FROM FIBONACCI NUMBERS TO CENTRAL LIMIT TYPE THEOREMS

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Abstract

A beautiful theorem of Zeckendorf states that every integer can be written uniquely as a sum of non-consecutive Fibonacci numbers $\{F_n\}_{n=1}^{\infty}$. Lekkerkerker [Lek] proved the average number of summands for integers in $[F_n, F_{n+1})$ is $n/(\varphi^2+1)$, with φ the golden mean. This has been generalized: given nonnegative integers $c_1, c_2, \dots, c_L$ with $c_1, c_L > 0$ and recursive sequence $\{H_n\}_{n=1}^{\infty}$ with $H_1 = 1$, $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1$ $(1 \le n < L)$ and $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n+1-L}$ $(n \ge L)$, every positive integer can be written uniquely as $\sum a_i H_i$ under natural constraints on the a_i's, the mean and variance of the numbers of summands for integers in $[H_n, H_{n+1})$ are of size n, and as $n \to \infty$ the distribution of the number of summands converges to a Gaussian. Previous approaches used number theory or ergodic theory. We convert the problem to a combinatorial one. In addition to re-deriving these results, our method generalizes to other problems (in the sequel paper [BGM] we show how this perspective allows us to determine the distribution of gaps between summands). For example, it is known that every integer can be written uniquely as a sum of the $\pm F_n$'s, such that every two terms of the same (opposite) sign differ in index by at least 4 (3). The presence of negative summands introduces complications and features not seen in previous problems. We prove that the distribution of the numbers of positive and negative summands converges to a bivariate normal with computable, negative correlation, namely $-(21 - 2\varphi)/(29 + 2\varphi) \approx -0.551058$.

1. Introduction

1.1. **History.** The Fibonacci numbers have intrigued mathematicians for hundreds of years. One of their most interesting properties is the Zeckendorf decomposition. Zeckendorf [Ze] proved that every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers (called the *Zeckendorf decomposition*), where the Fibonacci numbers[1] are $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5$, $\dots$. Lekkerkerker [Lek] extended this result and proved that the average number of summands needed to represent an integer in $[F_n, F_{n+1})$ is $\frac{n}{\varphi^2+1} + O(1) \approx 0.276n$, where $\varphi = \frac{\sqrt{5}+1}{2}$ is the golden mean. There is a related question: *how are the number of summands distributed about the mean for integers in* $[F_n, F_{n+1})$*?* This is a very natural question to ask. Both the question and the answer are reminiscent of the

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[1] If we use the standard counting, 1 appears twice and we lose uniqueness.

Erdős-Kac Theorem [EK], which states that as $n \to \infty$ the number of distinct prime divisors of integers on the order of size n tends to a Gaussian with mean $\log\log n$ and standard deviation $\sqrt{\log\log n}$.

We first set some notation before describing the previous results.

Definition 1.1. *We say a sequence $\{H_n\}_{n=1}^{\infty}$ of positive integers is a* ***Positive Linear Recurrence Sequence (PLRS)*** *if the following properties hold:*

(1) Recurrence relation: *There are non-negative integers $L, c_1, \dots, c_L$ such that*

$$H_{n+1} \;=\; c_1 H_n + \cdots + c_L H_{n+1-L},$$

with L, c_1 and c_L positive.

(2) Initial conditions: *$H_1 = 1$, and for $1 \le n < L$ we have*

$$H_{n+1} \;=\; c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1.$$

We call a decomposition $\sum_{i=1}^{m} a_i H_{m+1-i}$ of a positive integer N (and the sequence $\{a_i\}_{i=1}^{m}$) ***legal*** *if $a_1 > 0$, the other $a_i \ge 0$, and one of the following two conditions holds:*

Condition 1. *We have $m < L$ and $a_i = c_i$ for $1 \le i \le m$.*

Condition 2. *There exists $s \in \{1, \dots, L\}$ such that*

$$a_1 \;=\; c_1,\ a_2 \;=\; c_2,\ \cdots,\ a_{s-1} \;=\; c_{s-1} \text{ and } a_s < c_s, \tag{1.1}$$

$a_{s+1}, \dots, a_{s+\ell} \;=\; 0$ for some $\ell \ge 0$, and $\{b_i\}_{i=1}^{m-s-\ell}$ (with $b_i = a_{s+\ell+i}$) is legal.

If $\sum_{i=1}^{m} a_i H_{m+1-i}$ is a legal decomposition of N, we define the ***number of summands*** *(of this decomposition of N) to be $a_1 + \cdots + a_m$.*

Informally, a legal decomposition is one where we cannot use the recurrence relation to replace a linear combination of summands with another summand, and the coefficient of each summand is appropriately bounded; other authors [DG, Ste1] use the phrase G-ary decomposition for a legal decomposition, and sum-of-digits function for the number of summands. For example, if $H_{n+1} = 2H_n + 3H_{n-1} + H_{n-2}$, then $H_5 + 2H_4 + 3H_3 + H_1$ is legal, while $H_5 + 2H_4 + 3H_3 + H_2$ is not (we can replace $2H_4 + 3H_3 + H_2$ with H_5), nor is $7H_5 + 2H_2$ (as the coefficient of H_5 is too large).

The following probabilistic language will be convenient for stating some of the results.

Definition 1.2 (Associated Probability Space to a Positive Linear Recurrence Sequence)**.** *Let $\{H_n\}$ be a PLRS. For each n, consider the discrete outcome space $\Omega_n = \{H_n,\ H_n + 1,\ H_n + 2,\ \dots,\ H_{n+1} - 1\}$ with probability measure $\mathbb{P}_n(A) = \sum_{\omega \in A} \frac{1}{H_{n+1} - H_n}$ $(A \subset \Omega_n)$; in other words, each of the $H_{n+1} - H_n$ numbers is weighted equally. We define the random variable K_n by setting $K_n(\omega)$ equal to the number of summands of $\omega \in \Omega_n$ in its legal decomposition. Implicit in this definition is that each integer has a unique legal decomposition; we prove this in Theorem 1.1, and thus K_n is well-defined. We denote the cardinality of Ω_n by $\Delta_n = H_{n+1} - H_n$, and we set $p_{n,k}$ equal to the number of elements in $[H_n, H_{n+1})$ whose generalized Zeckendorf decomposition has exactly k summands; thus $p_{n,k} = \Delta_n \cdot \mathrm{Prob}(K_n = k)$.*

We first review previous results and methods, and then describe our new perspective and extensions. See [BCCSW, Ho, Ke, Len] for more on generalized Zeckendorf decompositions, [GT] for a proof of Theorems 1.1 and 1.2, and [DG, FGNPT, GTNP, LT, Ste1] for a proof and some generalizations of Theorem 1.3.

