AVOIDING 3-TERM GEOMETRIC PROGRESSIONS IN NON-COMMUTATIVE SETTINGS

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ABSTRACT. Several recent papers have considered the Ramsey-theoretic problem of how large a subset of integers can be without containing any 3-term geometric progressions. This problem has also recently been generalized to number fields and $\mathbb{F}_q[x]$. We study the analogous problem in two noncommutative settings, quaternions and free groups, to see how lack of commutivity affected the problem. In the quaternion case, we show bounds for the supremum of upper densities of 3-term geometric progression avoiding sets. In the free groups case, we calculate the decay rate for the greedy set in $\langle x, y : x^2 = y^2 = 1 \rangle$ avoiding 3-term geometric progressions.

CONTENTS

1. INTRODUCTION

Classically, there has been interest in how large a set can be while still avoiding arithmetic or geometric progressions. In a 1961 paper Rankin [Ran] introduced the idea of considering how large a set of integers can be without containing terms which are in geometric progression. He constructed a subset of the integers which avoids 3-term geometric progressions and has asymptotic density approximately 0.719745. Brown and Gordon [BG] noted that the set Rankin considered was the set obtained by greedily including integers subject to the condition that such integers don't create a progression involving integers already included in the set.

This question has been generalized to number fields [BHMMPTW] and polynomial rings over finite fields [AFGMMMM]. The purpose of [BHMMPTW] was to see how changing from subsets of $\mathbb{Z}$ to subsets of number fields affected the answer, while in [AFGMMMM] it was to see how the extra combinatorial structure of $\mathbb{F}_q[x]$ affected the tractability and features of the problem. In our case, we wish to see how non-commutativity affects the answer.

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The first half of this paper (Sections 2 through 5) is dedicated to studying the problem in the Hurwitz order quaternions, Q_{Hur} (see Section 2 for a review of their properties). We produce some bounds on the supremum of upper densities of sets avoiding 3-term geometric progressions, and use Rankin's greedy set to construct a high density set avoiding 3-term geometric progressions. We also discuss the peculiarities of this setting in Section 5. The second half (Section 6) is dedicated to studying the question in the setting of free groups. We arrive at the following results.

Theorem 3.1. *Let m_{Hur} be supremum of upper densities of subsets of Q_{Hur} containing no 3-term geometric progressions. Then*

$$.946589 \;\leq\; m_{\text{Hur}} \;\leq\; .952381. \tag{1.1}$$

Theorem 4.2. *Let Q_{Ran} be the set of Hurwitz quaternions with norm in Rankin's greedy set (avoiding 3-term geometric progressions in $\mathbb{Z}$). Let $A_3^*(\mathbb{Z})$ be the greedy set avoiding 3-term arithmetic progressions. The asymptotic density of Q_{Ran} is*

$$d(Q_{\text{Ran}}) \;=\; \left(\prod_{p \text{ odd}} \left[\sum_{n \in A_3^*(\mathbb{Z})} \frac{p^{n+3} - p^{n+2} - p^2 + 1}{p^2(p-1)p^{2n}}\right]\right) \cdot \left(\sum_{n \in A_3^*(\mathbb{Z})} \frac{2^2-1}{2^2 2^{2n}}\right). \tag{1.2}$$

Theorem 6.2. *Let $G = \langle x, y : x^2 = y^2 = 1\rangle$ be the free group on two generators each of order two. Order the group as $W = (I, x, y, xy, yx, xyx, yxy, xyxy, yxyx, \dots)$ and take the set G formed by greedily taking elements that don't form a 3-term progression with previously added ones. Then*

$$\frac{|G \cap \{w \in W : \text{length}(w) \leq 2 \cdot 3^n\}|}{|\{w \in W : \text{length}(w) \leq 2 \cdot 3^n|\}|} \;=\; \frac{2^{n+1}}{1 + 4 \cdot 3^n}, \tag{1.3}$$

and in general

$$\frac{|G \cap \{w \in W : \text{length}(w) \leq n\}|}{|\{w \in W : \text{length}(w) \leq n\}|} \;=\; \Theta\left((2/3)^{\log_3 n}\right). \tag{1.4}$$

2. Review of Hurwitz order quaterions

When considering the non-commutative analogue, the quaternions are a natural choice to look into first given they are one of the first non-commutative algebras one runs into, and because they have a norm. We will restrict our attention to the Hurwitz order of quaternions, due to the existence of prime factorization in the Hurwitz order.

Definition 2.1. *Quaternions constitute the algebra over the reals generated by units i, j, and k such that*

$$i^2 \;=\; j^2 \;=\; k^2 \;=\; ijk \;=\; -1.$$

Quaternions can be written as $a + bi + cj + dk$ for $a, b, c, d \in \mathbb{R}$.

Definition 2.2. *We say that $a + bi + cj + dk$ is in the Hurwitz order,* Hur, *if a, b, c, d are all in $\mathbb{Z}$ or all in $\mathbb{Z} + \frac{1}{2}$.*

Definition 2.3. *The Norm of a quaternion $Q = a + bi + cj + dk$ is given by* $\text{Norm}(Q) = a^2 + b^2 + c^2 + d^2$.

An element $P \in$ Hur is said to be prime if and only if its norm is prime.

Theorem 2.4. *Let $Q \in$ Hur. For every factorization $\text{Norm}(Q) = p_0 p_1 \dots p_k$ of the norm, there is a factorization*

$$Q = P_0 \dots P_k,$$

of Q into Hurwitz primes such that such that $N(P_i) = p_i$ for all $0 \leq i \leq k$. We call such a factorization modelled on the factorization $p_0 \dots p_k$ of $\text{Norm}(Q)$. Furthermore, any other factorization modelled on $\text{Norm}(Q) = p_0 \dots p_k$ is of the form:

$$Q = P_0 U_1 \cdot U_1^{-1} P_1 U_2 \cdot \dots \cdot U_k^{-1} P_k.$$

That is, the factorization is unique up to unit-migration, also known in this setting as metacommutation.

