ON NEAR PERFECT NUMBERS

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ABSTRACT. The study of perfect numbers (numbers which equal the sum of their proper divisors) goes back to antiquity, and is responsible for some of the oldest and most popular conjectures in number theory. We investigate a generalization introduced by Pollack and Shevelev: k-near-perfect numbers. These are examples to the well-known pseudoperfect numbers first defined by Sierpiński, and are numbers such that the sum of all but at most k of its proper divisors equals the number. We establish their asymptotic order for most integers $k \geq 4$, as well as some properties of related quantities.

1. INTRODUCTION

Let $\sigma(n)$ be the sum of all positive divisors of n. A natural number n is *perfect* if $\sigma(n) = 2n$. Perfect numbers have played a prominent role in classical number theory for millennia. A well-known conjecture claims that there are infinitely many even, but no odd, perfect numbers. Despite the fact that these conjectures remain unproven, there has been significant progress on studying the distribution of perfect numbers [Vo, HoWi, Ka, Er], as well as generalizations. One are the *pseudoperfect numbers*, which were introduced by Sierpiński [Si]. A natural number is pseudoperfect if it is a sum of some subset of its proper divisors. Erdös and Benkoski [Erd, BeEr] proved that the asymptotic density for pseudoperfect numbers, as well as that of abundant numbers that are not pseudoperfect (also called *weird numbers*), exist and are positive.

Pollack and Shevelev [PoSh] initiated the study of a subclass of pseudoperfect numbers called *near-perfect numbers*. A natural number is k-near-perfect if it is a sum of all of its proper divisors with *at most* k exceptions. Restriction on the number of exceptional divisors leads to asymptotic density 0. The number of 1-near-perfect numbers up to x is[1] at most $x^{3/4+o(1)}$, and in general for $k \geq 1$ the number of k-near-perfect numbers up to x is at most $x(\log\log x)^{k-1}/\log x$.

Our first result improves the count of k-near-perfect numbers.

Theorem 1.1. *Denote by $N(k;x)$ the set of k-near-perfect numbers at most x. For $k \geq 4$ with k not equal to $2^{s+2}-5$ or $2^{s+2}-6$ for some $s \geq 2$,*

$$\#N(k;x) \asymp_k \frac{x}{\log x}(\log\log x)^{\lfloor \frac{\log(k+4)}{\log 2} \rfloor - 3}. \tag{1.1}$$

We exclude integers of the forms $2^{s+2}-5$ and $2^{s+2}-6$ from the above theorem to keep our argument in the proof short and clean, and at the same time not to obscure the

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[1]This is a result stated in [AnPoPo]. In the original paper of Pollack and Shevelev [PoSh], the upper bound was given by $x^{5/6+o(1)}$.

main ideas behind. At the cost of additional work one could carry out the same kind of computations done in the proof of Theorem 1.1 in order to cover these two cases.

Our argument is based on a partition of the set $N(k;x)$ different from that of [PoSh]. This allows us to carry out an inductive argument and essentially reduces the count of $\#N(k;x)$ for large integers k to that for small integers k ($k = 4,5,6,7,8,9$). In fact, for these small integers, we even have precise asymptotic formulae.

Theorem 1.2. *For $4 \le k \le 9$, we have*

$$\#N(k;x) \ \sim \ c_k \frac{x}{\log x} \tag{1.2}$$

as $x \to \infty$, where

$$\begin{cases} c_4 = c_5 = \frac{1}{6}, \\ c_6 = \frac{17}{84} \ = \ \frac{1}{6} + \frac{1}{28} \\ c_7 = c_8 \ = \ \frac{493}{1260} \ = \ \frac{1}{6} + \frac{1}{12} + \frac{1}{18} + \frac{1}{20} + \frac{1}{28} \\ c_9 = \frac{179017}{360360} \ = \ \frac{1}{6} + \frac{1}{12} + \frac{1}{18} + \frac{1}{20} + \frac{1}{28} + \frac{1}{24} + \frac{1}{40} + \frac{1}{56} + \frac{1}{88} + \frac{1}{104}. \end{cases} \tag{1.3}$$

The numbers in the denominators of the reciprocal sums are 1-near-perfect. The presence of these numbers will become clear in the proof.

We can extend the notion of k-near perfect numbers with the constant k replaced by a positive strictly increasing function. For further discussion, see Appendix A.

Our last result is motivated by an open question raised in [BeEr]: can $\sigma(n)/n$ be arbitrarily large when n is a weird number?[2] We replace 'weirdness' by 'exact-perfectness', where a natural number is *k-exact-perfect* if it is a sum of all of its proper divisors with *exactly* k exceptions. Note the result below is conditional on there being no odd perfect numbers.

Theorem 1.3. *Let $\epsilon \in (0, 2/5)$. Denote by $E(k)$ the set of all k-exact-perfect numbers, $E(k;x) := E(k) \cap [1,x]$ and $E_\epsilon(k;x) := \{n \le x : n \in E(k),\ \sigma(n) \ge 2n + n^\epsilon\}$. Let M be the set of all natural numbers of the form $2q$, where q is a Mersenne prime[3]. If there are no odd perfect number, then for k sufficiently large and $k \notin M$, we have*

$$\lim_{x\to\infty} \frac{\#E_\epsilon(k;x)}{\#E(k;x)} \ = \ 1. \tag{1.4}$$

1.1. **Outline.** In Section 2 we introduce the necessary definitions and lemmata for our theorems. In Section 3, we set the stage for proving Theorem 1.1 and 1.2. In Section 4, 5 and 6 , we prove Theorem 1.2, 1.1 and 1.3 and respectively. A generalization of near-perfectness is discussed in Appendix A. We supply more detail to the calculations of Theorem 1.1 and 1.2 in Appendix B and C.

1.2. **Notations.** We use the following notations and definitions.

- We write $f(x) \asymp g(x)$ if there exist positive constants c_1, c_2 such that $c_1 g(x) < f(x) < c_2 g(x)$ for all sufficiently large x.
- We write $f(x) \sim g(x)$ if $\lim_{x\to\infty} f(x)/g(x) = 1$.
- We write $f(x) = O(g(x))$ or $f(x) \ll g(x)$ if there exists a positive constant C such that $f(x) < Cg(x)$ for all sufficiently large x.
- We write $f(x) = o(g(x))$ if $\lim_{x\to\infty} f(x)/g(x) = 0$.

[2] A number is *weird* if the sum of its proper divisors is greater than itself, but no subset of these divisors sums to the original number.

[3] Mersenne primes are primes of the form $2^p - 1$ for some prime p.