Theorem 1.1 (Generalized Zeckendorf's Theorem for PLRS)**.** *Let* $\{H_n\}_{n=1}^{\infty}$ *be a* Positive Linear Recurrence Sequence*. Then*

(a) *There is a unique legal decomposition for each positive integer* $N \geq 0$*.*

(b) *There is a bijection between the set* $\mathcal{S}_n$ *of integers in* $[H_n, H_{n+1})$ *and the set* $\mathcal{D}_n$ *(of cardinality* D_n*) of legal decompositions* $\sum_{i=1}^{n} a_i H_{n+1-i}$*.*

Theorem 1.2 (Generalized Lekkerkerker's Theorem for PLRS)**.** *Let* $\{H_n\}_{n=1}^{\infty}$ *be a* Positive Linear Recurrence Sequence*, let* K_n *be the random variable of Definition 1.2 and denote its mean by* μ_n*. Then there exist constants* $C > 0$*,* d *and* $\gamma_1 \in (0,1)$ *depending only on* L *and the* c_i*'s in the recurrence relation of the* H_n*'s such that*

$$\mu_n \ = \ Cn + d + o(\gamma_1^n). \tag{1.2}$$

Theorem 1.3 (Gaussian Behavior for PLRS)**.** *Let* $\{H_n\}_{n=1}^{\infty}$ *be a PLRS and let* K_n *be the random variable of Definition 1.2. The mean* μ_n *and variance* σ_n^2 *of* K_n *grow linearly in* n*, and* $(K_n - \mu_n)/\sigma_n$ *converges weakly to the standard normal* $N(0,1)$ *as* $n \to \infty$*.*

While the proof of Theorem 1.3 is technical in general, the special case $L = 1$ is straightforward, and suggests why the result holds. When $L = 1$, $H_n = c_1^{n-1}$. Thus our PLRS is just the geometric series $1, c_1, c_1^2, \dots,$ and a legal decomposition is just a base c_1 expansion. Hence every positive integer has a unique legal decomposition. Further, the distribution of the number of summands converges to a Gaussian by the Central Limit Theorem, as we essentially have the sum of $n-1$ independent, identically distributed discrete uniform random variables.

Previous approaches used number theory or ergodic theory, often requiring the analysis of certain exponential sums. We recast this as a combinatorial problem. We are able to re-derive the above results from a different perspective. Our method generalizes to other problems (in a sequel paper [BGM] we use the combinatorial vantage to determine the distribution of gaps between summands). For the main part of this paper, we concentrate on one particularly interesting situation where features not present in previous works arise.

Definition 1.4. *We call a sum of the* $\pm F_n$*'s a* ***far-difference representation*** *if every two terms of the same sign differ in index by at least 4, and every two terms of opposite sign differ in index by at least 3.*

Recently Alpert [Al] proved the analogue of Zeckendorf's Theorem for the far-difference representation. It is convenient to set

$$S_n \ = \ \begin{cases} \sum_{0 < n-4i \leq n} F_{n-4i} \ = \ F_n + F_{n-4} + F_{n-8} + \cdots & \text{if } n > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

Theorem 1.5 (Generalized Zeckendorf's Theorem for Far-Difference Representations)**.** *Every integer has a unique far-difference representation. For each* $N \in (S_{n-1} = F_n - S_{n-3} - 1, S_n]$*, the first term in its far-difference representation is* F_n*, and the unique far-difference representation of 0 is the empty representation.*

Most previous results concern only one quantity, the number of summands. An exception is [Ste2], where the standard Zeckendorf expansion (called the greedy expansion) and the lazy expansion (which uses as many summands as possible) are simultaneously considered. Steiner proves their joint distribution converges to a bivariate Gaussian with a correlation of $9 - 5\varphi \approx .90983$. Unlike the Zeckendorf expansions, the far-difference representations have both positive and negative summands, opening up the fascinating question of how the number of each are related. We find a non-zero correlation between the two types of summands.

Theorem 1.6 (Generalized Lekkerkerker's Theorem and Gaussian Behavior for Far-Difference Representations)**.** *Let $\mathcal{K}_n$ and $\mathcal{L}_n$ be random variables denoting the number of positive and negative summands in the far-difference representation for integers in $(S_{n-1}, S_n]$. As $n \to \infty$, $\mathbb{E}[\mathcal{K}_n] = \frac{1}{10}n + \frac{371-113\sqrt{5}}{40} + o(1)$, and is $\frac{\sqrt{5}+1}{4} = \frac{\varphi}{2}$ greater than $\mathbb{E}[\mathcal{L}_n]$; the variance of both is of size $\frac{15+21\sqrt{5}}{1000}n$ and the joint distribution of the standardized random variables converges weakly to a bivariate Gaussian with negative correlation $\frac{10\sqrt{5}-121}{179} = -\frac{21-2\varphi}{29+2\varphi} \approx -0.551$; and $\mathcal{K}_n + \mathcal{L}_n$ and $\mathcal{K}_n - \mathcal{L}_n$ are independent.*

1.2. **Sketch of Proofs.** By recasting the problem as a combinatorial one and using generating functions, we are able to re-derive and extend the previous results in the literature. The key techniques in our proof are generating functions, partial fractional expansions, differentiating identities and the method of moments. Unfortunately, in order to be able to handle a general Positive Linear Recurrence Sequence, the arguments become quite technical due to the fact that we cannot exploit any special properties of the coefficients of the recurrence relations, but rather must prove certain technical lemmas for *any* choice of the c_i's. We therefore quickly look at the special case of the Fibonacci numbers, as this highlights the main ideas of the method without many of the technicalities.[2] *In the rest of the paper, we provide details only for the results about far-difference representations, as the other results have been proved by other techniques. The reader interested in the details of applying our method to the known cases, or some of the standard algebra omitted below, should see* [MW] *for the details.*

We first derive a recurrence relation for the $p_{n,k}$'s, which in this case is the number of integers in $[F_n, F_{n+1})$ with precisely k summands in their legal decomposition (see Definition 1.2). We find $p_{n+1,k+1} = p_{n,k+1} + p_{n,k}$. Multiplying both sides of this equation by $x^k y^n$, summing over $n, k > 0$, and calculating the initial values of the $p_{n,k}$'s, namely $p_{1,1}$, $p_{2,1}$ and $p_{2,2}$, we obtain a formula for the generating function $\sum_{n,k>0} p_{n,k} x^k y^n$:

$$\mathscr{G}(x,y) \ := \ \sum_{n,k>0} p_{n,k} x^k y^n \ = \ \frac{xy}{1-y-xy^2}. \tag{1.4}$$

By partial fraction expansion, we write the right-hand side as

$$-\frac{y}{y_1(x)-y_2(x)}\left(\frac{1}{y-y_1(x)} - \frac{1}{y-y_2(x)}\right),$$

where $y_1(x)$ and $y_2(x)$ are the roots of $1 - y - xy^2 = 0$. Rewriting $\frac{1}{y-y_i(x)}$ as $-(1-\frac{y}{y_i(x)})^{-1}$ and using a power series expansion, we are able to compare the coefficients of y^n of both sides of (1.4). This gives an explicit formula for $g(x) = \sum_{k>0} p_{n,k} x^k$. Note that

$$g(1) \ = \ \sum_{k>0} p_{n,k},$$

[2] The proof can be simplified further for the Fibonacci numbers, as the key quantity $p_{n,k}$ equals $\binom{n-k}{k-1}/F_{n-1}$, which by Stirling's formula tends to the density of a normal random variable; see [KKMW] for details. Unfortunately this approach does not generalize, as the formulas for $p_{n,k}$ become far more involved.

which is $F_{n+1} - F_n$ by definition. Further, we have

$$g'(1) \;=\; \sum_{k>0} k p_{n,k} \;=\; \mathbb{E}[K_n](F_{n+1} - F_n) \;=\; \mathbb{E}[K_n] g(1).$$

Therefore, once we determine $g(1)$ and $g'(1)$, we know $\mathbb{E}[K_n]$.

Letting $\mu_n = \mathbb{E}[K_n]$, we define the random variable $K_n' = K_n - \mu_n$. We immediately obtain an explicit, closed form expression for $h_n(x) = g(x) - \mu_n$. Arguing as above we find $h_n(1) = F_{n+1} - F_n$ and $h_n'(1) = \mathbb{E}[K_n'] h_n(1)$. Furthermore, we get

$$(x h_n'(x))' \;=\; \mathbb{E}[{K_n'}^2] h_n(1), \quad \left(x\,(x h_n'(x))'\right)' \;=\; \mathbb{E}[{K_n'}^3] h_n(1), \quad \dots, \tag{1.5}$$

which allows us to compute the moments of K_n'.