Proof. See Chapter 5.2, Theorem 2 in [ConSm]. □

We will need a few facts about Hurwitz order quaternions to start calculating some of the densities and bounds. Namely, we want to know the number of Hurwitz quaternions up to a certain norm, number of Hurwitz quaternions of a particular norm, and the proportion of Hurwitz quaternions whose norm is divisible exactly by p^n. Readers with knowledge of the Hurwitz quaternions may wish to skip Section 2.1 and briefly skim Section 2.2. Section 2.1 will be used throughout, and Section 2.2 will specifically be useful for Section 4. For a more in-depth discussion of the Hurwitz order, see Chapter 5 in [ConSm].

2.1. **Quaternions up to a given norm.** We wish to count the number of quaternions with norm in $[0, M]$. Viewing the quaternions as $\mathbb{R}^4$, this reduces to counting points in 4-spheres.

Definition 2.5. *Let* $S(r) := \{h \in Q_{\text{Hur}} : N(h) \leq r\}$. *Similarly, for* $X \subseteq \mathbb{R}_{\geq 0}$, $S(X) := \{h \in Q_{\text{Hur}} : N(h) \in X\}$.

Lemma 2.6. *The number N(R) of lattice points in a 3-sphere of radius R is*

$$N(R) \;=\; V(R) \;+\; O(R^2\log^2 R) \;=\; \frac{\pi^2}{2}R^4 \;+\; O(R^2\log^2 R), \tag{2.1}$$

where $V(R)$ *is the volume of a 3-sphere.*

Proof. See [Mit]. □

This gives us the number of integer coordinate quaternions in a 3-sphere. We now turn to Hurwitz quaternions.

Lemma 2.7. *The number of Hurwitz quaternions with norm less than or equal to M is*

$$|S(M)| \;=\; 1/2^3 V(2\sqrt{M}) + O(M^{3/2}) \;=\; \pi^2 R^2 + O(M^{3/2}) \tag{2.2}$$

Proof. Since our norm is defined as the sum of the squares, the quaternions of norm up to M correspond to elements in the 4-sphere of radius $\sqrt{M}$. If we think of all the Hurwitz quaternions in a radius R 3-sphere, by multiplying their coordinates by 2 we can view them as lattice points in a sphere of radius $2R$. The Hurwitz quaternions will injectively get mapped to the lattice points with all coordinates even or all coordinates odd. The probability that a random quanternion will have all even or all odd coordinates is $1/2^3$.

However there are issues with points being missed around the boundary, which is 3-dimensional, so we get the rough count of

$$|S(M)| \;=\; 1/2^3 V(2\sqrt{M}) + O(M^{3/2}) \;=\; \pi^2 R^2 + O(M^{3/2}). \tag{2.3}$$

□

2.2. **Quaternions of a particular norm & prime divisors of norms.**

Lemma 2.8. *The number of Hurwitz quaternions of norm N is*

$$S(\{N\}) \;=\; 24 \sum_{2\nmid d|N} d, \tag{2.4}$$

the sum of the odd divisors of N multiplied by 24.

The proof of Lemma 2.8 can be found here [ConSm]. This fact allows us to offer an alternative proof of Lemma 2.7.

Proof. We can write the number of Hurwitz quaternions up to some norm M as

$$\begin{aligned} S(M) &= 24 \sum_{n<M} \sum_{\substack{d\mid n \\ 2\nmid d}} d \\ &= 24 \sum_{d<M} \sum_{\substack{e<\lfloor \frac{M}{d} \rfloor \\ 2\nmid e}} e \\ &= 24 \sum_{d<M} \frac{1}{4} \left(\left\lfloor \frac{M}{d} \right\rfloor + O(1) \right) \left\lfloor \frac{M}{d} \right\rfloor \\ &= \frac{24M^2}{4} \sum_{d<M} \frac{1}{d^2} + O\left(\sum_{d<M} \frac{M}{d} \right) \\ &= 6M^2 \frac{\pi^2}{6} + O\left(\sum_{d<M} \frac{M}{d} \right) \\ &= \pi^2 M^2 + O\left(\sum_{d<M} \frac{M}{d} \right). \end{aligned} \tag{2.5}$$

□

Lemma 2.9. *If p is odd, the proportion of Hurwitz quaternions whose norm is divisible by p^n but not p^{n+1} is*

$$\frac{p^{n+3} - p^{n+2} - p^2 + 1}{(p-1)p^2 p^{2n}}. \tag{2.6}$$

If $p = 2$, this proportion is instead

$$\frac{2^2 - 1}{2^2 2^{2n}} = \frac{3}{4 \cdot 2^{2n}}. \tag{2.7}$$

Proof. We first calculate the probability that the norm of a Hurwitz quaternion is divisible by p^n and not p^{n+1}. Consider the set $S(N)$, the set of Hurwitz quaternions with norm greater than or equal to N. Since we can always find a factorization of h based off any permutation of the prime factors of $N(h)$ (Theorem 2.4), we can always write

$$h = PH \tag{2.8}$$

where $N(P) = p^k$, with k being the largest power of p that divides $N(h)$. There are 24 ways to write h in this form, since $h = Pu \cdot u^{-1}H$ as well. Thus the proportion of elements of $S(N)$ that have at least a factor of p^n in their norm is

$$\frac{|S(\{p^n\})| \cdot |S(N/p^n)|}{24|S(N)|}. \tag{2.9}$$

From Lemma 2.8 we calculate that for an odd prime p,

$$S(\{p^n\}) = 1 + p + p^2 + \cdots + p^n = \frac{p^{n+1} - 1}{p - 1}. \tag{2.10}$$

Note that $|S(N)| = \pi^2 N^2 + O(N^{3/2})$ by Lemma 2.7 . Substituting all this information into Equation (2.9) yields

$$\frac{(p^{n+1} - 1)(\pi^2 (N/p^n)^2 + O(N^{3/2}))}{(p-1)(\pi^2 N^2 + O(N^{3/2}))}. \tag{2.11}$$

Subtracting the proportion of $S(N)$ of elements whose norm is at least divisible by p^{n+1} and taking the limit as $N \to \infty$, we get the proportion of elements of $S(N)$ whose norm is divisible by p^n but not divisible by p^{n+1}.

