \subset GL_n(\mathbb{R})$ of diagonal matrices with nonzero diagonal entries and its Lie algebra $\mathfrak{d}$ of diagonal matrices with entries in $\mathbb{R}$.
- The group $\mathrm{D}_1(\mathbb{R}) \subset GL_n(\mathbb{R})$ of diagonal matrices with determinant 1. The Lie algebra of D_1 is comprised of traceless diagonal matrices with entries in $\mathbb{R}$ which we denote by $\mathfrak{d}_1$.
- The group of upper triangular matrices $\mathrm{U}(\mathbb{R}) \subset GL_n(\mathbb{R})$ with nonzero diagonal entries and its Lie algebra $\mathfrak{u}$ of upper triangular matrices with entries in $\mathbb{R}$.

- The space of lower triangular matrices $\mathrm{L}(\mathbb{R}) \subset GL_n(\mathbb{R})$ with nonzero diagonal entries and its Lie algebra $\mathfrak{l}$ of lower triangular matrices with entries in $\mathbb{R}$.
- The orthogonal group: $\mathrm{O}_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) \mid A^T A = I\}$ and its Lie algebra $\mathfrak{o}_n(\mathbb{R}) = \{X \in M_n(\mathbb{R}) \mid F^T + F = 0\}$ of skew symmetric matrices.
- The unitary group $\mathrm{U}_n(\mathbb{C}) = \{U \in \mathrm{GL}_n(\mathbb{C}) \mid U^* U = I\}$ and its Lie algebra $\mathfrak{u}_n = \{W \in M_n(\mathbb{C}) \mid W + W^* = 0\}$ of skew-Heritian matrices.

The complex lie groups $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{O}_n(\mathbb{C})$, $\mathrm{U}(\mathbb{C})$, $\mathrm{L}(\mathbb{C})$, $\mathrm{D}(\mathbb{C})$, $\mathrm{D}_1(\mathbb{C})$ are defined analogously.

Appendix B. Haar measure on $\mathrm{SL}_2(\mathbb{R})$ is B-Benford in each component

.

The goal of this section is to provide a geometric proof of Theorem 1.12 in two dimensions. We start with a useful, classical result.

Lemma B.1. *The area of the hyperbolic cone*

$$C([a,b])\{(t, t/x) : t \in [0,1],\ 0 < a \leq x \leq b\} \tag{B.1}$$

is equal to $\log(b) - \log(a)$.

Proof. The region under the curve $1/x$ has area $\log(b) - \log(a) = \log(b/a)$, and one can form the sector from this region by first attaching the triangle with corners $(0,0), (a,0), (a,1/a)$ and then removing the triangle with corners $(0,0), (b,0), (b,1/b)$. Both triangles have area $1/2$. □

As a quick corollary to Lemma B.1, we determine the leading digit law for the hyperbola $v^2 - w^2 = 1$ in each coordinate. The measure of a hyperbolic arc S assigns the area of the cone on S.

Corollary B.2. *For the hyperbola $v^2 - w^2 = 1$ in $\mathbb{R}^2$ we have*

$$\mathrm{Prob}(a < v < b) = \frac{1}{2}\left(\log\left(\frac{b + \sqrt{b^2-1}}{a + \sqrt{a^2-1}}\right) + \log\left(\frac{a - \sqrt{a^2-1}}{b - \sqrt{b^2-1}}\right)\right) \quad (1 \leq a < b < B), \tag{B.2}$$

and

$$\mathrm{Prob}(a < w < b) = \frac{1}{2}\left(\log\left(\frac{b + \sqrt{b^2+1}}{a + \sqrt{a^2+1}}\right) + \log\left(\frac{-a + \sqrt{a^2+1}}{-b + \sqrt{b^2+1}}\right)\right) \quad (1 \leq a < b < B). \tag{B.3}$$

Consequently, the digit law in the u^{th} and v^{th} coordinates are B-Benford for all bases $B > 1$.