Let σ_n denote the variance of K_n (which is of course also the variance of K_n'), and recall that the $2m^{\text{th}}$ moment of the standard normal is $(2m-1)!! = (2m-1)(2m-3)\cdots 1$. To show that K_n converges to being normally distributed with mean μ_n and variance σ_n, it suffices to show that the $2m^{\text{th}}$ moment of K_n'/σ_n converges to $(2m-1)!!$ and the odd moments converge to 0. We are able to prove this through (1.5), which are repeated applications of differentiating identities to our partial fraction expansion of the generating function.

We prove the Gaussian behavior for the far-difference representation in Section 2. We conclude with some natural problems to consider.

2. Far-difference Representation

We now apply the generating function approach to study the distributions of the numbers of positive and negative summands in the far-difference representation of integers (see Definition 1.4), proving that as $n \to \infty$ these two random variables converge to a bivariate Gaussian with a computable, negative correlation. We do not need to prove that a generalization of Zeckendorf's theorem holds for far-difference representations, as this was done by Alpert [Al] (see Theorem 1.5).

2.1. **Generating Function of the Probability Density.** Let $p_{n,k,l}$ $(n > 0)$ be the number of far-difference representations of integers in $(S_{n-1}, S_n]$ with k positive summands and l negative summands, and set $D_n = S_n - S_{n-1}$. We have the following formula for the generating function $\hat{\mathscr{G}}(x,y,z) = \sum_{n>0,k>0,l\geq 0} p_{n,k,l} x^k y^l z^n$.

Theorem 2.1. *We have*

$$\hat{\mathscr{G}}(x,y,z) \;=\; \frac{xz + xyz^4}{1 - z - (x+y)z^4 - xyz^6 - xyz^7}. \tag{2.1}$$

Proof. We first derive the recurrence relation

$$p_{n,k,l} \;=\; p_{n-1,k,l} + p_{n-4,k-1,l} + p_{n-3,l,k-1},\ n \geq 5, \tag{2.2}$$

by a combinatorial approach. Next we get the generating function. To achieve that, we need a recurrence relation with all terms of form $p_{n-n0,k-k_0,l-l_0}$ with n_0, k_0 and l_0 constant. We solve this by using the proceeding recurrence relation with repeated substitutions.

Let us prove (2.2) first. Clearly, $p_{n,k,l} = 0$ if $k \leq 0$ or $l < 0$. For every far-difference representation $N = \sum_{j=1}^m a_j F_{i_j} \in [S_{n-1}+1, S_n]$, $N' := \sum_{j=2}^m a_j F_{i_j}$ is also a far-difference representation. Theorem 1.5 states that $i_1 = n$ and $a_1 = 1$, therefore $N' \in [S_{n-1}+1-F_n, S_n -$

$F_n]$. From (1.3) and the recurrence for the F_n's it readily follows that $S_{n-1}+1-F_n=-S_{n-3}$. Thus $p_{n,k,l}$ is the number of far-difference representations of integers in $[-S_{n-3},S_{n-4}]$ with $k-1$ positive summands and l negative summands.

Let $n\geq 5$. We have two cases: $(k-1,l)\neq(0,0)$ and $(k-1,l)=(0,0)$. We do the first as the second follows similarly. Since $(k-1,l)\neq(0,0)$, $N'=N-a_1F_{i_1}\neq 0$. Let $N(J,k,l)$ be the number of far-difference representations of integers in the interval J with k positive summands and l negative summands. Thus

$$\begin{aligned} p_{n,k,l} &= N((0,S_{n-4}],k-1,l)+N([-S_{n-3},0),k-1,l) \\ &= N((0,S_{n-4}],k-1,l)+N((0,S_{n-3}],l,k-1) \ = \ \sum_{i=1}^{n-4}p_{i,k-1,l}+\sum_{i=1}^{n-3}p_{i,l,k-1}, \end{aligned}$$

which implies the claim by telescoping.

Let $n\geq 9$. By further straightforward manipulations (see [MW] for the details) we find

$$\begin{aligned} p_{n,k,l} &= 2p_{n-1,k,l}-p_{n-2,k,l}+p_{n-4,k-1,l}+p_{n-4,k,l-1}-p_{n-5,k-1,l} \\ &\quad -p_{n-5,k,l-1}+p_{n-6,k-1,l-1}-p_{n-8,k-1,l-1},\ \ n\geq 9. \end{aligned} \tag{2.3}$$

The claimed formula for $\hat{\mathscr{G}}(x,y,z)$ now follows by straightforward algebra. □

To show that $\mathcal{K}_n$ and $\mathcal{L}_n$ are asymptotically bivariate Gaussian, it suffices to prove the Gaussian behavior of $a\mathcal{K}_n+b\mathcal{L}_n$ for any a, b with $(a,b)\neq(0,0)$. Note that the coefficient of z^n in $\hat{\mathscr{G}}(x,y,z)$ is $\sum_{k>0,l\geq 0}p_{n,k,l}x^ky^l$; we denote this by $\langle z^n\rangle\hat{\mathscr{G}}(x,y,z)$. Setting $(x,y)=(w^a,w^b)$ and using differentiating identities will give the moments of $a\mathcal{K}_n+b\mathcal{L}_n$.

We first prove a generalized Lekkerkerker's Theorem and Gaussian behavior for $a\mathcal{K}_n+b\mathcal{L}_n$, which is a slight generalization of Theorem 1.6. This suffices to deduce Theorem 1.6 as $\mathrm{cov}(\mathcal{K}_n,\mathcal{L}_n)=\frac14\mathrm{var}(\mathcal{K}_n+\mathcal{L}_n)-\frac14\mathrm{var}(\mathcal{K}_n-\mathcal{L}_n)$.

Theorem 2.2. *For any real numbers $(a,b)\neq(0,0)$, we have*

(a) *The mean of $a\mathcal{K}_n+b\mathcal{L}_n$ is*

$$\frac{a+b}{10}n+\frac{371-113\sqrt{5}}{40}\ a+\frac{361-123\sqrt{5}}{40}\ b+o(\hat{\gamma}_{a,b}^n)\ \text{for some}\ \hat{\gamma}_{a,b}\in(0,1), \tag{2.4}$$

(b) *The variance of $a\mathcal{K}_n+b\mathcal{L}_n$ is*

$$\frac{\sqrt{5}-1}{200}\left[10\left(a^2+b^2\right)-\frac{20-\sqrt{5}}{5}(a+b)^2\right]n+q_{a,b}+o(\hat{\tau}_{a,b}^n)\ \text{for some}\ \hat{\tau}_{a,b}\in(0,1), \tag{2.5}$$

with $q_{a,b}$ constant depending on only a and b; further, the joint distribution of the standardized random variables of $a\mathcal{K}_n+b\mathcal{L}_n$ converges weakly to a Gaussian; in other words, $\mathcal{K}_n$ and $\mathcal{L}_n$ are asymptotically bivariate Gaussian as $n\to\infty$.

2.2. **Generalized Lekkerkerker's Theorem.** As the mean is a crucial input in the proof of Gaussian behavior, we isolate this calculation first.

Proof of Theorem 2.2(a). Denote $\hat{g}(w)$ the coefficient of z^n in $\hat{\mathscr{G}}(w^a,w^b,z)$, i.e.,

$$\hat{g}(w) \ = \ \sum_{k>0,l\geq 0}p_{n,k,l}w^{ak+bl}. \tag{2.6}$$

As $p_{n,k,l}/\hat{g}(1)$ is the probability that $a\mathcal{K}_n + b\mathcal{L}_n$ equals $ak + bl$, $\hat{g}(x)/\hat{g}(1)$ is the probability generating function of the random variable $a\mathcal{K}_n + b\mathcal{L}_n$, and thus

$$\mu_n \ := \ \mathbb{E}[a\mathcal{K}_n + b\mathcal{L}_n] \ = \ \frac{\hat{g}'(1)}{\hat{g}(1)} \ = \ \frac{\hat{g}'(1)}{D_1} \tag{2.7}$$

(since $\hat{g}(1) = \sum_{k>0,l\geq 0} p_{n,k,l} = D_n$). Therefore the proof of Part (a) reduces to finding $\hat{g}'(1)$.