$$\frac{(p^{n+1} - 1)(\pi^2 (N/p^n)^2 + O(N^{3/2}))}{(p-1)(\pi^2 N^2 + O(N^{3/2}))} - \frac{(p^{n+2} - 1)(\pi^2 (N/p^{n+1})^2 + O(N^{3/2}))}{(p-1)(\pi^2 N^2 + O(N^{3/2}))}. \tag{2.12}$$

Taking the limit $N \to \infty$ gives the proportion of Q_{Hur} whose norm is exactly divisible by p^n.

$$\lim_{n\to\infty} \frac{(p^{n+1}-1)(\pi^2(N/p^n)^2) - (p^{n+2}-1)(\pi^2(N/p^{n+1})^2) + O(N^{3/2})}{(p-1)(\pi^2N^2) + O(N^{3/2})} \;=\; \frac{p^{n+3}-p^{n+2}-p^2+1}{(p-1)p^2p^{2n}}. \quad (2.13)$$

An analogous calculation for $p = 2$, using that $S(\{2^k\}) = \sum_{2\nmid d|2^k} d = 1$, gives us that the proportion of elements whose norm is exactly divisibly by 2^n is

$$\frac{2^2-1}{2^2 2^{2n}} \;=\; \frac{3}{4 \cdot 2^{2n}}. \quad (2.14)$$

□

3. Bounds on the supremum of the upper densities

3.1. **Lower bound.** One will note that working in a space that looks geometrically like $\mathbb{R}^4$ tends to up the exponents in our formulas, causing lower order terms to go to zero quicker and pushing the estimations and bounds we caluclate closer to 1 than in previous settings.

Viewing the quaternions as $\mathbb{R}^4$, we use the fact that the smallest non-unit ratio between quaternions is 2 to construct a union of six hyperspheric annuli that does not contain any geometric progressions. We choose the norm ranges, i.e., unions of intervals in $\mathbb{R}_{\geq 0}$, that induce these annuli to avoid geometric progressions in the norm, which implies we avoid geometric progressions in the quaternion elements themselves.

This construction is done in [McN] for the integers. Since Hurwitz quaternions can have any integer as a norm, the intervals chosen there also work in our case.

For $S(M)$, M large, consider $S((M/4, M])$. Since the smallest non-unit ratio for a geometric progression is 2, this set has no 3-term progressions in the norms, and thus cannot have any 3-term progressions in its elements. Thus, the proportion of elements in $S((M/4, M])$ compared to $S(M)$ is

$$\frac{|S((M/4,M])|}{|S(M)|} \;=\; \frac{(1/8)V(2\sqrt{M}) - (1/8)V(2\sqrt{M/4}) + O(M^{3/2})}{(1/8)V(2\sqrt{M}) + O(M^{3/2})} \;=\; \frac{\pi^2M^2 - \pi^2(M/4)^2 + O(M^{3/2})}{\pi^2M^2 + O(M^{3/2})}. \quad (3.1)$$

As $M \to \infty$ this proportion goes to $1 - 1/16 = 15/16$. We can get a higher proportion by including more annuli. We define

$$T_M \;:=\; \left(\frac{M}{48}, \frac{M}{45}\right] \cup \left(\frac{M}{40}, \frac{M}{36}\right] \cup \left(\frac{M}{32}, \frac{M}{27}\right] \cup \left(\frac{M}{24}, \frac{M}{12}\right] \cup \left(\frac{M}{9}, \frac{M}{8}\right] \cup \left(\frac{M}{4}, M\right]. \quad (3.2)$$

Fix $N = 1$ and $N_i = 48^2 N_{i-1}^2$ for $i \geq 2$. Consider

$$S_N \;=\; \bigcup_{M\in\mathbb{N}} T_M. \quad (3.3)$$

The proof that S_N avoids geometric progressions can be found in Theorem 3.1 of [McN]. The upper density of this set is the proportion $|T_M|/|S(M)|$ as $M \to \infty$:

$$\begin{aligned}
\overline{d}(S_N) \;&=\; \lim_{M\to\infty} \frac{|T_M|}{|S(M)|} \\
&=\; \lim_{M\to\infty} \frac{1}{|S(M)|} \cdot \Big[S((M/48, M/45]) + S((M/40, M/36]) + S((M/32, M/27]) + \\
&\qquad S((M/24, M/12]) + S((M/9, M/8]) + |S((M/4, M])|\Big] \\
&=\; \frac{1}{M^2}\Big[[M^2 - (M/4)^2] + [(M/8)^2 - (M/9)^2] + [(M/12)^2 - (M/24)^2] + \\
&\qquad [(M/27)^2 - (M/32)^2] + [(M/36)^2 - (M/40)^2] + [(M/45)^2 - (M/48)^2]\Big] \\
&=\; \Big[[1 - 1/4^2] + [1/8^2 - 1/9^2] + [1/12^2 - 1/24^2] + \\
&\qquad [1/27^2 - 1/32^2] + [1/36^2 - 1/40^2] + [1/45^2 - 1/48^2]\Big],
\end{aligned} \quad (3.4)$$

which, to six decimal places, is .946589.

3.2. **Upper bound.** We generalize a construction done in [McN] where we show a certain proportion of elements are forced to be removed to avoid three-term progressions. Namely, we look at disjoint 3-tuples (b, rb, r^2b), from which one element must be excluded. We pick r to have the smallest norm, 2, to get a large number of exclusions.

By Lemma 2.8, there is one prime r of norm 2 up to unit multiples on either side. As an analogue of "coprime", by Lemma 2.9, 3/4 of Hurwitz quaternions have no power of 2 in their norm, and thus contain no factors of r in their factorization.

Fix r a prime of norm 2. Consider $S(M)$ for large M. Then if $b \in S(M)$, $N(b) \leq M/4$, and b has no power of 2 in its norm, then b, rb, rb^2 forms a progression, and all such sequences are disjoint for different b.