- In all cases, subscripts indicate dependence of implied constants on other parameters.
- Denote by $[a, b]_{\mathbb{Z}}$ the set of all integers n such that $a \le n \le b$.
- Denote by $\log_k x$ the k^{th} iterate of logarithm. Thus $\log_1 x = \log x$, $\log_2 x = \log\log x$, and so on.
- Let $y \ge 2$. A natural number n is said to be y-smooth if all of its prime factors are at most y.
- Let $x \ge y \ge 2$. Denote by $\Phi(x, y)$ the set of all y-smooth numbers up to x.
- We use p and p_i to denote primes, and $P^+(n)$ to denote the largest prime factor of n.
- Denote by $\tau(n)$ the number of positive divisors of n.

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2. Preparations

In this section, we collect the necessary lemmata for our theorems. We begin with two estimations of the number of y-smooth numbers up to x.

Lemma 2.1. *Let*

$$u \;=\; \frac{\log x}{\log y}. \tag{2.1}$$

Uniformly for $x \ge y \ge 2$, we have the following bound on the size of $\Phi(x, y)$, the y-smooth numbers up to x:

$$\#\Phi(x, y) \;\ll\; x \exp(-u/2). \tag{2.2}$$

Uniformly for $(\log x)^3 \le y \le x$, we have

$$\#\Phi(x, y) \;=\; x \exp\left(-u \log u + O(u \log\log u)\right). \tag{2.3}$$

Proof. See Theorem 9.5, Theorem 9.15 and Corollary 9.18 of [DeKLu]. □

Our next lemma is a standard result from sieve theory.

Lemma 2.2. *Suppose A is a finite set of natural numbers, P is a set of primes, $z > 0$ and $P(z)$ is the product of primes in P not greater than z. Let*

$$S(A, P, z) \;:=\; \{n \in A : (n, P(z)) = 1\}$$

and

$$A_d \;:=\; \{a \in A : d \mid a\}.$$

Assume the following conditions.

(1) Suppose g is a multiplicative function satisfying

$$0 \;\le\; g(p) \;<\; 1 \;\text{ for }\; p \in P \;\text{ and }\; g(p) = 0 \;\text{ for }\; p \notin P,$$

and there exists constants $B > 0$ and $\kappa \ge 0$ such that

$$\prod_{y \le p \le w} (1 - g(p))^{-1} \;\le\; \left(\frac{\log w}{\log y}\right)^{\kappa} \exp\left(\frac{B}{\log y}\right)$$

for $2 \leq y < w$.

(2) *Let* $X > 0$. *For any square-free number* d *with all of its prime factors in* P, *define*

$$r_d \ := \ \#A_d - Xg(d).$$

Assume that r_d *satisfies the inequality*

$$\sum_{\substack{d|P(z)\\ d\leq X^\theta}} |r_d| \ \leq \ C\frac{x}{(\log x)^\kappa}.$$

for some $\theta > 0$.

Then for $2 \leq z \leq X$, *we have*

$$\#S(A,P,z) \ \ll_{\kappa,\theta,C,B} \ XV(z), \tag{2.4}$$

where

$$V(z) \ := \ \prod_{\substack{p\leq z\\ p\in P}} (1-g(p)). \tag{2.5}$$

Proof. For example, see [FoHa]. □

In the proof of Theorem 1.1, an estimate is needed for the size of the set

$$\Phi_j(x,y) \ := \ \{n \leq x : n = p_1 \cdots p_j m_j, P^+(m_j) \leq y < p_j < \cdots < p_1\}, \tag{2.6}$$

where $j \geq 1$ and $x \geq y \geq 2$. The following lemma follows from Lemmas 2.1 and 2.2.

Lemma 2.3. *Suppose* $x \geq y \geq 2$ *and* $y \leq x^{o(1)}$. *For every* $j \geq 1$, *we have*

$$\#\Phi_j(x,y) \ \ll \ \frac{x \log y}{\log x} (\log\log x)^{j-1}. \tag{2.7}$$

Proof. We follow the settings and notations of Lemma 2.2. Let A be the set of all natural numbers up to x, P be the set of primes in $(y, x^{1/(j+1)}]$, $z := x^{1/(j+1)}$, $X := x$ and $g(d) := 1/d$. The set $S(A,P,z)$ is all natural numbers up to x whose prime factors are at most y or exceed $x^{\frac{1}{j+1}}$. (Note that at most j prime factors can be larger than $x^{\frac{1}{j+1}}$.)

By Merten's estimates, we can see that all of the assumptions of Lemma 2.2 are satisfied and hence we have

$$\#S(A,P,z) \ \ll \ \frac{x \log y}{\log x}. \tag{2.8}$$

Therefore,

$$\begin{aligned}\#Q^{(j)}(x) \ &:= \ \#\{n \leq x : n = p_1 \cdots p_j m_j, P^+(m_j) \leq y < x^{\frac{1}{j+1}} < p_j < \cdots < p_1\} \\ &\ll \frac{x \log y}{\log x}.\end{aligned} \tag{2.9}$$

For $1 \leq i \leq j-1$, denote by $Q^{(i)}(x)$ the set

$$\left\{n \leq x : n = p_1 \cdots p_j m_j, P^+(m_j) \leq y < p_j < \cdots < p_{i+1} \leq x^{\frac{1}{j+1}} < p_i < \cdots < p_1\right\} \tag{2.10}$$

and by $Q^{(0)}(x)$ the set

$$\left\{n \leq x : n = p_1 \cdots p_j m_j, P^+(m_j) \leq y < p_j < \cdots < p_1 \leq x^{\frac{1}{j+1}}\right\}. \tag{2.11}$$

For $1 \leq i \leq j-1$, we use the same kind of estimate of $S(A, P, z)$ with the same choices of parameters as above, except this time we choose

$$X \;:=\; \frac{x}{p_{i+1}\cdots p_j} \tag{2.12}$$

and A the set of all natural numbers up to X, for some fixed choices of primes $p_{i+1}, \ldots p_j$. Hence