Let $\hat{A}_w(z)$ be the denominator of $\hat{\mathscr{G}}(w^a, w^b, z)$, namely

$$\hat{A}_w(z) \ = \ 1 - z - (w^a + w^b)z^4 - w^{a+b}z^6 - w^{a+b}z^7,$$

and $e_1(w), e_2(w), \dots, e_7(w)$ the roots of $\hat{A}_w(z)$ (i.e., regarding $\hat{A}_w(z)$ as function of z). We want to write $\frac{1}{\hat{A}_w(z)}$ as a linear combination of the $\frac{1}{z-e_i(w)}$'s, i.e., the partial fraction expansion, as we can use power series expansion to find $\hat{g}(w)$, the coefficient of z^n in $\hat{\mathscr{G}}(w^a, w^b, z)$. In fact, we have the following proposition.

Proposition 2.3. *There exists $\varepsilon \in (0,1)$ such that for any $w \in I_\varepsilon = (1-\varepsilon, 1+\varepsilon)$,*

(a) *The 7 roots of $\hat{A}_w(z)$ are nonzero and distinct.*

(b) *There exists a root $e_1(w)$ such that $|e_1(w)| < 1$ and $|e_1(w)| < |e_i(w)|$, $1 < i \leq 7$.*

(c) *Each root $e_i(w)$ $(1 \leq i \leq 7)$ is continuous and ℓ-times differentiable for any $\ell \geq 1$, and*

$$e_i'(w) \ = \ -\frac{\left(aw^{a-1} + bw^{b-1}\right)e_i^4(w) + (a+b)w^{a+b-1}[e_i^6(w) + e_i^7(w)]}{1 + 4(w^a + w^b)e_i^3(w) + 6w^{a+b}e_i^5(w) + 7w^{a+b}e_i^6(w)}. \tag{2.8}$$

(d)

$$\frac{1}{\hat{A}_w(z)} \ = \ -\frac{1}{w^{a+b}}\sum_{i=1}^{7}\frac{1}{(z - e_i(w))\prod_{j\neq i}(e_j(w) - e_i(w))}. \tag{2.9}$$

Proof. Clearly, 0 is not a root of $\hat{A}_w(z)$. When $w = 1$, we have

$$\hat{A}_1(z) \ = \ 1 - z - 2z^4 - z^6 - z^7 = -(z^2 + z - 1)(z^2 + 1)(z^3 + 1). \tag{2.10}$$

Thus $\hat{A}_1(z)$ has no multiple roots; moreover, except $\frac{\sqrt{5}-1}{2}$, any other root z of $\hat{A}(z)$ satisfies $|z| \geq 1 > |\frac{\sqrt{5}-1}{2}|$. Hence (a), (b) hold for $w = 1$.

Note that when $w \neq 0$, the leading coefficient of $\hat{A}_w(z)$ is nonzero, and the coefficients of $\hat{A}(z)$ are polynomials in one variable and hence continuous, thus the roots of $\hat{A}_w(z)$ are continuous with respect to w (see Appendix A of [MW]). Since (a), (b) hold for $w = 1$, they also hold for a sufficiently small neighborhood I_ε of 1. Parts (c) and (d) follow from algebraic manipulations (see Appendix E of [MW] for the details). □

Assume $w \in I_\varepsilon$. Combining (2.1) and Proposition 2.3(d), we get

$$\hat{\mathscr{G}}(w^a, w^b, z) \ = \ -(z + w^b z^4)\sum_{i=1}^{7}\frac{1}{w^b(z - e_i(w))\prod_{j\neq i}(e_j(w) - e_i(w))},$$

which yields

$$\hat{g}(w) \ = \ \sum_{i=1}^{7}\frac{w^{-b} + e_i^3(w)}{e_i^n(w)\prod_{j\neq i}(e_j(w) - e_i(w))}.$$

Let

$$\hat{q}_i(w) \;=\; \frac{w^{-b}+e_i^3(w)}{w\prod_{j\neq i}\left(e_j(w)-e_i(w)\right)}, \quad \hat{\alpha}_i(w) \;=\; \frac{1}{e_i(w)} \text{ and } \hat{g}_i(w) \;=\; w\hat{q}_i(w)\hat{\alpha}_i^n(w). \tag{2.11}$$

Then $\hat{g}(w)=\sum_{i=1}^{7} w\hat{q}_i(w)\hat{\alpha}_i^n(w)=\sum_{i=1}^{7}\hat{g}_i(w)$. Since $e_i(w)$ is nonzero and ℓ-times differentiable for all ℓ and i, so are $\hat{q}_i(w)$ and $\hat{\alpha}_i(w)$. Further, it follows from Proposition 2.3(b) that $|\hat{\alpha}_1(w)|>1$ and $|\hat{\alpha}_1(w)|>|\hat{\alpha}_i(w)|$, $1<i\le 7$. Hence for fixed ℓ, we have

$$\hat{g}^{(\ell)}(w) \;=\; \hat{g}_1^{(\ell)}(w)+\sum_{i=2}^{7}\left[w\hat{q}_i(w)\hat{\alpha}_i^n(w)\right]^{(\ell)} \;=\; \hat{g}_1^{(\ell)}(w)+o(\hat{\gamma}_\ell^n(w))\hat{\alpha}_1^n(w) \tag{2.12}$$

for some $\hat{\gamma}_\ell(w)\in(0,1)$. Taking $w=1$ yields

$$\hat{g}^{(\ell)}(1) \;=\; \hat{g}_1^{(\ell)}(1)+o(\hat{\gamma}_\ell^n)\hat{\alpha}_1^n(1), \tag{2.13}$$

where $\hat{\gamma}_\ell=\hat{\gamma}_\ell(1)\in(0,1)$. Applying (2.11), (2.12) and (2.13) with $\ell=1$, by (2.7) $\hat{\mu}_n$ is of the form

$$\hat{\mu}_n \;=\; \hat{C}_{a,b}n+\hat{d}_{a,b}+o(\hat{\gamma}_{a,b}^n), \tag{2.14}$$

with

$$\hat{C}_{a,b} \;=\; \frac{\hat{\alpha}_1'(1)}{\hat{\alpha}_1(1)} \;=\; -\frac{e_1'(1)}{e_1(1)} \quad\text{and}\quad \hat{d}_{a,b} \;=\; 1+\frac{\hat{q}_1'(1)}{\hat{q}_1(1)}. \tag{2.15}$$

Here we used the definition that $\hat{\alpha}_1(w)=1/e_1(w)$ (see 2.11).