If b has no power of 2 in its norm and $N(b) \leq M/2^5$, then a similar argument follows with r^3b, r^4b, r^5b. Looking at the norms of r^3b, r^4b, r^5b, these sequences are disjoint from the b', rb', r^2b' sequences from before. So for each b (up to units on the left) with no power of 2 in its norm, and $N(b) \leq M/2^5$, we need to made an additional exclusion to avoid three-term progressions. Taking $M \to \infty$ we get an upper bound of

$$\begin{aligned}
\lim_{M\to\infty} 1 - \left(\frac{3}{4}\right)\left(\frac{|S(M/2^2)| + |S(M/2^5)|}{|S(M)|}\right) &= \lim_{M\to\infty} 1 - \left(\frac{3}{4}\right)\left(\frac{\pi^2M^2/2^4 + \pi^2M^2/2^{10} + O(M^{3/2})}{\pi^2M2 + O(M^{3/2})}\right) \\
&= 1 - \frac{3}{2^6} - \frac{3}{2^{12}} \\
&\approx .952393. \qquad (3.5)
\end{aligned}$$

We can improve this bound further by considering more b. Looking at b's in $S(M/2^2), S(M/2^5), S(M/2^8), \ldots$ we get an upper bound of

$$\begin{aligned}
\lim_{M\to\infty} 1 - \left(\frac{3}{4}\right)\left(\frac{|S(M/2^2)| + |S(M/2^5)| + \cdots + |S(M/2^{2+3i})| + \ldots}{|S(M)|}\right) &= 1 - \frac{3}{4}\cdot\frac{1}{2^4}\cdot\sum_{i=0}^{\infty}\frac{1}{2^{6i}} \\
&= 1 - \frac{3}{2^6}\left(\frac{1}{1-1/2^6}\right) \\
&= 1 - \frac{3}{2^6-1} \\
&\approx .952381. \qquad (3.6)
\end{aligned}$$

From the two subsections, we get the following.

Theorem 3.1. *Let m_{Hur} be supremum of upper densities of subsets of Q_{Hur} containing no 3-term geometric progressions. Then:*

$$.946589 \leq m_{\mathrm{Hur}} \leq .952381. \qquad (3.7)$$

4. Density of Rankin's quaternion greedy set

Consider the set $G_3^*(\mathbb{Z}) = \{1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, \ldots\}$, which we will refer to as Rankin's (geometric) greedy set. $G_3^*(\mathbb{Z})$ is the set formed by greedily including integers that do not form 3-term geometric progressions with the previous elements. Since geometric progressions give arithmetic progressions in their terms' prime powers, $G_3^*(\mathbb{Z})$ is the set of elements whose prime factors' exponents are all in $A_3^*(\mathbb{Z}) = \{0, 1, 3, 4, 8, 10, 12, 13\ldots\}$, the set formed by greedily taking integers that do not form an arithmetic progression. $A_3^*(\mathbb{Z})$ is also the set of integers whose ternary expansion does not contain the digit 2.

Definition 4.1. *We define Q_{Ran} as the set of Hurwitz quaternions whose norm is in Rankin's greedy set:*

$$Q_{\mathrm{Ran}} := \{h \in Q_{\mathrm{Hur}} : N(h) \in G_3^*(\mathbb{Z})\}. \qquad (4.1)$$

Since this set avoids progressions in the norms of its elements, it avoids progressions in its quaternion elements. We wish to deduce the density of this set. We will do this be calculating the probably that an element has norm divisible by a suitable power of p.

Theorem 4.2. *The asymptotic density of Q_{Ran} is*

$$d(Q_{\text{Ran}}) = \left(\prod_{p \text{ odd}} \left[\sum_{n\in A_3^*(\mathbb{Z})} \frac{p^{n+3}-p^{n+2}-p^2+1}{p^2(p-1)p^{2n}}\right]\right) \cdot \left(\sum_{n\in A_3^*(\mathbb{Z})} \frac{2^2-1}{2^2 2^{2n}}\right). \tag{4.2}$$

Proof. By Lemma 2.9 the probability that the norm of a Hurwitz quaternion has norm exactly divisible by p^n, for p odd, is

$$\frac{p^{n+3}-p^{n+2}-p^2+1}{(p-1)p^2p^{2n}}, \tag{2.6}$$

and $(2^2-1)/(2^2 2^{2n})$ for $p = 2$. So the probability that the norm of a Hurwitz quaternion has a proper power of an odd p (that is, a power of p in $A_3^*(\mathbb{Z})$) is

$$\sum_{n\in A_3^*(\mathbb{Z})} \frac{p^{n+3}-p^{n+2}-p^2+1}{(p-1)p^2p^{2n}}. \tag{4.3}$$

Note that in Equation (2.12), with respect to the p factors the expression is $\sim 1/p^n$, so even with the error terms we should get proper convergence of the sum. The proportion of Hurwitz quaternions with a proper power of 2 in their norm is

$$\left(\sum_{n\in A_3^*(\mathbb{Z})} \frac{2^2-1}{2^2 2^{2n}}\right). \tag{4.4}$$

By a Chinese Remainder Theorem-type argument, we get the desired product:

$$d(Q_{\text{Ran}}) = \left(\prod_{p \text{ odd}} \left[\sum_{n\in A_3^*(\mathbb{Z})} \frac{p^{n+3}-p^{n+2}-p^2+1}{p^2(p-1)p^{2n}}\right]\right) \cdot \left(\sum_{n\in A_3^*(\mathbb{Z})} \frac{2^2-1}{2^2 2^{2n}}\right). \tag{4.2}$$

□

5. The quaternion greedy set

In a similar style to Rankin, we can form a greedy set of quaternions, call it $G_3^*(\text{Hur})$, if we include quaternions that do not form geometric progression. This greedy inclusion will result in a set similar to Q_{Ran}. However, due to some properties of quaternions, discrepancies between $G_3^*(\text{Hur})$ and Q_{Ran} will arise. For example, due to an argument relying on the fact that 7 cannot be written as the sum of three squares, we find that not all quaternions of norm 49 can be written as the square of a quaternion of norm 7 multiplied by a unit on the left. This results in some elements of norm 49 being included in $G_3^*(\text{Hur})$, whereas 49 is not in $G_3^*(\mathbb{Z})$ and therefore no elements of norm 49 are contained in Q_{Ran}. As a result, some items of norm 343 will form a geometric progression in $G_3^*(\text{Hur})$ and will thus be excluded while all Hurwitz quaternions of norm 343 are included in Q_{Ran}. This sequence of inclusions and exclusions will continue for all powers of 7.