$$\begin{aligned}
\#Q^{(i)}(x) \;&=\; \sum_{y<p_j<\cdots<p_{i+1}\leq x^{\frac{1}{j+1}}} \;\sum_{\substack{P^+(m_j)\leq y\\ p_1>\cdots>p_i>x^{\frac{1}{j+1}}\\ p_1\cdots p_i m_j\leq x/(p_{i+1}\cdots p_j)}} 1\\
&\ll \sum_{y<p_j<\cdots<p_{i+1}\leq x^{\frac{1}{j+1}}} \frac{x}{p_{i+1}\cdots p_j}\frac{\log y}{\log x}\\
&\leq \frac{x\log y}{\log x}\left(\sum_{p\leq x^{\frac{1}{j+1}}}\frac{1}{p}\right)^{j-i} \;\ll\; \frac{x\log y}{\log x}(\log\log x)^{j-i},
\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}
\#Q^{(0)}(x) \;&=\; \sum_{y<p_j<\cdots<p_1\leq x^{\frac{1}{j+1}}} \;\sum_{\substack{P^+(m_j)\leq y\\ m_j\leq x/(p_1\cdots p_j)}} 1\\
&\ll \sum_{y<p_j<\cdots<p_1\leq x^{\frac{1}{j+1}}} \frac{x}{p_1\cdots p_j}\exp\left(-\frac{\log(x/p_1\cdots p_j)}{2\log y}\right)\\
&\leq \sum_{y<p_j<\cdots<p_1\leq x^{\frac{1}{j+1}}} \frac{x}{p_1\cdots p_j}\exp\left(-\frac{1}{2(j+1)}\frac{\log x}{\log y}\right)\\
&\leq x\exp\left(-\frac{1}{2(j+1)}\frac{\log x}{\log y}\right)\left(\sum_{p\leq x^{\frac{1}{j+1}}}\frac{1}{p}\right)^{j}\\
&\ll x(\log\log x)^j\exp\left(-\frac{1}{2(j+1)}\frac{\log x}{\log y}\right).
\end{aligned} \tag{2.14}$$

We thus have

$$\#\Phi_j(x,y) \;=\; \sum_{i=0}^{j}\#Q^{(i)}(x) \;\ll\; \frac{x\log y}{\log x}(\log\log x)^{j-1}, \tag{2.15}$$

which completes the proof. □

Remark 2.4. *Since*

$$\{n\leq x : n = p_1\cdots p_j m_j, m_j\leq y<p_j<\cdots<p_1\} \;\subset\; \Phi_j(x,y), \tag{2.16}$$

it follows that

$$\begin{aligned}
\#\Phi_j(x,y) &\geq \sum_{m_j \leq y} \sum_{\substack{n_j \leq \frac{x}{m_j} \\ n_j = p_1 \cdots p_j \\ \text{for some } p_1 > \cdots p_j > y}} 1 \\
&\gg \sum_{m_j \leq y} \frac{x/m_j}{\log(x/m_j)} \left(\log\log \frac{x}{m_j}\right)^{j-1} \\
&\geq \frac{x}{\log x} \left(\log\log \frac{x}{y}\right)^{j-1} \sum_{m_j \leq y} \frac{1}{m_j} \\
&\gg \frac{x \log y}{\log x} \left(\log\log \frac{x}{y}\right)^{j-1}. \qquad (2.17)
\end{aligned}$$

Below we state some simple observations about near-perfect numbers.

Lemma 2.5. *Prime powers cannot be k-near-perfect for any $k \geq 0$.*

Proof. This follows directly from the definition of near-perfect numbers and the uniqueness of q-ary representation. □

Let $\tau(m)$ be the number of positive divisors of the positive integer m. From Lemma 2.5 we immediately deduce the following.

Lemma 2.6. *If $\tau(m)$ is prime, then m cannot be k-near-perfect for any $k \geq 0$. If m is a k-near-perfect number for some $k \geq 0$, then $\tau(m) \geq 4$.*

Finally, a direct computation yields

Lemma 2.7. *If m is a k-near-perfect number for some $k \geq 0$ and $\tau(m) = 4$ or $\tau(m) = 6$, then $m \in \{6, 12, 18, 20, 28\}$.*

The following is a complete classification of 1-near-perfect numbers with two distinct prime factors.

Lemma 2.8. *A 1-near-perfect number which is not perfect and has two distinct prime factors is of the form*

(1) $2^{t-1}(2^t - 2^k - 1)$, *where* $2^t - 2^k - 1$ *is prime,*
(2) $2^{2p-1}(2^p - 1)$, *where* p *is a prime such that* $2^p - 1$ *is also a prime.*
(3) $2^{p-1}(2^p - 1)^2$, *where* p *is a prime such that* $2^p - 1$ *is also a prime.*
(4) 40.

Proof. See [ReCh]. □

Denote by $\Omega(n)$ the number of prime factors of n *counting multiplicities* and let $\Omega(r;x) := \{n \leq x : \Omega(n) = r\}$. We have the following result, due to Landau (Theorem 10.3 of [DeKLu]).

Lemma 2.9. *Fix an integer $r \geq 1$. As $x \to \infty$, we have*

$$\#\Omega(r;x) \sim \frac{1}{(r-1)!} \frac{x}{\log x} (\log\log x)^{r-1}. \qquad (2.18)$$

The equation $\sigma(n) = \ell n + k$ has been studied by a number of authors in the past decades; for more detail, see [AnPoPo, Po, Po1, Po2, PoPo, PoSh, PoPoTh]. In this article, we only need the case of $\ell = 2$ and adopt following definitions from the aforementioned literature.

Definition 2.10 (Regular / Sporadic Solutions)**.** *The solutions of* $\sigma(n) = 2n + k$ *of the form*

$$n = pm', \text{ where } p \nmid m',\ \sigma(m') = 2m',\ \sigma(m') = k, \tag{2.19}$$

are called regular. All other solutions are called sporadic.

Lemma 2.11. *The number of sporadic solutions in* $[1, x]$ *is at most* $x^{3/5+o(1)}$ *as* $x \to \infty$.

Proof. See [PoPoTh]. □

3. Outline of Theorem 1.1 and 1.2

Let us first recall the settings in [PoSh]. In order to estimate the size of the set $N(k; x)$, one may partition it into the following three subsets and estimate each respectively:

$$\begin{aligned} N_1(k;x) &:= \{n \in N(k;x) : P^+(n) \le y\}, \\ N_2(k;x) &:= \{n \in N(k;x) : P^+(n) > y \text{ and } P^+(n)^2 | n\}, \\ N_3(k;x) &:= \{n \in N(k;x) : P^+(n) > y \text{ and } P^+(n) \mid\mid n\}, \end{aligned} \tag{3.1}$$

where $y = y(x)$ is some parameter to be chosen later.

In [PoSh] they further partitioned $N_3(k; x)$ according to whether $\tau(m)$ is at most k or not. They bounded the contribution from $\tau(m) \le k$ simply by $\frac{x}{\log x}(\log\log x)^{k-1}$, i.e., Lemma 2.9. Instead, if one considers the normal order of $\log \tau(n)$, which is $(\log 2)\log\log n$, one obtains the bound $\frac{x}{\log x}(\log\log x)^{\lfloor \frac{\log k}{\log 2}\rfloor}$ for $N(k; x)$. More work is needed, though, as this is still not the correct order for $\#N(k; x)$; we thus have to partition $N_3(k; x)$ more carefully. This is explained as follows.