Setting $w=1$ in (2.8) and using $e_1(1)=\Phi$ (with $\Phi=(\sqrt{5}-1)/2$), we get $\hat{C}_{a,b}=-e_1'(1)/e_1(1)=(a+b)/10$. It is harder to calculate $\hat{d}_{a,b}$, but still tractable. We prove

$$\hat{d}_{a,b} \;=\; \frac{371-113\sqrt{5}}{40}\,a+\frac{361-123\sqrt{5}}{40}\,b.$$

Recall from (2.11) that

$$\hat{q}_1(w) \;=\; \frac{w^{-b}+e_1^3(w)}{w\hat{E}(w)} \quad\text{with}\quad E(w) \;:=\; \prod_{j\neq 1}\left(e_j(w)-e_1(w)\right),$$

Thus

$$\hat{d}_{a,b} \;=\; 1+\frac{\hat{q}_1'(1)}{\hat{q}_1(1)} \;=\; 1+\frac{-bw^{-b-1}+3e_1^2(w)e_1'(w)}{w^{-b}+e_1^3(w)}-\frac{\hat{E}(w)+w\hat{E}'(w)}{w\hat{E}(w)}.$$

Setting $x=1$ and using $e_1(1)=\Phi$ and $e_1'(1)=-(a+b)\Phi/10$, we get

$$\hat{d}_{a,b} \;=\; \frac{-b-\frac{3}{10}(a+b)\Phi^3}{1+\Phi^3}-\frac{\hat{E}'(1)}{\hat{E}(1)} \;=\; -\frac{\sqrt{5}+1}{4}b-\frac{9-3\sqrt{5}}{40}(a+b)-\frac{\hat{E}'(1)}{\hat{E}(1)}. \tag{2.16}$$

Thus it remains to evaluate $\hat{E}(1)$ and $\hat{E}'(1)$. Consider $\hat{A}_w(e'+e_1(w))$:

$$\hat{A}_w(e'+e_1(w)) \;=\; 1-e'-e_1(w)-(w^a+w^b)(e'+e_1(w))^4-w^{a+b}(e'+e_1(w))^6-w^{a+b}(e'+e_1(w))^7. \tag{2.17}$$

On the other hand, we have

$$\hat{A}_w(e'+e_1(w)) \;=\; -w^{a+b}\prod_{j\neq 1}(e'+e_1(w)-e_j(w)). \tag{2.18}$$

Comparing the coefficients of e' in (2.17) and (2.18) gives

$$\hat{E}(w) \;=\; \prod_{j\neq 1}(e_1(w)-e_j(w)) \;=\; w^{-(a+b)}+4(w^{-b}+w^{-a})e_1^3(w)+6e_1^5(w)+7e_1^6(w). \quad (2.19)$$

Setting $w=1$, we get

$$\hat{E}(1) \;=\; 1+8\Phi^3+6\Phi^5+7\Phi^6 \;=\; 10\Phi^2.$$

Differentiating both sides of (2.19) yields

$$\hat{E}'(x) \;=(a+b)w^{-(a+b+1)}-4\left(aw^{-a-1}+bw^{-b-1}\right)e_1^3(w)+30e_1^4(w)e_1'(w)+42e_1^5(w)e_1'(w).$$

Setting $x=1$ and plugging in $e_1(1)=\Phi$ and $e_1'(1)=-(a+b)\Phi/10$ yields

$$\frac{\hat{E}'(1)}{\hat{E}(1)} \;=\; \frac{29\sqrt{5}-95}{10}(a+b). \quad (2.20)$$

Plugging (2.20) into (2.16) yields

$$\hat{d}_{a,b} \;=\; \frac{371-113\sqrt{5}}{40}\,a+\frac{361-123\sqrt{5}}{40}\,b. \quad (2.21)$$

This completes the proof of Theorem 2.2(a). □

2.3. **Gaussian Behavior.** We prove $\mathcal{M}_n = a\mathcal{K}_n + b\mathcal{L}_n$ converges weakly to a Gaussian by calculating its centralized moments and using Markov's Method of Moments. Its variance is a special case, and is determined below. Note that the proof of Theorem 2.2(a) yielded

$$\mathbb{E}[a\mathcal{K}_n+b\mathcal{L}_n] \;=\; \hat{C}_{a,b}n+\hat{d}_{a,b}+o(\hat{\gamma}_{a,b}^n) \text{ and } \operatorname{var}(a\mathcal{K}_n+b\mathcal{L}_n) \;=\; \hat{h}'(1)n+\hat{q}_1''(1)+o(\hat{\tau}_{a,b}^n) \quad (2.22)$$

with

$$\hat{C}_{a,b} \;=\; -\frac{e_1'(1)}{e_1(1)},\ \hat{d}_{a,b} \;=\; 1+\frac{\hat{q}_1'(1)}{\hat{q}_1(1)},\ \hat{h}(w) \;=\; -\frac{we_1'(w)}{e_1(w)}-\hat{C}_{a,b}$$

and constants $\hat{\gamma}_{a,b},\hat{\tau}_{a,b}\in(0,1)$ and $\hat{q}_1''(1)$ depending on only a and b.

Let $\hat{\sigma}_n$ be the standard deviation of $\mathcal{M}_n = a\mathcal{K}_n+b\mathcal{L}_n$. First we centralize and normalize $\mathcal{M}$ to $\mathcal{M}_n^{(c)} = (\mathcal{M}_n-\hat{\mu}_n)/\hat{\sigma}_n$. Thus it suffices to show that $\mathcal{M}_n^{(c)}$ converges to the standard normal. According to Markov's Method of Moments, we only need to show that each moment of $\mathcal{M}_n^{(c)}$ tends to that of the standard normal distribution, which is equivalent to the following.

Theorem 2.4. *Let $\hat{\mu}_n(m)$ be the m^{th} moment of $\mathcal{M}_n-\hat{\mu}_n$, then for any integer $u\geq 1$,*

$$\frac{\hat{\mu}_n(2u-1)}{\hat{\sigma}_n^{2u-1}}\to 0 \text{ and } \frac{\hat{\mu}_n(2u)}{\hat{\sigma}_n^{2u}}\to(2u-1)!!,\ \text{ as } n\to\infty. \quad (2.23)$$

In the proof, we first point out that it suffices to prove the same result for $\mathcal{M}_n-\tilde{\mu}_n$ with $\tilde{\mu}_n=\hat{C}_{a,b}n+\hat{d}_{a,b}$ and $\hat{C}_{a,b}$, $\hat{d}_{a,b}$ defined in (2.15). Then we show that the m^{th} moment $\tilde{\mu}_n(m)$ of $\mathcal{M}_n-\hat{\mu}_n$ equals $\tilde{g}_m(1)/D_n$ for polynomials $\tilde{g}_m(x)$ with

$$\tilde{g}_0(x) \;=\; \sum_{k,l}p_{n,k,l}w^{ak+bl-\tilde{\mu}_n-1} \;=\; \frac{\hat{g}(x)}{x^{\tilde{\mu}_n+1}},\quad \tilde{g}_{j+1}(x) \;=\; (x\tilde{g}_j(x))',\ j\geq 1. \quad (2.24)$$

By Definitions (2.6) and (2.24), we prove by induction that the main term of $\tilde{g}_m(1)$ is $\hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n}\sum_{i=0}^m f_{i,m}(x)n^i$ for some functions $f_{i,m}(x)$'s and thus conclude that $\tilde{\mu}_n(m)=\frac{1}{q_1(1)}\sum_{i=0}^m f_{i,m}(1)n^i+o(\tau_m^n)$ for some $\tau_m\in(0,1)$. Finally, we evaluate the $f_{i,m}(1)$'s to obtain (2.23).