This occurrence is also not specific to 7 but all integers that cannot be written as the sum of three squares. Furthermore, any integer with an odd divisor greater than 23 poses the same problem.

Lemma 5.1. *The set of quaternions with norm, n, divisible by significantly high integers cannot completely realize all quaternions of norm n^2 by squaring and multiplying by a unit on the left.*

Proof. Lemma 2.8 allows us to write $S(\{n\})$ as $24\sum_{2\nmid d|n} d$ and $S(\{n^2\})$ as $24\sum_{2\nmid d|n^2} d$. Then the number of possibilities for a square of norm n multiplied by a unit on the left is $24 * S(\{n\})$. Let D be the greatest odd divisor of n. Then we have

$$24 * D * \sum_{2\nmid d|n} d \le 24 * \sum_{2\nmid d|n^2} d.$$

Thus, if $D > 23$, then

$$\begin{aligned} 24 * D * \sum_{2\nmid d|n} d \ &\leq \ 24 * \sum_{2\nmid d|n^2} d \\ &\leq \ 24 * 24 * \sum_{2\nmid d|n} d. \end{aligned} \tag{5.1}$$

Therefore by a simple counting argument, the set of quaternions with norm n, where n has an odd divisor greater than 23, cannot square to realize all quaternions of norm n^2. □

6. Free groups

6.1. **Introduction to the free groups case.** We now consider the case of subsets of free groups containing no three-term geometric progressions. Due to the nature of free groups and not being able to "space out" geometric progressions as in the integers, this case acts much more arithmetically. In fact we get an analogue of Szemerédi's theorem: any subset of a freegroup with positive natural density (where the limit is taken over the length of an element) has arbitrarily long geometric progressions. We instead consider an often overlooked question.

Question 6.1. *In "greedily formed sets" avoiding three-term geometric progressions, which are generally the best candidate for high-density or large sets avoiding progressions, what is the rate of decay for the density? That is, how quickly is the density limit*

$$\lim_{\text{length}(g)\leq n} \frac{|S \cap \{g : \text{length}(g) \leq n\}|}{|\{g : \text{length}(g) \leq n\}|} \tag{6.1}$$

going to zero?

The combinatorics involved quickly becomes quite tedious, but we are able to calculate this in the case of the free group on two generators of order two. The rest of the paper will be spent resolving this case, resulting in the following theorem.

Theorem 6.2. *Let $G = \langle x, y : x^2 = y^2 = 1\rangle$ be the free group on two generators each of order two. Order the group as $W = (I, x, y, xy, yx, xyx, yxy, xyxy, yxyx, \dots)$ and take the set G formed by greedily taking elements that don't form a 3-term progression with previously added ones. Then*

$$\frac{|G \cap \{w \in W : \text{length}(w) \leq 2\cdot 3^n\}|}{|\{w \in W : \text{length}(w) \leq 2\cdot 3^n|\}|} \ = \ \frac{2^{n+1}}{1+4\cdot 3^n}, \tag{6.2}$$

and in general

$$\frac{|G \cap \{w \in W : \text{length}(w) \leq n\}|}{|\{w \in W : \text{length}(w) \leq n\}|} \ = \ \Theta\left((2/3)^{\log_3 n}\right). \tag{6.3}$$

6.2. **The case of the free group on two generators of order two.** Consider $W = \langle x, y : x^2 = y^2 = 1\rangle$, the free group generated by two elements, x and y, both of order two. Order the group

$$W = (I, x, y, xy, yx, xyx, yxy, xyxy, yxyx, \dots),$$

by word length, with $x < y$. Let w_n be the n^{th} element in the set, so $w_1 = I, w_2 = x$, and so on.

Definition 6.3. *Let $G_1 = \{I\}$, and recursively define G_n to be $G_{n-1} \cup \{w_n\}$ if w_n does not form a geometric progression with the elements of G_{n-1}, and set it to be G_{n-1} otherwise. Define*

$$G \ =: \ \bigcup_{i=1}^{\infty} G_n. \tag{6.4}$$

Then G is the set formed by greedily taking elements from W that do not form 3-term progressions with the previous elements.

Our next few propositions will make clear the arithmetic nature of this set. First, we order the integers alternatingly as $\mathbb{Z}_A = (0, 1, -1, 2, -2, 3, \dots)$ and use z_n to denote the n^{th} element. We similarly define $A_1 = \{0\}$ and define $A_{n+1} = A_n \cup \{z_{n+1}\}$ if s_n does not form a 3-term arithmetic progression with the other elements, and set $A_{n+1} = A_n$ otherwise. Then

$$A := \bigcup_{n=1}^{\infty} A_n, \tag{6.5}$$

is the greedy set constructed so that it has no 3-term arithmetic progressions.

Note that for an element $x \in W$ with odd length, $x, 1, x$ is a progression with ratio x. Thus G contains no odd length elements. If one wishes to exclude cases like this, accounting for odds becomes similar to the even case. In our case, we choose to count $x, 1, x$ as a valid progression and hence all odd elements are excluded from G.

Proposition 6.4. *Denote the (ordered) subgroup of W generated by xy, yx by W_2. Then*

$$\begin{aligned} W_2 &\to \mathbb{Z}_A \\ xy &\mapsto 1 \end{aligned} \tag{6.6}$$

is an isomorphism of groups that preserves the orderings on each.

Thus it's enough to work with A and to determine the density of A in $\mathbb{Z}_A$.

Theorem 6.5. *The set A consists precisely of the following:*

(1) zero,
(2) positive integers whose ternary expansions include a single 1, with only 0s to the right of the 1,
(3) negative integers whose ternary expansions do not have a 1.

Proof. We proceed by induction. Since $A = \cup A_m$, it suffices to show that at the m^{th} step, i.e., when we try to add in the m^{th} element z_m of $\{0, 1, -1, 2, -2, 3, -3, \dots\}$, that it is included if and only if it has the properties above. The base case can quickly be checked: $0, 1, -2, 3, -6, \dots$, the first few elements of A, follow the pattern.