Suppose $n \in N_3(k; x)$. There exists a set of proper divisors D_n of n with $\#D_n \le k$ such that

$$\sigma(n) = 2n + \sum_{d \in D_n} d, \tag{3.2}$$

and n is of the form

$$n = pm, \text{ where } p > \max\{y, P^+(m)\}. \tag{3.3}$$

Divide D_n into two subsets:

$$\begin{aligned} D_n^{(1)} &:= \{d \in D_n : p \nmid d\}, \\ D_n^{(2)} &:= \{d/p : d \in D_n,\ p \mid d\}. \end{aligned} \tag{3.4}$$

It is clear that

$$\sigma(m) - \sum_{d \in D_n^{(1)}} d \ \ge\ 0, \tag{3.5}$$

and $D_n^{(2)}$ consists of proper divisors of m.

Now also suppose

$$\sigma(m) - \sum_{d \in D_n^{(1)}} d \ =\ 0. \tag{3.6}$$

From

$$(1+p)\sigma(m) \;=\; \sigma(pm) \;=\; 2pm + \sum_{d\in D_n^{(1)}} d + p\sum_{d\in D_n^{(2)}} d, \tag{3.7}$$

we see that

$$\sigma(m) \;=\; 2m + \sum_{d\in D_n^{(2)}} d. \tag{3.8}$$

Since $\#D_n^{(1)} = \tau(m)$ and $\#D_n^{(1)} + \#D_n^{(2)} = \#D_n \le k$, we have $\tau(m) \le k$ and $\#D_n^{(2)} \le k - \tau(m)$. It follows that

$$m \in N(k-\tau(m)). \tag{3.9}$$

To facilitate discussion that follows, we introduce the following notations:

$$\begin{aligned} N_3^{(1)}(k;x) \;&:=\; \{n\in N(k;x) : n = pm,\ p > \max\{y, P^+(m)\}, \tau(m)\le k \\ &\qquad \text{and } m \in N(k-\tau(m))\}, \\ N_3^{(2)}(k;x) \;&:=\; N_3(k;x)\setminus N_3^{(1)}(k;x). \end{aligned} \tag{3.10}$$

Denote by $M(k)$ the set of $n\in N(k)$ such that $n = pm$, $p > P^+(m)$ and

$$\sigma(m) - \sum_{d\in D_n^{(1)}} d \;>\; 0. \tag{3.11}$$

Let $M(k;x) = M(k)\cap[1,x]$.

We carry out the above partition (into $N_1, N_2, N_3^{(1)}, N_3^{(2)}$) recursively in the next section. At each step, we show that the contributions from $N_1, N_2, N_3^{(2)}$ are of acceptable sizes.[4] After this is done, we analyze $N_3^{(1)}(k;x)$ carefully for small integers k. In this way, we improve upon the bound $\frac{x}{\log x}(\log\log x)^{\lfloor \frac{\log k}{\log 2}\rfloor}$ and establish Theorem 1.1. The proof of Theorem 1.2 is simpler. It follows quite directly from the partition as in Theorem 1.1 without encountering complications of the recursive process. We start with its proof in the next section.

4. Proof of Theorem 1.2

Firstly, observe that the contributions from $N_1(k;x)$, $N_2(k;x)$ and $N_3^{(2)}(k;x)$ are acceptable. Indeed, take $y = (\log x)^{3k+10}$ and similar to [PoSh], one has

$$\#N_1(k;x), \#N_2(k;x) \;\ll\; \frac{x}{(\log x)^2}, \tag{4.1}$$

and

$$\#N_3^{(2)}(k;x) \;\ll\; \frac{x}{y}(\log x)^{3k+1} \;\ll\; \frac{x}{(\log x)^2}. \tag{4.2}$$

Consider $n\in N_3^{(1)}(k;x)$, i.e., $n = pm$, $p > \max\{y, P^+(m)\}$ and $m \in N(k-\tau(m))$.

For $k = 4, 5$, $\tau(m) = 4$ and $m\in N(1)$ by Corollary 2.6. By Lemma 2.7, we have $m = 6$. By the Prime Number Theorem, we have

$$\#N_3^{(1)}(4;x) \;=\; \pi(x/6) - \pi(x^{1/\log\log x}) \;\sim\; \frac{1}{6}\frac{x}{\log x}, \tag{4.3}$$

[4] We say that the size of a quantity is *acceptable* if it is not greater than that of the main term, e.g., $\frac{x}{\log x}(\log\log x)^{\lfloor \frac{\log(k+4)}{\log 2}\rfloor - 3}$ in Theorem 1.1 and $x/\log x$ in Theorem 1.2.

and so

$$\#N(4;x) \sim \frac{1}{6}\frac{x}{\log x}. \tag{4.4}$$

For $k=6$, we have $\tau(m)\in\{4,6\}$.

- If $\tau(m)=4$, then $m\in N(2)$. We have $m=6$.
- If $\tau(m)=6$, then $m\in N(0)$. We have $m=28$.

Therefore,

$$\#N(6;x) \sim \frac{17}{84}\frac{x}{\log x}. \tag{4.5}$$

For the rest of the cases claimed in Theorem 1.2, the argument is similar but is more involved. For the case of $k=9$, we have to use the classification result of [ReCh] (Lemma 2.8). The details are included in Appendix B. This completes the proof of Theorem 1.2.

Before we end this section, we record the following observation. It was established in [PoSh] that

$$\#N(k;x) \ll x\exp(-(c_k+o(1))\sqrt{\log x\log\log x}), \tag{4.6}$$

where $c_2=\sqrt{6}/6\approx 0.4082$ and $c_3=\sqrt{2}/4\approx 0.3535$. This follows easily by the method above, with c_k being improved to $1/2$ and now one can take $o(1)$ to be $O(\log_3 x/\log_2 x)$.