We now give the proof. We will interrupt the proof to prove some simple, needed results. Noting that $\hat{\mu}_n = \tilde{\mu}_n + o(\gamma_1^n)$ by (2.14), by simple approximations (see Appendix E.2 of [MW])

$$\hat{\mu}_n(m) \ = \ \tilde{\mu}_n(m) + o(\beta_m^n) \tag{2.25}$$

for some $\beta_m \in (0,1)$. In the special case of $m=2$, we have $\hat{\sigma}_n^2 = \hat{\mu}_n(2) = \tilde{\mu}_n(2) + o(\tau_m^n)$, therefore (2.23) is equivalent to

$$\frac{\tilde{\mu}_n(2u-1)}{\tilde{\mu}_n^{u-\frac{1}{2}}(2)} \to 0 \text{ and } \frac{\tilde{\mu}_n(2u)}{\tilde{\mu}_n^u(2)} \to (2u-1)!!, \text{ as } u \to \infty. \tag{2.26}$$

We calculate the moments $\tilde{\mu}_n(m)$. By (2.24)

$$\begin{aligned} \tilde{g}_1(x) \ &= \ (x\tilde{g}_0(x))' \ \text{ so } \ \tilde{g}_1(1) \ = \ \sum_{k,l} p_{n,k,l}(ak+bl-\tilde{\mu}_n)x^{ak+bl-\tilde{\mu}_n-1} \\ \tilde{g}_2(x) \ &= \ (x\tilde{g}_1(x))' \ \text{ so } \ \tilde{g}_2(1) \ = \ \tilde{\mu}_n(2)D_n, \end{aligned} \tag{2.27}$$

and since $\tilde{g}_0(x)/D_m$ is the probability generating function of $\mathcal{M}_n - \mu_n$, we have for general m that

$$\tilde{g}_m(x) \ = \ \sum_{k,l} p_{n,k,l}(ak+bl-\tilde{\mu}_n)^m x^{ak+bl-\tilde{\mu}_n-1} \text{ and } \tilde{g}_m(1) \ = \ \tilde{\mu}_n(m)D_n. \tag{2.28}$$

Returning to the proof of Theorem 2.4, denote

$$\tilde{g}_{0,i}(x) \ = \ \frac{\hat{q}_i(x)\hat{\alpha}_i^n(x)}{x^{\tilde{\mu}_n}}, \text{ and } \tilde{g}_{j+1,i}(x) \ = \ (x\tilde{g}_{j,i}(x))' \tag{2.29}$$

for $x \in I_\varepsilon$ if $1 < i \leq 7$ and for $x \in I_\varepsilon \cup \{1\}$ if $i=1$. By Definition (2.29) and using (2.24) and the same approach as in proving (2.12), there is a $\tau_j \in (0,1)$ such that

$$\forall x \in I_\varepsilon: \ \tilde{g}_j(x) \ = \ \sum_{i=1}^{L} \tilde{g}_{j,i}(x) \ = \ \tilde{g}_{j,1}(x) + o(\tau_j^n)\hat{\alpha}_1^n(x). \tag{2.30}$$

Denoting $\tilde{g}_{j,1}(x)$ by $F_j(x)$, then

$$F_0(x) \ = \ \hat{q}_1(x)\hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n} \text{ and } F_{j+1}(x) \ = \ (xF_j(x))'. \tag{2.31}$$

Note that $\hat{q}_1(x)$ and $\hat{\alpha}_1(x)$ are ℓ-times differentiable for all ℓ. Thus when $j=0$, we get

$$F_1(x) \ = \ (xF_0(x))' \ = \ \hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n}\left[h(x)\hat{q}_1(x)n + d'\hat{q}_1(x) + x\hat{q}_1'(x)\right], \tag{2.32}$$

where $h(x)$ and d' are defined as

$$h(x) \ = \ \frac{x\hat{\alpha}_1'(x)}{\hat{\alpha}_1(x)} - \hat{C}_{a,b} \text{ and } d' = 1 - \hat{d}_{a,b} \ = \ -\frac{\hat{q}_1'(1)}{\hat{q}_1(1)} \tag{2.33}$$

(see (2.15) for the definition of $\hat{d}_{a,b}$). By (2.15), we have

$$h(1) \ = \ 0. \tag{2.34}$$

Moreover, since $\hat{\alpha}_1(x)$ is ℓ-times differentiable at 1 and $\hat{\alpha}_1(1) \neq 0$, we have

$$h(x) \text{ is } \ell\text{−times differentiable at 1 for any } \ell \geq 1. \tag{2.35}$$

From (2.31) and (2.32), we observe that $F_m(x)$ can be written as a product of $\hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n}$ and a sum of other functions of n and x. In fact, we have the following.

Proposition 2.5. *For any $m \geq 0$,*

(a) *We have $F_m(x)$ is of the form*

$$F_m(x) \;=\; \hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n} \sum_{i=0}^{m} f_{i,m}(x)n^i, \tag{2.36}$$

where the $f_{i,m}$'s are functions of x and $\hat{\alpha}_1(x)$ but independent of n.

(b) *The $f_{i,m}$'s are ℓ-times differentiable at $x \in I_\varepsilon$ for any $\ell \geq 1$.*

(c) *Define*

$$f_{i,m}(x) \;=\; 0 \text{ if } i > m \text{ or } i < 0 \text{ or } m < 0, \tag{2.37}$$

then for $m > 0$, we have the following recurrence relation:

$$f_{i,m}(x) \;=\; h(x)f_{i-1,m-1}(x) + d' f_{i,m-1}(x) + x f'_{i,m-1}(x). \tag{2.38}$$

Proof. We proceed by induction on m. We first do $m = 0$ and 1 for all three cases. (a) holds because of (2.31) and (2.32). Further, (2.31) and (2.32) give the expressions of $f_{0,0}$, $f_{0,1}$ and $f_{1,1}$:

$$f_{0,0}(x) \;=\; \hat{q}_1(x), f_{0,1}(x) \;=\; d'\hat{q}_1(x) + x\hat{q}_1'(x), f_{1,1}(x) \;=\; h(x)\hat{q}_1(x). \tag{2.39}$$

Thus they are differentiable ℓ-times at $x \in I_\varepsilon$ for any $\ell \geq 1$. Hence (b) holds for $m = 0$ and 1. Finally, with (2.39), it is easy to verify that (c) holds for $m = 0$ and 1.

Standard algebra and induction gives (a). For (b) and (c), we find

$$\begin{aligned} f_{m+1,m+1}(x) &= h(x)f_{m,m}(x) \\ f_{i,m+1}(x) &= h(x)f_{i-1,m}(x) + d' f_{i,m}(x) + x f'_{i,m}(x),\ 1 \leq i \leq m \\ f_{0,m+1}(x) &= d' f_{0,m}(x) + x f'_{0,m}(x). \end{aligned} \tag{2.40}$$

By Definition (2.37), we can combine (2.40) into one recurrence relation (2.38) (with m replaced by $m + 1$). With this recurrence relation, (2.35) and the induction hypothesis of (b) for m, we see that (b) also holds for $m + 1$, completing the proof. □

Proposition 2.6. *We have*

$$\tilde{\mu}_n(m) \;=\; \frac{1}{\hat{q}_1(1)} \sum_{i=0}^{m} f_{i,m}(1)n^i + o(\tau_m^n) \text{ for some } \tau_m \in (0, 1).$$

Proof. This follows from (2.28), (2.30), (2.7), (2.13) with $\ell = 0$, the definition $F_m(x) = \tilde{g}_{m,1}(x)$ and Proposition 2.5, and some straightforward algebra. □

From Proposition 2.6, we see that the main term of $\tilde{\mu}_n(m)$ only depends on $\hat{q}_1(1)$ and the $f_{i,m}(1)$'s. Note that to prove (2.26), it suffices to find the main term of $\tilde{\mu}_n(m)$. Thus the problem reduces to finding the $f_{i,m}(1)$'s. We first calculate the variance, namely $\tilde{\mu}_n(2)$.

Proposition 2.7. *The variance of $\mathcal{M}_n - \tilde{\mu}_n$ is*

$$\tilde{\mu}_n(2) \;=\; h'(1)n + \hat{q}_1''(1) + o(\tau_2^n) \tag{2.41}$$

with $h'(1) \neq 0$, $\hat{q}_1''(1)$ and $\tau_2 \in (0, 1)$ constant depending on only L and the c_i's.