Proof. Under the change of coordinates $v = x + y, w = x - y$, the hyperbola is the graph of $y = 1/(4x)$. By Lemma B.1, the measure of the hyperbolic arcs lying in the region $-a < v < a$ is easily determined to be

$$\frac{1}{2}\left(\log(a + \sqrt{a^2-1}) - \log(a - \sqrt{a^2-1})\right). \tag{B.4}$$

Similarly, the measure of the two hyperbolic arcs lying in the region $-a < w < a$ is

$$\frac{1}{2}\left(\log(a + \sqrt{a^2+1}) - \log(-a + \sqrt{a^2+1})\right), \tag{B.5}$$

and the first part of the corollary follows. We now show that the digit law in the v^{th} coordinate is B-Benford for all bases $B > 1$; the hyperbola relation immediately yields the claim for the other coordinate. From Definition 1.3 we see

$$\operatorname{Prob}(a < S_B(v) < b) \;=\; \lim_{k\to\infty} \frac{\sum_{\ell=1}^{k} \frac{1}{2}\left(\log\left(\frac{b\cdot B^\ell + \sqrt{(b\cdot B^\ell)^2-1}}{a\cdot B^k + \sqrt{(a\cdot B^\ell)^2-1}}\right) + \log\left(\frac{a\cdot B^\ell - \sqrt{(a\cdot B^\ell)^2-1}}{b\cdot B^\ell - \sqrt{(b\cdot B^\ell)^2-1}}\right)\right)}{\sum_{\ell=1}^{k} \frac{1}{2}\left(\log\left(\frac{B^{\ell+1} + \sqrt{(B^{2\ell+2})-1}}{B^\ell + \sqrt{B^{2\ell}-1}}\right) + \log\left(\frac{B^\ell - \sqrt{B^{2\ell}-1}}{B^{\ell+1} - \sqrt{B^{2\ell+2}-1}}\right)\right)}. \tag{B.6}$$

Notice the summand in the numerator of (B.6) converges to $\log(b/a)$ as $k \to \infty$ and the denominator converges to $\log(B)$ as $k \to \infty$. Thus the limit of the ratio of the sums converges to $\log_B(b/a)$, implying the v^{th} coordinate is B-Benford for all bases $B > 1$. □

Lemma B.1 states that the Haar measure of a set $A \subset \mathbb{R}^+$ is equal to the area of the cone $C(A)$ on A. Generalizing this observation to $\mathrm{SL}_2(\mathbb{R})$ forms the basis of the proof of our next result.

Theorem B.3. *The* $(1,1)$ *component of* $\mathrm{SL}_2(\mathbb{R})$ *with Haar measure is* B*-Benford.*

Proof. Let μ be a Haar measure on $\mathrm{SL}_2(\mathbb{R})$, λ the Lebesgue measure on $\mathbb{R}^4$. Write $\mathrm{SL}_2(\mathbb{R})$ as

$$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;:\; ad - bc = 1 \right\}. \tag{B.7}$$

Treat an element

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

as a point on the graph $d = (1+bc)/a$. Given a Haar measurable subset $A \subset \mathrm{SL}_n(\mathbb{R})$ construct the cone on A, $C(A) \subset \mathbb{R}^4$, by

$$C(A) \;=\; \{tx \mid t \in [0,1], x \in A\}; \tag{B.8}$$

$C(A)$ is Lebesgue measurable. By embedding $\mathbb{R}^4$ as 2×2 square matrices one sees that the $\mathrm{SL}_2(\mathbb{R})$-action on $\mathbb{R}^4$ leaves λ invariant. By uniqueness of Haar (see §15 of [HR]) the Haar measure of $A \subset \mathrm{SL}_n(\mathbb{R})$ equals the volume of the cone $C(A) \subset \mathbb{R}^4$ up to positive constant. We give a series of statements that simplify the proof but create no loss of generality. Clearly $(G, \mu, p_{1,1})$ is B-Benford if and only if $(G, c\mu, p_{1,1})$ is B-Benford (for every base $B > 1$) in the $(1,1)$ entry. We take the Haar measure on $\mathrm{SL}_2(\mathbb{R})$ as the Lebesgue measure on cones $C(A) \subset \mathbb{R}^4$. Let $a_{11} = a$; notice that $a = 0$ is a zero measure subset of μ. We treat a matrix element $x \in \mathrm{SL}_2(\mathbb{R})$ as a point on the graph of $d = (1+bc)/a$. By symmetry, it suffices to prove the theorem when $A \subset SL_n(\mathbb{R})^+$. We may further restrict A to lie on the graph of $d = (1+bc)/a$ defined over a rectangular domain $D = [1, x) \times [-\epsilon, \epsilon] \times [-\epsilon, \epsilon]$. Up to positive constant, $\mu(A) = \mu(\mathrm{graph}(d)) = \lambda(C(\mathrm{graph}(d)))$ is the volume of the cone consisting of all line segments between the origin and points on the graph of d. Consider the solid $S := S(\mathrm{graph}(d))$ bounded below by the graph of d whose volume is

$$\lambda(S) \;=\; \iiint_D \left(\frac{1+bc}{a}\right) \, da \, db \, dc. \tag{B.9}$$