By Corollary 2.6, $N_3^{(1)}(3;x)$ is an empty set. So,

$$\begin{aligned}\#N(3;x) &= \#N_1(3;x)+\#N_2(3;x)+\#N_3^{(2)}(3;x)\\ &\ll x\exp(-u\log u+O(u\log\log u))+\frac{x}{y}(\log x)^{10},\end{aligned} \tag{4.7}$$

where $u=(\log x)/(\log y)$. Choose $\log y=\sqrt{\log x\log\log x}$. Then

$$u=\sqrt{\frac{\log x}{\log\log x}} \quad\text{and}\quad u\log u = \frac{1}{2}\sqrt{\frac{\log x}{\log\log x}}(\log\log x-\log\log\log x)\asymp \log y. \tag{4.8}$$

Therefore

$$\#N(3;x) \ll x\exp\left(-\frac{1}{2}\sqrt{\log x\log\log x}\left(1+O\left(\frac{\log_3 x}{\log_2 x}\right)\right)\right). \tag{4.9}$$

5. Proof of Theorem 1.1

As in the proof of Theorem 1.2,

$$\#N_1(k;x),\#N_2(k;x),\#N_3^{(2)}(k;x) \ll \frac{x}{(\log x)^2}. \tag{5.1}$$

Therefore, it now suffices to consider n is of the form p_1m_1 with $p_1>\max\{y,P^+(m_1)\}$, $\tau(m_1)\le k$ and $m_1\in N(k-\tau(m_1))$. Repeat the partition to m_1, i.e., for $R=N_1,N_2,N_3^{(1)},N_3^{(2)}$, consider the sets

$$\left\{n\le x: n=p_1m_1, p_1>\max\{y,P^+(m_1)\}, m_1\in R\left(k-\tau(m_1);\frac{x}{y}\right)\right\}. \tag{5.2}$$

- When $R=N_1,N_2,N_3^{(2)}$, the set (5.2) will be shown to be of acceptable size $O(\frac{x}{\log x}\log\log x)$ in Sections 5.1, 5.2 and 5.3 (in more general settings).

- When $R = N_3^{(1)}$, the set (5.2) is contained in

$$\left\{n \leq x : n = p_1 p_2 m_2,\ p_1 > p_2 > \max\{y, P^+(m_2)\}, m_2 \in N\left(k - 3\tau(m_2); \frac{x}{y^2}\right)\right\}. \tag{5.3}$$

Repeat the partition to m_2, so on and so forth.

In general, let $j \geq 2$ be any integer. Suppose we have carried out this partitioning procedure for $j-1$ times. We arrive at the tasks of estimating the sizes of the sets

$$\left\{n \in Q_{j-1}(x) : m_{j-1} \in R\left(k - (2^{j-1}-1)\tau(m_{j-1}); \frac{x}{y^{j-1}}\right)\right\} \tag{5.4}$$

for $R = N_1, N_2, N_3^{(2)}$, where $Q_{j-1}(x)$ denotes the set of natural numbers in $[1, x]$ of the form $p_1 \cdots p_{j-1} m_{j-1}$ with $p_1 > \cdots > p_{j-1} > \max\{y, P^+(m_{j-1})\}$. This will be done in Sections 5.1, 5.2 and 5.3.

5.1. **Estimation for $R = N_1$.** From Lemma 2.3, we immediately have

$$\begin{aligned} &\#\left\{n \in Q_{j-1}(x) : m_{j-1} \in N_1\left(k - (2^{j-1}-1)\tau(m_{j-1}); \frac{x}{y^{j-1}}\right)\right\} \\ \leq\ &\#\Phi^{(j-1)}(x, y) \\ \ll_k\ &\frac{x}{\log x}(\log\log x)^{j-1}. \end{aligned} \tag{5.5}$$

5.2. **Estimation for $R = N_2$.** From our previous analysis, we have

$$\begin{aligned} &\#\left\{n \in Q_{j-1}(x) : m_{j-1} \in N_2\left(k - (2^{j-1}-1)\tau(m_{j-1}); \frac{x}{y^{j-1}}\right)\right\} \\ \leq &\sum_{\substack{p_1 > \cdots > p_{j-1} > y \\ p_1 \cdots p_{j-1} \leq x}} \sum_{\substack{m_{j-1} \leq x/p_1 \cdots p_{j-1} \\ P^+(m_{j-1})^2 | m_{j-1} \\ P^+(m_{j-1}) > y}} 1 \\ \leq &\sum_{\substack{p_1 > \cdots > p_{j-1} > y \\ p_1 \cdots p_{j-1} \leq x}} \frac{x}{y} \frac{1}{p_1 \cdots p_{j-1}} \\ \leq &\frac{x}{y}\left(\sum_{p \leq x} \frac{1}{p}\right)^{j-1} \ \ll\ \frac{x}{y}(\log\log x)^{j-1} \ \ll\ \frac{x}{(\log x)^{3k+10}}(\log\log x)^{j-1}. \end{aligned} \tag{5.6}$$

5.3. **Estimation of $N_3^{(2)}$.** From [PoSh], we have

$$\#\{n \leq x : n \in M(k), P^+(n) > y\} \ \ll_k\ \frac{x}{y}(\log x)^{3k+1}. \tag{5.7}$$

Then

$$\begin{aligned} \#M(k; x) \ &= \ \#\{n \leq x : n \in M(k), P^+(n) \leq y\} + \#\{n \leq x : n \in M(k), P^+(n) > y\} \\ &\ll_k \ \#\Phi(x, y) + \frac{x}{y}(\log x)^{3k+1} \ \ll_k \ \frac{x}{(\log x)^2}. \end{aligned} \tag{5.8}$$

It follows from partial summation that

$$\sum_{n \in M(k)} \frac{1}{n} \ < \ \infty.$$

Therefore,

$$
\begin{aligned}
&\#\left\{n \in Q_{j-1}(x) : m_{j-1} \in N_3^{(2)}\left(k-(2^{j-1}-1)\tau(m_{j-1}); \frac{x}{y^{j-1}}\right)\right\}\\
\le\ & \sum_{\substack{p_1>\cdots>p_{j-1}>y\\ p_1\cdots p_{j-1}\le\sqrt{x}}}\ \sum_{\substack{m_{j-1}\le x/p_1\cdots p_{j-1}\\ m_{j-1}\in M(k)}} 1 + \sum_{\substack{m_{j-1}\le\sqrt{x}\\ m_{j-1}\in M(k)}}\ \sum_{\substack{p_1>\cdots>p_{j-1}>y\\ p_1\cdots p_{j-1}\le x/m_{j-1}}} 1\\
\ll_k\ & \sum_{\substack{p_1>\cdots>p_{j-1}>\\ p_1\cdots p_{j-1}\le\sqrt{x}}} \frac{\frac{x}{p_1\cdots p_{j-1}}}{(\log\frac{x}{p_1\cdots p_{j-1}})^2} + \sum_{\substack{m_{j-1}\le\sqrt{x}\\ m_{j-1}\in M(k)}} \frac{\frac{x}{m_{j-1}}}{\log\frac{x}{m_{j-1}}}\left(\log\log\frac{x}{m_{j-1}}\right)^{j-2}\\
\ll\ & \frac{x}{(\log x)^2}\sum_{\substack{p_1>\cdots>p_{j-1}>y\\ p_1\cdots p_{j-1}\le\sqrt{x}}}\frac{1}{p_1\cdots p_{j-1}} + \frac{x}{\log x}(\log\log x)^{j-2}\sum_{\substack{m_{j-1}\le\sqrt{x}\\ m_{j-1}\in M(k)}}\frac{1}{m_{j-1}}\\
\ll\ & \frac{x}{(\log x)^2}\left(\sum_{p\le\sqrt{x}}\frac{1}{p}\right)^{j-1} + \frac{x}{\log x}(\log\log x)^{j-2}\\
\ll\ & \frac{x}{\log x}(\log\log x)^{j-2}. \qquad (5.9)
\end{aligned}
$$