With the estimation (2.25), it follows immediately that the variance of $\mathcal{M}_n = a\mathcal{K}_n + b\mathcal{L}_n$ is

$$\hat{\mu}_n(2) \;=\; h'(1)n + \hat{q}_1''(1) + o(\tau_2'^n),$$

with $h'(1) \neq 0$, $\hat{q}_1''(1)$ and $\tau_2 \in (0,1)$ constant depending on only a and b, with

$$\hat{h}'(1) \;=\; \frac{\sqrt{5}-1}{200}\left[10\left(a^2+b^2\right) - \frac{20-\sqrt{5}}{5}(a+b)^2\right]. \tag{2.42}$$

Proof of Proposition 2.7. If $m = 2$, by (2.40) and (2.34) we get $f_{2,2}(1) = h(1)f_{1,1}(1) = 0$. Applying (2.38) to $(i,m) = (1,2)$ and plugging in (2.39) yields

$$\begin{aligned} f_{1,2}(x) &= h(x)f_{0,1}(x) + d'f_{1,1}(x) + xf_{1,1}'(x) \\ &= h(x)f_{0,1}(x) + d'h(x)\hat{q}_1(x) + xh(x)\hat{q}_1'(x) + xh'(x)\hat{q}_1(x). \end{aligned}$$

Setting $x = 1$ and using $h(1) = 0$ (see (2.34)) yields

$$f_{1,2}(1) \;=\; h(1)f_{0,1}(1) + d'h(1)\hat{q}_1(1) + h(1)\hat{q}_1'(1) + h'(1)\hat{q}_1(1) = h'(1)\hat{q}_1(1).$$

Using (2.40) and (2.38), we can find $f_{0,2}(x)$ as follows:

$$f_{0,2}(x) \;=\; d'f_{0,1}(x) + xf_{0,1}'(x) \;=\; d'^2\hat{q}_1(x) + d'x\hat{q}_1'(x) + d'x\hat{q}_1(x) + x\hat{q}_1'(x) + x^2\hat{q}_1''(x).$$

Setting $x = 1$ and substituting d' by $-\frac{\hat{q}_1'(1)}{\hat{q}_1(1)}$ (see (2.33)) yields $f_{0,2}(1) = \hat{q}_1''(1)$. Combining the above results with Proposition 2.6 gives (2.41). We can derive a formula for $\hat{h}'(w)$ in terms of $e_1(w)$ by using (2.8). Then (2.42) follows by $e_1(1) = \Phi$. We can verify that $\hat{h}'(1) \neq 0$ by simple quadratic inequalities (details can be found in Appendix E.4 of [MW]). This completes the proof, and proves (2.5) of Theorem 2.2. □

From Propositions 2.6 and 2.7, we see that (2.26) (which is what we need to show to finish the proof of Theorem 2.4) is equivalent to

$$f_{i,2u-1}(1) \;=\; 0,\; i \geq u, \tag{2.43}$$

$$f_{i,2u}(1) \;=\; 0,\; i > u, \tag{2.44}$$

and

$$f_{u,2u}(1) \;=\; (2u-1)!!\hat{q}_1(1)\left(h'(1)\right)^u. \tag{2.45}$$

For convenience, we denote $t_{i,m}^{(\ell)} = f_{i,m}^{(\ell)}(1)$, $\ell \geq 0$. Note that if $\ell = 0$, then the definition is just $t_{i,m} = f_{i,m}(1)$.

Proposition 2.8. *For any $0 \leq m < 2i$ and $\ell \geq 0$, we have*

$$t_{i,m-\ell}^{(\ell)} \;=\; f_{i,m-\ell}^{(\ell)}(1) \;=\; 0. \tag{2.46}$$

Proof. If $\ell > m$ or $i > m - \ell$, according to Definition (2.37), we have $f_{i,m-\ell}(x) = 0$. Thus $f_{i,m-\ell}^{(\ell)}(x) = 0$ and (2.46) follows. Therefore, it suffices to prove for $0 \leq \ell \leq m < 2i$ and $i \leq m - \ell$, which follows by induction. □

Corollary 2.9. *For any $u \geq 1$, we have (2.43) and (2.44), i.e.,*

$$t_{i,2u-1} \;=\; 0,\; i \geq u \text{ and } t_{i,2u} \;=\; 0,\; i > u.$$

Proof. Use Proposition 2.8 with $(i,m,\ell) = (i, 2u-1, 0)$. □

Thus it remains to show (2.45).

Proposition 2.10. *For any $u \geq 1$ we have*

(a)$f_{u,u+v}(x)$ *with* $0 \leq v \leq u$ *is of the form*

$$f_{u,u+v}(x) \;=\; r_{u,v}\hat{q}_1(x)x^v h^{u-v}(x)\left(h'(x)\right)^v + s_{u,v}(x)h^{u+1-v}(x), \tag{2.47}$$

where $r_{u,v}$ *is a constant determined by* u *and* v*,* $s_{u,v}(x)$ *is a polynomial of the* $h^{(\ell)}(x)$*'s and the* $\hat{q}_1^{(\ell)}(x)$*'s* $(\ell \geq 0)$ *with coefficients polynomials of* x*.*

(b) $r_{u,0} = 1$ *and*

$$r_{u,v} \;=\; r_{u-1,v} + (u-v+1)r_{u,v-1},\; r_{u,u} \;=\; r_{u,u-1},\; 1 \leq v < u. \tag{2.48}$$

(c)

$$r_{u,u} \;=\; (2u-1)!!. \tag{2.49}$$

Proof. We proceed by induction on $u+v$. By (2.39) and (2.40), we get

$$f_{u,u}(x) \;=\; \hat{q}_1(x)h^u(x),\; u \geq 1.$$

Hence (a) holds for $v = 0$ and $r_{u,0} = 1$.

Since the only (u,v) with $u+v = 1$ and $0 \leq v \leq u$ is $(0,1)$, (a) holds for $u+v = 1$. Assume that (a) holds for $u+v \leq t$ $(t \geq 1)$. We simultaneously prove (a) and (b). If $u+v = t+1$, we have shown that the statement holds for $v = 0$. For $1 \leq v \leq u$, we have three cases: $v = 1$, $1 < v < u$ and $1 < v = u$.

When $1 \leq v < u$, applying (2.38) to $(i,m,\ell) = (u, u+v, 0)$ and using the induction hypothesis for $(u-1,v),(u,v-1)$, we get

$$\begin{aligned} f_{u,u+v}(x) \;&=\; h(x)f_{u-1,u+v-1} + d'f_{u,u+v-1} + xf'_{u,u+v-1} \\ &=\; r_{u-1,v}\hat{q}_1(x)x^v h^{u-v}(x)\left(h'(x)\right)^v + [s_{u-1,v}(x) \\ &\quad + d'r_{u,v-1}\hat{q}_1(x)x^{v-1}\left(h'(x)\right)^{v-1} + d's_{u,v-1}(x)h(x)]h^{u+1-v}(x) \\ &\quad + x\left[r_{u,v-1}\hat{q}_1(x)x^{v-1}h^{u-v+1}(x)\left(h'(x)\right)^{v-1} + s_{u,v-1}(x)h^{u+2-v}(x)\right]'. \end{aligned} \tag{2.50}$$

Denote the last line of (2.50) by W. There are three cases: $v = 1$, $1 < v < u$, and $1 < v = u$. We prove the third case, as the others follow similarly (or see [MW]).