We wish to relate $\lambda(C(\mathrm{graph}(d)))$ to $\lambda(S(\mathrm{graph}(d)))$. By our restriction to positive coordinates, we see that d is decreasing along each ray emanating from the origin in a direction of D. As we are assuming $\mathrm{graph}(d) > 0$ on D, $\lambda(C(\mathrm{graph}(d))$ can be found by appending to S the three pyramidal regions whose bases are the three 3-dimensional facets of S, given by

$$S \cap \{a = 1\}, S \cap \{b = -\epsilon\}, S \cap \{c = -\epsilon\}, \tag{B.10}$$

then removing the pyramids whose bases are the three facets

$$S \cap \{a = x\}, S \cap \{b = \epsilon\}, S \cap \{c = \epsilon\}. \tag{B.11}$$

The apex for all 6 pyramids is the origin. Thus

$$\begin{aligned}\lambda(C(\mathrm{graph}(d))) &= \lambda(S) + \lambda(C(S \cap \{a = 1\})) - \lambda(C(S \cap \{a = x\})) \\ &\quad + \lambda(C(S \cap \{b = -\epsilon\})) - \lambda(C(S \cap \{b = \epsilon\})) \\ &\quad + \lambda(C(S \cap \{c = -\epsilon\})) - \lambda(C(S \cap \{c = \epsilon\})). \end{aligned} \tag{B.12}$$

Recall that the 4-dimensional volume of a pyramid is $1/4$ the volume of the base times the height of its perpendicular, and the volume of the base of each pyramid is simply the double integral over the appropriate slice. Thus

$$\begin{aligned}\lambda(C(\mathrm{graph}(d))) \;=\; & \iiint_D \frac{1+bc}{a}\, da\, db\, dc \\ &+\; \frac{1}{4}\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon} \frac{1+bc}{1}\, dc\, db - \frac{x}{4}\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon} \frac{1+bc}{x}\, dc\, db \\ &+\; \frac{\epsilon}{4}\int_{1}^{x}\int_{-\epsilon}^{\epsilon} \frac{1-\epsilon c}{a}\, dc\, da - \frac{\epsilon}{4}\int_{1}^{x}\int_{-\epsilon}^{\epsilon} \frac{1+\epsilon c}{a}\, dc\, da \\ &+\; \frac{\epsilon}{4}\int_{1}^{x}\int_{-\epsilon}^{\epsilon} \frac{1-b\epsilon}{a}\, db\, da - \frac{\epsilon}{4}\int_{1}^{x}\int_{-\epsilon}^{\epsilon} \frac{1+b\epsilon}{a}\, db\, da. \end{aligned} \tag{B.13}$$

Of the seven terms listed in (B.13), notice that the second and third terms cancel. The five integrals that remain are separable, with the same limits of integration on a. Further, the fourth and sixth terms are equal as are the fifth and seventh terms. Therefore, if we let $F(\epsilon)$ be the quantity

$$\begin{aligned} F(\epsilon) &= \int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon} (1+bc)\, db\, dc \\ &\quad + \frac{1}{4}\left(\int_{-\epsilon}^{\epsilon} (\epsilon(1-\epsilon c) - \epsilon(1+\epsilon c))\, dc\right) \\ &\quad + \frac{1}{4}\left(\int_{-\epsilon}^{\epsilon} (\epsilon(1-b\epsilon) - \epsilon(1+b\epsilon))\, db\right), \end{aligned} \tag{B.14}$$

then

$$\mu(\exp(U_\epsilon([0,x)X))) \;=\; \lambda(C(\mathrm{graph}(d))) \;= F(\epsilon)\int_1^x \frac{1}{a}\, da \;=\; \log(x)F(\epsilon). \tag{B.15}$$

By Definition 1.3,

$$\mathrm{Prob}(S_B(a) < x) \;=\; \frac{\log(x)F(\epsilon)}{\log(B)F(\epsilon)} \;=\; \log_B(x), \tag{B.16}$$

which is independent of ϵ. Letting $\epsilon \to 0$ proves the theorem. □

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