For the inductive step, suppose we are at the m^{th} step and trying to add in an element n. Then n is included provided one of the following holds:

(1) n is positive and has only zeros after the first 1 in its ternary expansion, or
(2) n is negative and has only 2s in its ternary expansion,

while n is not included if one of the following holds:

(3) n is positive and its ternary expansion does not include a 1,
(4) n is positive and its ternary expansion has anything besides zeros after the initial 1, or
(5) n is negative and includes a 1 in its expansion.

These cases are exhaustive, and are sufficient to prove the conjecture.

Since n has size greater than or equal to elements of currently in the set, if it forms any potential progressions, it will either be the first of third term. By taking the negative of the ratio depending on if n is the first or last, we can always consider n to be the third term.

We will begin with item (1). Suppose n is positive and contains only 0's after the first 1 in its ternary expansion. Suppose n forms a progression a, b, n with elements in A_{m-1}. a and n will have the same parity, which means a must be odd and thus positive. So $0 < a < b < n$ and by the inductive hypothesis a, b have a one in their ternary expansion with only zeros following. We will split item (1) into two cases: n is of the form $10\dots0$ or n is of the form $2\dots10\dots0$.

- Suppose $n = 1c_k \dots {}_1$ with $c_i = 0$ for all i. If $n = 1$ we are done. Otherwise, b must be at least $n/2$, otherwise a will be negative. The only way for b to be greater than $n/2$ is if $b = 2b_{k-2} \dots b_1$, i.e., have a 2 in its k^{th} decimal place. Let j denote the decimal place of the 1 in the expansion of b. The difference between b and n is

$$r = r_{k-1} \dots r_1 \tag{6.7}$$

where $r_i = 2$ if $b_i = 0$ or 1 and $i \geq j$, and $r_i = 0$ otherwise. This can be verified by looking at the 1 in the expansion of b – grouping the 1 with the 2 added in from r and then "grouping upward" to higher powers of 3, one gets $10\dots0 = n$ as the sum $b + r$. For example, the difference between 10000 and 02021 is 00202.

Let b_ℓ be the digit that is the first 2 in b's expansion that is followed by a 1, or the digit of the 1 in b's expansion if the first condition cannot be met. Note that one of the conditions must be met as b is positive. If we solve for the first term in the sequence, $a = b - r$, subtracting r results in the ℓ^{th} digit of a being a 1 (since we will be subtracting a $2 \cdot 3^{j-1}$). There will be nonzero terms following the ℓ^{th} digit of a, however. Thus a cannot be in the set A. Thus, n cannot form a progression, which means it is included.

- Suppose $n = 2c_k \dots c_1$. Since $b > n/2$, and b cannot have anything following the 1 in its expansion, b must be of the form $b = 2b_k \dots b_1$. a cannot have any digits past the $(k+1)^{\text{th}}$ place, otherwise it is bigger than n. a cannot have a 0 or 1 in its $(k+1)^{\text{th}}$ decimal place, otherwise n would take more than $k+1$ digits to write. So a is also of the form $a = 2a_k \dots a_1$. Consider the translated sequence

$$a - 2\underbrace{0\dots0}_{k\text{ times}},\ b - 20\dots0,\ n - 20\dots0 \ = \ a_k \dots a_1,\ b_k \dots b_1,\ c_k \dots c_1. \tag{6.8}$$

The elements of the above sequence have size less than n, and all have only zeros following the 1 in their ternary expansions. So they are all in A_{m-1}. But this contradicts the construction of A. Thus, n cannot form a progression, and it is included in A_m.

Next we'll prove item (2). Suppose n is negative and contains only 2's in its ternary expansion. Again, suppose n forms a progression a, b, n with $a, b \in A_{m-1}$. Since the first and last elements of a 3-term progression have the same parity, a is also even and thus must be negative and by the inductive hypothesis, must have only 2's in its ternary expansion. Thus b must be negative and have the same conditions on its ternary expansion.

Write the ternary expansion of n as

$$n \ = \ -2c_k, \dots, c_1. \tag{6.9}$$

Note that $|b|$ must be at least $|n|/2$ otherwise a will be positive. The only way to do this and still have b satisfy the inductive hypothesis is if b is also of the form $b = -2b_k \dots b_1$. Now consider the $(k+1)^{\text{th}}$ ternary digit of a. It cannot be 0: since b and n both have a 2 in their $(k+1)^{\text{th}}$ decimal place, the $(k+1)^{\text{th}}$ digit of $a = n - 2(n-b)$ must be 2 to be in the set A_{m-1}. Since $|a| < |b| < |c|$, a has no nonzero digits above the $(k+1)^{\text{st}}$ place. Write $a = -2a_k \dots a_1$. Then consider the translated sequence

$$a + 2\underbrace{0\dots0}_{k\text{ times}},\ b + 20\dots0,\ n + 20\dots0 \ = \ -a_1 \dots a_k,\ -b_1 \cdots_k,\ -c_1 \dots_k, \tag{6.10}$$

the sequence formed by removed the 2 in the $k + 1^{\text{th}}$ place of a, b, n. All the terms of the above sequence are negative and are thus in A_{m-1} (since their ternary expansions have all 2s or 0s and they have absolute values less than $|b|$). But this contradicts the inductive hypothesis. Thus, n cannot form a progression with the elements in A_{m-1}, which implies n is included.

We quickly prove item (3). Consider a positive $n = a_k \cdots a_1 2 b_\ell \cdots b_1$ where the a_i are 0 or 2 and the b_j are all 0. Note that it is possible that there are no a_i or b_j. Now consider $a = a_k \cdots a_1 1 b_\ell \cdots b_1$. First note that $|n - a|, |n - 2a| < n$. Further, $n - a$ is positive, contains a 1 and has nothing but zeros following the initial 1. Thus by the inductive hypothesis $n - a \in A_{m-1}$. $n - 2a$ is negative and does not include a 1 so likewise $n - 2a \in A_{m-1}$. So $n - 2a, n - a, n$ is a three-term arithmetic progression with $n - 2a, n - a \in A_{m-1}$ and thus $n \notin A_m$.

Now we prove item (4). We split into two cases: n is odd or n is even. Note that the parity of n depends on the number of 1's in its ternary expansion.