As $1 < v = u$, we have $u \geq 2$. From the recurrence relation (2.38) and the initial condition (2.39), we see that each $f_{i,m}$ is a polynomial of the $h^{(\ell)}(x)$'s and the $\hat{q}_1^{(\ell)}(x)$'s $(\ell \geq 0)$ with coefficients polynomials of x. By (2.50) and the induction hypothesis (2.47) for $(u,v) = (u,u-1)$, after some algebra we get $f_{u,u+v}(x)$ is of the form (2.47) and (2.48) holds, completing the proof of (a) and (b). We use generating functions to prove (c). The proof of (c) is an immediate consequence of Lemma 2.11 (see Remark 2.1 for the details). □

Lemma 2.11. *Define*

$$T_v(x) \;=\; \sum_{u=v}^{\infty} r_{u,v}x^{u-v},\; v \geq 0. \tag{2.51}$$

Then we have

(a)

$$T_v(x) \;=\; \frac{T'_{v-1}(x)}{1-x},\; v \geq 1. \tag{2.52}$$

(b)

$$T_0(x) \;=\; \frac{1}{1-x} \text{ and } T_v(x) \;=\; \frac{(2v-1)!!}{(1-x)^{2v+1}},\; v \geq 1. \tag{2.53}$$

Proof. (a) From Definition (2.51) and the recurrence relation, we find

$$(1-x)T_v(x) \;=\; \sum_{u=v}^{\infty}(u-v+1)r_{u,v-1}x^{u-v}. \tag{2.54}$$

On the other hand, taking the derivative of both sides of Definition (2.51), we see that $T'_{v-1}(x)$ also equals (2.54). Therefore (2.52) holds.

(b) Since $r_{u,0}=1$ (see Proposition 2.10(b)), we obtain $T_0(x)=\frac{1}{1-x}$. The claimed expressions for T_v follow from (a) by induction. ☐

Remark 2.1. *The proof of part (c) of Proposition 2.10 is immediate, as any $u\geq 1$,*

$$r_{u,u} \;=\; T_u(0) \;=\; (2u-1)!!$$

by Definition (2.51) and Lemma 2.11.

Setting $v=u$ and $x=1$ in Proposition 2.10(a) and using (2.34) and (2.49), we get

$$f_{u,2u}(1) \;=\; r_{u,u}\hat{q}_1(1)\left(h'(1)\right)^u = (2u-1)!!\hat{q}_1(1)\left(h'(1)\right)^u,$$

as desired, completing the proof of Theorem 2.4 and Theorem 2.2(b). ☐

Applying Theorem 2.2 to the special cases $(a,b)=(1,0)$ and $(0,1)$ yields

Theorem 2.12. *The expected values and variances of $\mathcal{K}_n$ and $\mathcal{L}_n$ are*

$$\begin{aligned}\mathbb{E}[\mathcal{K}_n] \;&=\; \frac{1}{10}n+\frac{371-113\sqrt{5}}{40}+o(\hat{\gamma}_{1,0}^n), \quad \mathrm{var}(\mathcal{K}_n) \;=\; \frac{29\sqrt{5}-25}{1000}n+O(1),\\ \mathbb{E}[\mathcal{L}_n] \;&=\; \frac{1}{10}n+\frac{361-123\sqrt{5}}{40}+o(\hat{\gamma}_{0,1}^n), \quad \mathrm{var}(\mathcal{L}_n) \;=\; \frac{15+21\sqrt{5}}{1000}n+O(1).\end{aligned}$$

Additionally, we have

$$\mathbb{E}[\mathcal{K}_n]-\mathbb{E}[\mathcal{L}_n] \;=\; \frac{1+\sqrt{5}}{4}+o(\hat{\gamma}'^n) \;=\; \frac{\varphi}{2}+o(\hat{\gamma}'^n)\approx 0.809016994 \text{ for some } \hat{\gamma}'\in(0,1).$$

In words, on average there are approximately 0.809 more positive terms than negative terms in the far-difference representation.

Applying Theorem 2.2 to $a=b=1$, we get

$$\mathrm{var}(\mathcal{K}_n+\mathcal{L}_n) \;=\; \frac{2\sqrt{5}}{125}n+O(1), \text{ and } \mathrm{var}(\mathcal{K}_n-\mathcal{L}_n) \;=\; \frac{\sqrt{5}-1}{10}n+O(1). \tag{2.55}$$

Hence the covariance is approximately $-0.0219574275n+O(1)$, as

$$\mathrm{cov}(\mathcal{K}_n,\mathcal{L}_n) \;=\; \frac{\mathrm{var}(\mathcal{K}_n+\mathcal{L}_n)-\mathrm{var}(\mathcal{K}_n-\mathcal{L}_n)}{4} \;=\; \frac{25-21\sqrt{5}}{1000}n+O(1). \tag{2.56}$$

With Theorem 2.12 and (2.56), we compute the correlation between $\mathcal{K}_n$ and $\mathcal{L}_n$:

$$\mathrm{corr}(\mathcal{K}_n,\mathcal{L}_n) \;=\; \frac{10\sqrt{5}-121}{179}+o(1) \;\approx\; -0.551057655+o(1).$$

Since $\mathrm{var}(\mathcal{K}_n)$ and $\mathrm{var}(\mathcal{L}_n)$ are of size n and have the same coefficients of n, we have

$$\mathrm{cov}(\mathcal{K}_n+\mathcal{L}_n,\mathcal{K}_n-\mathcal{L}_n) \;=\; \mathrm{var}(\mathcal{K}_n)-\mathrm{var}(\mathcal{L}_n) \;=\; O(1).$$

Further, we have the values of $\text{var}(\mathcal{K}_n + \mathcal{L}_n)$ and $\text{var}(\mathcal{K}_n - \mathcal{L}_n)$ from (2.55), thus

$$\text{corr}(\mathcal{K}_n + \mathcal{L}_n, \mathcal{K}_n - \mathcal{L}_n) \;=\; \frac{\text{cov}(\mathcal{K}_n + \mathcal{L}_n, \mathcal{K}_n - \mathcal{L}_n)}{\sqrt{\text{var}(\mathcal{K}_n + \mathcal{L}_n)\text{var}(\mathcal{K}_n - \mathcal{L}_n)}} \;=\; o(1).$$

Since $\mathcal{K}_n$ and $\mathcal{L}_n$ are asymptotically bivariate Gaussian, $\mathcal{K}_n+\mathcal{L}_n$ and $\mathcal{K}_n-\mathcal{L}_n$ are independent as $n \to \infty$.

3. Conclusion and Future Research

Our combinatorial perspective has extended previous work, allowing us to prove Gaussian behavior for the number of summands for a large class of expansions in terms of solutions to linear recurrence relations. This is just the first of many questions one can ask. Others, which we hope to return to at a later date, include:

(1) What happens for linearly recursive sequences with arbitrary integer coefficients?
(2) What if either uniqueness fails, or some numbers are not representable?
(3) What if we only care about how many distinct H_i's occur in the decomposition?
(4) What is true about gaps between summands?

The last question has been solved in some cases in [BGM], and is currently being generalized to additional recurrence relations. They prove

Theorem 3.1 (Base B Gap Distribution)**.** *For base B decompositions, as $n \to \infty$ the probability of a gap of length 0 between summands for numbers in $[B^n, B^{n+1})$ tends to $\frac{(B-1)(B-2)}{B^2}$, and for gaps of length $k \geq 1$ to $\frac{(B-1)(3B-2)}{B^2}B^{-k}$.*

Theorem 3.2 (Zeckendorf Gap Distribution)**.** *For Zeckendorf decompositions, for integers in $[F_n, F_{n+1})$ the probability of a gap of length $k \geq 2$ tends to $\frac{\varphi(\varphi-1)}{\varphi^k}$ for $k \geq 2$, with $\varphi = \frac{1+\sqrt{5}}{2}$ the golden mean. A similar result holds for Tribonacci and other recurrence sequences.*

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