5.4. **Analyzing $R = N_3^{(1)}$.** Suppose that for some $j = j(k) \ge 1$ there are only finitely many m_j such that $(2^j-1)\tau(m_j) \le k$ and $m_j \in N(k-(2^j-1)\tau(m_j))$. It follows from Lemma 2.9 that

$$
\begin{aligned}
&\#\left\{n \in Q_{j-1}(x) : m_{j-1} \in N_3^{(1)}(k-(2^{j-1}-1)\tau(m_{j-1}))\right\}\\
\le\ & \#\left\{n \in Q_j(x) : m_j \in N(k-(2^j-1)\tau(m_j))\right\}\\
\ll_k\ & \frac{x}{\log x}(\log\log x)^{j-1}. \qquad (5.10)
\end{aligned}
$$

Putting the estimates (5.5), (5.6), (5.9) and (5.10) together, we have

$$\#N(k;x) \ll_k \frac{x}{\log x}(\log\log x)^{j-1}. \qquad (5.11)$$

Therefore, it remains to determine what $j = j(k)$ one can take such that the above argument is valid. This will be done by a case-by-case study. For simplicity, we just demonstrate one such case below. The analysis of the rest of the cases are similar and we include the detail in Appendix C for the sake of completeness .

Let $\ell \ge -4$. Consider k of the form $2^{s+2}+\ell$ with

$$s > \frac{\log(\ell+6)}{\log 2} - 1. \qquad (5.12)$$

Take $j = s$, i.e., repeat the partitioning processes for $s-1$ times. From (5.12),

$$4 \le \tau(m_s) \le \frac{k}{2^s-1} = \frac{2^{s+2}+\ell}{2^s-1} < 6. \qquad (5.13)$$

By Lemma 2.5, 2.6 and 2.7, $\tau(m_s) = 4$, $m_s \in N(\ell+4)$ and then $m_s = 6$. It follows that

$$\#N(k;x) \ll_k \frac{x}{\log x}(\log\log x)^{s-1}. \qquad (5.14)$$

From (5.12), we have

$$s \ < \ \frac{\log(k+4)}{\log 2} - 2 \ < \ s+1. \tag{5.15}$$

Hence

$$s \ = \ \left\lfloor \frac{\log(k+4)}{\log 2} \right\rfloor - 2 \tag{5.16}$$

and

$$\#N(k;x) \ \ll_k \ \frac{x}{\log x}(\log\log x)^{\lfloor \frac{\log(k+4)}{\log 2} \rfloor - 3} \tag{5.17}$$

for

$$k \in \bigcup_{\ell \geq -4} \left\{ 2^{s+2} + \ell : s > \frac{\log(\ell+6)}{\log 2} - 1 \right\} \ = \ \bigcup_{r \geq 1} [4 \cdot 2^r - 4, 6 \cdot 2^r - 7]_{\mathbb{Z}}. \tag{5.18}$$

Taking account of the cases to be handled in Appendix C as well, Theorem 1.1 holds for

$$k \geq 4 \text{ and } k \neq 2^{s+2} - 6, 2^{s+2} - 5 \text{ for } s \geq 2. \tag{5.19}$$

On the other hand, the lower bound for $\#N(k;x)$ is obvious. Consider simply the natural numbers of the form $6p_1 \cdots p_s = p_1 \cdots p_s + 2p_1 \cdots p_s + 3p_1 \cdots p_s$. By Lemma 2.9, one has

$$\liminf_{x \to \infty} \frac{\#N(k;x)}{\frac{x}{\log x}(\log\log x)^{r-2}} \ \geq \ \frac{1}{6(r-2)!}, \tag{5.20}$$

where $r \geq 2$. This completes the proof of Theorem 1.1.

6. PROOF OF THEOREM 1.3

Let $\epsilon \in (0, 2/5)$. By Lemma 2.11,

$$\begin{aligned} \#(E(k;x) \setminus E_\epsilon(k;x)) \ \leq \ & \#\{n \leq x : n \in E(k), n = pm', p \nmid m', \sigma(m') = 2m'\} \\ & + O(x^{3/5+\epsilon+o(1)}). \end{aligned} \tag{6.1}$$

For $n \in E(k)$ with $n = pm'$, $p \nmid m'$ and $\sigma(m') = 2m'$, we have

$$pm' \ = \ \sum_{d_1 \in D_1} d_1 + p \sum_{d_2 \in D_2} d_2, \tag{6.2}$$

where D_1 is a subset of positive divisors of m', D_2 is a subset of proper divisors of m' with $\#D_1 + \#D_2 = \tau(pm') - 1 - k = 2\tau(m') - 1 - k$.

Suppose that $D_1 \neq \emptyset$. Then

$$1 \ \leq \ \sum_{d_1 \in D_1} d_1 \ \leq \ \sigma(m') \ = \ 2m'. \tag{6.3}$$

Reducing (6.2) modulo p, we have

$$p \ \Big| \ \sum_{d_1 \in D_1} d_1. \tag{6.4}$$

The number of possible values for p is $O(\log 2m') = O(\log x)$. Thus the number of possible values for such n is $O(x^{o(1)} \log x)$ by the Hornfeck-Wirsing Theorem ([HoWi]), which is acceptable.

Now suppose that $D_1 = \emptyset$. Then $\#D_2 = 2\tau(m') - 1 - k$ and

$$m' \;=\; \sum_{d_2 \in D_2} d_2. \tag{6.5}$$

Since $\sigma(m') = 2m'$, we have $\#D_2 = \tau(m') - 1$. Therefore, $\tau(m') - 1 = 2\tau(m') - 1 - k$, i.e., $\tau(m') = k$.