- Suppose n is odd, so n has an odd number of 1's in its ternary expansion. We want to find $a, b \in A_{m-1}$ such that a, b, n is an arithmetic progression. Again, a, n have the same parity. Therefore, the a we choose must be positive, and thus b must be positive as well.

Note that if $a_k \dots a_1, b_k \dots b_1, c_k \dots c_1$ forms a progression with $a_k \dots a_1, b_k \dots b_1$ in A and ratio r, then

$$t_1 \dots t_\ell a_k \dots a_1,\ t_1 \dots t_\ell b_k \dots b_1,\ t_1 \dots t_\ell c_k \dots c_1 \tag{6.11}$$

(where the t_i's are all 0 or 2) forms a progression with ratio r and $t_1 \dots t_\ell a_k \dots a_1,\ t_1 \dots t_\ell b_k \dots b_1 \in A$. Therefore, it is enough to consider the n whose first digit is a 1. Likewise, we may suppose the 1's place digit is nonzero.

We first demonstrate the construction of desired a, b for n that have only zeros and ones in their expansion, and then do the general construction. Write $n = c_k \dots c_1$, where $c_1, c_k = 1$ and all the c_i's are 1 or 0. Let $i_{2\ell+1} > \cdots > i_1$ be the decimal places of the 1's in the expansion of n. Consider

$$b \;=\; b_k \dots b_1 \tag{6.12}$$

where the b_i's are chosen in the following way:

(i) $b_1 = b_{i_1} = 1$.
(ii) $b_i = 2$ if $i_{2q+1} > i \geq i_{2q}$ for some $\ell \geq q \geq 0$, and $i \neq 1$.
(iii) $b_i = 0$ else.

What this construction says is that if i^{th} digit of n is between a pair of 1's at the i_{2q+1}, i_{2q} digits, the i^{th} digit of b becomes a 2, modulo some exceptions. For example, if $n = 110111$, b is 020021. By construction, b is in A and $b < n$. The ratio $n - b := r$ is

$$r_k \dots r_1 \tag{6.13}$$

where $r_i = 2$ if $i = i_{2q}$ for some q, and 0 otherwise. For example, the ratio between $n = 110111$ and $b = 020021$ is $r = n - b = 020020$. Again we use a similar idea to part 1 where the 2's being added from r "group upwards" to a 1 in the right location, to see that r indeed satisfies $b + r = n$.

Whenever r has a 2 in its expansion, b has a 2 in its expansion. So $a := b - r$ is still in A_{m-1}. Therefore, a, b, n forms a progression with ratio r, so n is not in the greedy set A.

Now we will do the general construction. Again, write $n = c_k \dots c_1$, and let $i_{2\ell+1} > \cdots > i_1$ be the decimal places of the 1's in the expansion of n. Consider

$$b \;=\; b_k \dots b_1 \tag{6.14}$$

where the b_i's are chosen as such:

(i) $b_{i_1} = 1$,
(ii) $b_i = 2$ if $i_{2q+1} > i \geq i_{2q}$ for some $\ell \geq q \geq 0$, and $i \neq 1$,
(iii) $b_i = 0$ otherwise.

The ratio $r := n - b$ is then

$$r \;=\; r_k \dots r_1, \tag{6.15}$$

where $r_i = 2$ if $i = i_{2q}$ for some q *or* if $c_i = 2$ (and $r_i = 0$ otherwise). Let L be the decimal place of the rightmost 2 in the expansion of n, or set $L = i_1$ if no such thing digit exists. Then $a := b - r$ is

$$a_k \dots a_1, \tag{6.16}$$

where

(i) $a_i = 2$ if $b_i = 2, r_i \neq 2$, and $i > i_1$,
(ii) $a_i = 0$ if $b_i = 2, r_i = 2$, and $i > i_1$ (these first two deal with the digits before the last 1 in n, and should evoke the first construction),
(iii) $a_L = 1$ (these next two steps come from the part of n to the right of the last 1 and deal with any 2's past that rightmost 1),
(iv) $a_i = 2$ if $r_i = 0$, and $i_q > i > L$,
(v) $a_1 = 0$ if $r_i = 2$ and $i_q > i > L$,
(vi) $a_i = 0$ otherwise.

To illustrate, the sequence for $n = 10112$ is $02001, 02210, 10112$, and the ratio is $r = 002002$. The sequence for $n = 120101$ is $002001, 022201, 120101$ with ratio $r = 020200$. The construction is more straightforward if one works out a few examples where n has only 0's and 1's and then tries to generalize to include 2's, which mainly involves dealing with the "tail" for numbers n possessing 2's past their rightmost 1.

Such b, a are in A. Since $a < b < n$, a, b are in the construction of A up to the $m - 1$th stage. Therefore, n is not included in the greedy set.

- Suppose n is even, so that n has an even number of 1's in its ternary expansion. Note that a, n have the same parity, so a is even and thus must be negative.

  Let $n = c_k \dots c_1$. Similar to before, we may assume n starts with a 1 and that its last digit is nonzero. Let $i_{2\ell} > \cdots > i_1$ be the decimal places of the 1's in the expansion of n. Consider

$$b \; := \; b_k \cdots b_1 \tag{6.17}$$

  where
  (i) $b_{i_1} = 1$,
  (ii) $b_i = 2$ if $i = i_{2q-1}$ for some $\ell \geq i > 1$,
  (iii) $b_i = 2$ if $c_i = 2$ and $i > i_1$,
  (iv) $b_i = 0$ otherwise.

  Then the ratio $r := n - b$ is $r = r_k \dots r_1$ where
  (i) $r_i = 2$ if $i_{2q} > i \geq i_{2q-1}$ and $i \neq i_1$,
  (ii) $r_i = 2$ if $c_i = 2$ and $i_1 > i$ (so the i^{th} place is to the right of the $i_1{}^{\text{th}}$ place),
  (iii) $r_i = 1$ if $i = i_2$,
  (iv) $r_i = 0$ otherwise.