By the hypothesis of non-existence of odd perfect number and the Euclid-Euler Theorem, we have $m' = 2^{q'-1}(2^{q'} - 1)$ for some Mersenne prime q'. So $k = \tau(m') = 2q' \in M$. Hence if $k \notin M$, then we have a contradiction and

$$\#(E(k;x) \setminus E_\epsilon(k;x)) \;=\; O(x^{o(1)} \log x) + O(x^{3/5+\epsilon+o(1)}) \;=\; O(x^{3/5+\epsilon+o(1)}). \tag{6.6}$$

It was shown in [PoSh], by using a form of the Prime Number Theorem of Drmota, Mauduit and Rivat, that for all large k the number of k-exactly-perfect numbers up to x is $\gg_k x/\log x$. Therefore

$$\frac{\#(E(k;x) \setminus E_\epsilon(k;x))}{\#E(k;x)} \;\ll_k\; \frac{\log x}{x^{2/5-\epsilon-o(1)}} \tag{6.7}$$

and

$$\lim_{x\to\infty} \frac{\#E_\epsilon(k;x)}{\#E(k;x)} \;=\; 1. \tag{6.8}$$

Remark 6.1. *Suppose $k \in M$. Then $k = 2q$ for some Mersenne prime q. Let $m = 2^{q-1}(2^q - 1)$. Then $q' = q$ and so $m' = m$ in the above argument. By the Prime Number Theorem,*

$$\limsup_{x\to\infty} \frac{\#(E(k;x) \setminus E_\epsilon(k;x))}{x/\log x} \;\le\; \frac{1}{m}. \tag{6.9}$$

On the other hand, since m is perfect, the number of proper divisors of m is $\tau(m) - 1 = 2q - 1$. Hence pm is a sum of $2q - 1$ of its proper divisors. The number of proper divisors of pm is $\tau(pm) - 1 = 4q - 1$. So, pm is a sum of all of its proper divisors with exactly $(4q-1) - (2q-1) = 2q$ exceptions, i.e., $pm \in E(k)$. Clearly $\sigma(pm) - 2pm < (pm)^\epsilon$ if $p > (2m^{1-\epsilon})^{1/\epsilon}$ and $p \nmid m$. It follows that

$$\liminf_{x\to\infty} \frac{\#(E(k;x) \setminus E_\epsilon(k;x))}{x/\log x} \;\ge\; \frac{1}{m}. \tag{6.10}$$

As a result,

$$\lim_{x\to\infty} \frac{\#(E(k;x) \setminus E_\epsilon(k;x))}{x/\log x} \;=\; \frac{1}{m}. \tag{6.11}$$

Appendix A. Generalizations and Phase-Changes of Near-Perfectness

Throughout this section, k is a positive strictly increasing function. A natural number n is said to be k-*near-perfect* if n is a sum of all of its proper divisors with at most $k(n)$ exceptions, i.e., we allow k to increase with n and larger natural numbers n have more exceptional divisors.

It is well-known that

$$\limsup_{n\to\infty} \frac{\log \tau(n)}{\log n/\log\log n} \;=\; \log 2. \tag{A.1}$$

Take

$$k_0(y) \;:=\; \exp\left(\frac{\log y}{\log\log y}\right). \tag{A.2}$$

The set $N(k_0)$ is simply the set of all pseudoperfect numbers. It suffices to consider k satisfying

$$\limsup_{y\to\infty}\frac{k(y)}{k_0(y)} \ < \ 1. \tag{A.3}$$

It is not hard to observe that if

$$\liminf_{y\to\infty}\frac{k(y)}{(\log y)^{\delta}} \ > \ 0 \tag{A.4}$$

for some $\delta \ > \ \log 2$, then $N(k)$ has the asymptotic density of the set of pseudoperfect numbers. This has little to do with the structure of k-near-perfectness. Indeed, since the normal order of $\log\tau(n)$ is $(\log 2)\log\log n$,

$$\frac{\#N(k_0;x)}{x} \ = \ \frac{\#N(k;x)}{x}+O\left(\frac{1}{x}\#\{n\le x:\tau(n)\ge k(n)\}\right) \ = \ \frac{\#N(k;x)}{x}+o(1)$$

as $x\to\infty$. The result follows.

Lemma A.1. *Uniformly for* $\alpha\in(0,1)$*,*

$$\#\{n\le x:\log\tau(n)\le\alpha\log 2\log\log x\} \ \ll \ x(\log x)^{-B(\alpha)} \tag{A.5}$$

where $B(\alpha)=\alpha\log\alpha-\alpha+1$.

Proof. See for example Theorem 3.7 of Chapter III.3 of [Ten] for the corresponding results for the prime-divisor-counting functions $\omega(n)$ and $\Omega(n)$. The results for $\tau(n)$ follows from $2^{\omega(n)}\le\tau(n)\le 2^{\Omega(n)}$. □

The following result is also not unexpected.

Proposition A.2. *If*

$$\limsup_{y\to\infty}\frac{k(y)}{(\log y)^{\epsilon}} \ < \ 1, \tag{A.6}$$

for some $\epsilon\in(0,\log 2)$*, then* $N(k)$ *has asymptotic density* 0*:*

$$\#N(k;x) \ \ll_{\epsilon} \ \frac{x}{(\log x)^{r(\epsilon)}}, \tag{A.7}$$

where

$$r(\epsilon) \ := \ 1-\frac{\epsilon(1+\log_2 2-\log\epsilon)}{\log 2}\in(0,1). \tag{A.8}$$

Proof. Use the same partition as in [PoSh]: $N_1(k;x), N_2(k;x), N_3'(k;x), N_3''(k;x)$, where $N_3'(k;x)$ consists of the natural numbers in $N_3(k;x)$ such that $\tau(n)\le k$ and $N_3''(k;x)$ otherwise. We now replace certain estimates done in [PoSh].

$$\begin{aligned}
\#N_3''(k;x) \ &\ll \ \frac{x}{y}(\log x)(1+(\log x)^3)^{k(x)}\\
&\ll \ x\log x\exp\left(-\frac{\log x}{\log\log x}\right)\exp(k(x)\log(1+(\log x)^3))\\
&= \ x\log x\exp\left(-\frac{\log x}{\log\log x}\right)\exp\left(3k(x)\log\log x+O\left(\frac{k(x)}{(\log x)^3}\right)\right)\\
&\ll \ x\log x\exp\left(-\frac{\log x}{\log\log x}+3(\log x)^{\log 2}\log\log x\right)\\
&\ll \ x\log x\exp\left(-\frac{\log x}{2\log\log x}\right) \ \ll \ \frac{x}{(\log x)^2}.
\end{aligned} \tag{A.9}$$

By Lemma A.1,

$$\begin{aligned}\#N'(k;x) &\le \#N'((\log y)^{\epsilon};x)\\ &\ll \frac{x}{\log x}(\log x)^{\epsilon(1+\log_2 2)/\log 2}\exp\left(\left(1+\frac{\epsilon\log_2 x}{\log 2}\right)(\log_3 x-\log_2 2(\log x)^{\epsilon})\right)\\ &\ll_{\epsilon} \frac{x}{\log x}(\log x)^{\epsilon(1+\log_2 2)/\log 2}\exp\left(-\log\epsilon\left(1+\frac{\epsilon\log_2 x}{\log 2}\right)\right)\\ &\ll_{\epsilon} \frac{x}{(\log x)^{r(\epsilon)}}, \end{aligned}\tag{A.10}$$

where

$$r(\epsilon) := 1-\frac{\epsilon(1+\log_2 2-\log\epsilon)}{\log 2} \in (0,1). \tag{A.11}$$

□

Appendix B. Cases for $k=7,8,9$ for Theorem 1.2

For $k=7$, we have $\tau(m)\in\{4,6\}$. For $k=8$, $\tau(m)\in\{4,6,8\}$.

- If $\tau(m)=4$, then $m\in N(3)$. We have $m=6$.
- If $\tau(m)=6$, then $m\in N(1)$. We have $m\in\{12,18,20,28\}$.
- If $k=8$ and $\tau(m)=8$, then $m\in N(0)$ and m has at most 3 prime factors. It follows that m cannot be an odd perfect number. [5]. Therefore, m is even and by the Euclid-Euler Theorem, m is of the form $2^{p-1}(2^p-1)$ for some prime p such that 2^p-1 is also a prime. Then $8=\tau(m)=2p$, which is a contradiction. Hence there is no such m.

Therefore, we have

$$\#N(7;x),\#N(8;x) \sim \frac{493}{1260}\frac{x}{\log x}. \tag{B.1}$$

For $k=9$, $\tau(m)\in\{4,6,8,9\}$. Again if $\tau(m)=4$ or 6, $m\in\{6,12,18,20,28\}$.

- If $\tau(m)=8$, then $m\in N(1)$. By the discussion in the case $k=8$, m cannot be perfect. By Lemma 2.5, we have m is of the form q^3r or qrs, where q,r,s are distinct primes. For the first case we have $m\in\{24,40,56,88,104\}$ by using Lemma 2.8. For the second case, we consider the following set of equations

$$\begin{aligned}(1+q)(1+r)(1+s)&=2qrs+1,\\ (1+q)(1+r)(1+s)&=2qrs+q,\\ (1+q)(1+r)(1+s)&=2qrs+qr,\end{aligned}\tag{B.2}$$

  in which it is easy to check all of them have no solution.
- If $\tau(m)=9$, then $m\in N(0)$. By a similar discussion as that in the case of $k=8$, there is no such m.

Therefore, we have

$$\#N(9;x) \sim \frac{179017}{360360}\frac{x}{\log x}. \tag{B.3}$$

---

[5] In fact, it is now known that an odd perfect number must have at least 10 distinct prime factors. This is due to Nielsen [Niel]. The proof of an odd perfect has at least 4 distinct prime factors is completely elementary.

APPENDIX C. THE REST OF THE CASES OF THEOREM 1.1

(1) Let $\ell > 8$. Consider k of the form $2^{s+2} - \ell$ for

$$s \ \geq \ \frac{\log(\ell - 4)}{\log 2} - 1. \tag{C.1}$$

Take $j = s - 1$. Then

$$4 \ \leq \ \tau(m_{s-1}) \ \leq \ \frac{2^{s+2} - \ell}{2^{s-1} - 1} \ = \ 8 - \frac{\ell - 8}{2^{s-1} - 1} \ < \ 8. \tag{C.2}$$

By Lemma 2.5, 2.6 and 2.7, $\tau(m_{s-1}) = \{4, 6\}$ and $m_{s-1} \in \{6, 12, 18, 20, 28\}$. Hence

$$\#N(k; x) \ \ll_k \ \frac{x}{\log x} (\log \log x)^{s-2}. \tag{C.3}$$

By the following sets of elementary inequalities:

$$\frac{\log(k + 4)}{\log 2} \ = \ \frac{\log(2^{s+2} + 4 - \ell)}{\log 2} \ < \ \frac{\log(2^{s+2} - 4)}{\log 2} \ < \ s + 2, \tag{C.4}$$

$$2^{s+1} \ \geq \ \ell - 4 \ = \ 2^{s+2} - k - 4, \tag{C.5}$$

and

$$\frac{\log(k + 4)}{\log 2} \ \geq \ s + 1, \tag{C.6}$$

we have

$$\left\lfloor \frac{\log(k + 4)}{\log 2} \right\rfloor \ = \ s + 1. \tag{C.7}$$

Therefore,

$$\#N(k; x) \ \ll_k \ \frac{x}{\log x} (\log \log x)^{\lfloor \frac{\log(k+4)}{\log 2} \rfloor - 3} \tag{C.8}$$

for

$$k \in \bigcup_{\ell > 8} \left\{ 2^{s+2} - \ell : s \geq \frac{\log(\ell - 4)}{\log 2} - 1 \right\} \ = \ \bigcup_{r \geq 2} [3 \cdot 2^r - 6, 4 \cdot 2^r - 9]_{\mathbb{Z}}. \tag{C.9}$$

(2) For $k = 2^{s+2} - 8$ and $s \geq 2$, we have $4 \leq \tau(m_{s-1}) \leq 8$ and $m_{s-1} \in N(2^{s+2} - 8 - (2^{s-1} - 1)\tau(m_{s-1}))$. This was settled in the case of $k = 8$ in Appendix B. Therefore,

$$\#N(k; x) \ \ll_k \ \frac{x}{\log x} (\log \log x)^{s-2} \ = \ \frac{x}{\log x} (\log \log x)^{\lfloor \frac{\log(k+4)}{\log 2} \rfloor - 3}. \tag{C.10}$$

(3) For $k = 2^{s+2} - 7$ and $s \geq 3$, we have $4 \leq \tau(m_{s-1}) \leq 8$ and $m_{s-1} \in N(2^{s+2} - 7 - (2^{s-1} - 1)\tau(m_{s-1}))$. This was settled in the case of $k = 9$ in Appendix B . Therefore,

$$\#N(k; x) \ \ll_k \ \frac{x}{\log x} (\log \log x)^{s-2} \ = \ \frac{x}{\log x} (\log \log x)^{\lfloor \frac{\log(k+4)}{\log 2} \rfloor - 3}. \tag{C.11}$$

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