  For example, if $n = 11011$ then $b = 02001$ and $r = n - b = 02010$. Consider $a = b - r$. That is, a is the number such that $r = b - a$. Since our choice of b satisfies $b < n/2$, we have that a is always negative. So a is the unique number such that $b + |a| = r$. So a is $-a_k \dots a_1$, where
  (i) $a_i = 2$ if $r_i = 2$ and $b_i = 0$, and $i > i_2$,
  (ii) $a_i = 0$ if $r_i = 0$ and $b_i = 2$ and $i > i_2$,
  (iii) $a_i = 2$ if $r_i = 2$ and $i_1 > i$,
  (iv) $a_i = 2$ if $i_2 > i \geq i_1$ and $b_i = 0$,
  (v) $a_i = 0$ if $i_2 > i \geq i_1$ and $b_i = 2$,
  (vi) $a_i = 0$ otherwise.

  Note that the case of $i_2 > i \geq i_1$ and $r_i \neq 0$ will never happen, which is why the $i_2 > i \geq i$ cases above have that symmetry. Since $|a|, |b| < n$ and have the proper form, a, b are in A_{m-1}. Therefore, n is not included in the greedy set.

  For examples: if $n = 1112111$, this method generates the progression

$$-\,0000002, 0202201, 1112111 \tag{6.18}$$

  with ratio $r = 0202210$. If $n = 112$, this generates the progression $-022, 010, 112$ with ratio 102. As in the odd case, trying out this construction with only 0's and 1's and then generalizing to n with 2's in the expansion may illuminate the process to the reader.

We conclude with item 5. Note that n has at least one 1 in its ternary expansion. Again we split into two cases: n is even, and thus has an even number of ones in its expansion, or n is odd, and hence has an odd number of ones in its expansion. We will construct a, b such that a, b are already in A and a, b, n forms an arithmetic progression.

- Suppose n has an even number of 1's in its ternary expansion. Note that a will have to be $n + 2r$ for some ratio r, so the a we construct must be even and therefore negative. Write $n = c_k \dots c_1$, its ternary expansion .Let $i_{2\ell} > \cdots > i_1$ be the decimal places in the ternary expansion of n in which a 1 appears. Then define $b = -b_k \dots b_1$ where:
  (i) $b_i = 2$ if $c_i = 2$,
  (ii) $b_i = 2$ if $i_{2q} > i \geq i_{2q-1}$ for some q,
  (iii) $b_i = 0$ otherwise.

  That is, b has 2's "inbetween" pairs of ones. b is in A_{m-1} at this point since it is negative, has only twos in its expansion, and has absolute value less than n. The ratio between n and b is calculated using the usual approach of $\underbrace{2\dots2}_{k \text{ times}} + 2 = 1 \underbrace{0\dots0}_{k-1 \text{ times}} 1$. In this case the ratio is equal to $r = -r_k \dots r_1$

  where
  (i) $r_i = 2$ if $c_i = 2$,
  (ii) $r_i = 2$ if $i = i_{2q-1}$ for some q,
  (iii) $r_i = 0$ otherwise.

Note that r_i is two whenever b_i is, and $|r| < |b|$, so $a := b - r$ is negative and has only twos in its ternary expansion. So a, b, n forms an arithmetic progression with $a, b \in A_{m-1}$. Therefore, n is not included in the greedy set. As an example, for $n = -2201221$ this process creates the sequence $-2200000, -2200222, -2201221$ with ratio -0000222.

- Suppose n has an odd number of 1's in its ternary expansion. In this case, the a we want to construct must be odd and hence positive.

  Write $n = c_k \dots c_1$ out as the ternary expansion. Let $i_{2\ell+1} > \cdots > i_1$ be the decimal places in the ternary expansion of n in which a 1 appears. Let L be the decimal place of the rightmost 2 in n, or $L = i_1$ if no such thing exists.

  Then define $b = -b_k \dots b_1$ where
  (i) $b_i = 2$ if $c_i = 2$ and $i_{2\ell+1} > i$,
  (ii) $b_i = 2$ if $i = 2q$ for some q,
  (iii) $b_i = 0$ otherwise.

  Then the ratio between b and n is $r = -r_k \dots r_1$ where
  (i) $r_i = 2$ if $i_{2q+1} > i \geq i_{2q}$ for $q \geq 1$,
  (ii) $r_i = 2$ if $c_i = 2$ and $i \geq i_{2\ell+1}$,
  (iii) $r_i = 2$ if $c_i = 0$ and $i_1 > i \geq L$,
  (iv) $r_L = 1$,
  (v) $r_i = 0$ otherwise.

  Consider $a := b - r$. Then $a - b = -r$, so $a + |b| = |r|$. Then $a = a_k \dots a_1$, where
  (i) $a_i = 2$ if $b_i = 0$ and $i_{2q+1} > i \geq i_{2q}$ for $q \geq 1$,
  (ii) $a_i = 2$ if $c_i = 2$ and $i \geq i_{2\ell+1}$,
  (iii) $a_i = 2$ if $c_i = 0$ and $i_1 < i < L$,
  (iv) $a_L = 1$,
  (v) $r_i = 0$ otherwise.

  So a, b, n forms an arithmetic progression. By construction $|a|, |b| < n$, so $a, b \in A_{m-1}$. Therefore, n is not included in the greedy set.

  For some examples, for $n = -22010211$ this construction produces the sequence $2202001, -00000220, -22010211$ with ratio -22002221. For $n = -1202$, this construction produces $021, -202, -1202$ with ratio $r = -1000$.

$\square$

Consider the non-negative integers up to 3^n. The number of elements in this set with only 2s or 0s in their ternary expansion is 2^n. The number of elements with a single 1 in its ternary expansion and only 0s following it is likewise 2^n. As a corollary of Theorem 6.5 we get the following.

Corollary 6.6. *The proportion of elements included in A_{3^n} is*

$$\frac{|A_{3^n}|}{|\{m \in \mathbb{Z} : |m| \leq 3^n\}|} = \frac{2^{n+1}}{1 + 2 \cdot 3^n}, \tag{6.19}$$

and in general we have

$$\frac{|A_n|}{|\{m \in \mathbb{Z} : |m| \leq n\}|} = \Theta((2/3)^{\log_3 n}). \tag{6.20}$$

As n tends to infinity, this proportion goes to zero.

Theorem 6.2 follows by including the odd-length elements in the count for the denominator.

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