VIRUS DYNAMICS ON STARLIKE GRAPHS

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ABSTRACT. The field of epidemiology has presented fascinating and relevant questions for mathematicians, primarily concerning the spread of viruses in a community. The importance of this research has greatly increased over time as its applications have expanded to also include studies of electronic and social networks and the spread of information and ideas. We study virus propagation on a non-linear hub and spoke graph (which models well many airline networks). We determine the long-term behavior as a function of the cure and infection rates, as well as the number of spokes n. For each n we prove the existence of a critical threshold relating the two rates. Below this threshold, the virus always dies out; above this threshold, all non-trivial initial conditions iterate to a unique non-trivial steady state. We end with some generalizations to other networks.

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1. INTRODUCTION

1.1. **Previous Work.** The general problem of studying the propagation of a node-state within a large interconnected network of nodes has a wide range of applications across domains, such as studying computer virus propagation in computer science, studying the penetration of a meme or product in marketing and sociology, and studying the propagation of an infection in epidemiology. Many of the earliest investigations [Ba, KeWh, McK] assume a homogenous network, where each node has identical connections to all other nodes: for such networks, the rate of virus propagation

Date: November 5, 2011.

2010 *Mathematics Subject Classification.* 94C15 (primary), (secondary) 82B26, 92E10.

Key words and phrases. virus propagation, star networks, SIS model.

The first, second and sixth named authors were partially supported by Williams College and NSF grant DMS0850577, and the fourth named author was partly supported by NSF grant DMS0970067. It is a pleasure to thank Andres Douglas Castroviejo, Amitabha Roy, and our colleagues from the Williams College 2011 SMALL REU program for many helpful conversations.

was then shown to be determined by the density of infected nodes. While mathematically tractable, the results in [FFF, RiDo, RiFoIa] also suggested that such homogenous models fail to represent many real networks. There has thus also been work on alternatives to this strict homogeneous model. For instance, [P-SV1, P-SV2, P-SV3, P-SV4, MP-SV] study power law networks, where the probability of a node having k neighbors is proportional to $k^{-\gamma}$ for some exponent $\gamma > 0$. Although more realistic, [WKE] shows that even this model is not well-suited for many real networks. Moreover, an issue with these results is that their models, describing the propagation of node-states, themselves are dependent on the network topology. In contrast to these, [WDWF] proposes a more natural topology-agnostic model that relies on local node interactions. Specifically, their proposed SIS (Susceptible Infected Susceptible) model is a discrete-time model where each node is either Susceptible (S) or Infected (I). A susceptible node is currently healthy, but at any time step can be infected by its infected neighbors. At any time step moreover, an infected node can be cured and go back to being susceptible. The model parameters are β, the probability at any time step that an infected node infects its neighbors, and δ, the probability at any time step that an infected node is cured. A central set of questions given this model for propagation of a node-state through the network are:

(1) Given a set of model parameters and a particular initial state, does the system then reach a steady state?

(2) If the system does reach a steady state, what are the characteristics of that state?

(3) What is the dynamical behavior (rate of convergence) of the system?

For the SIS model, Wang et al. [WDWF] gave a heuristic argument for a sufficient criterion for the node infection probabilities to converge to a trivial solution, so that the infection dies out. Using a reasonable approximation to eliminate lower order terms, they conjecture a sufficient condition for the virus to die out. For star graphs, this condition is $b \leq (1-a)/\sqrt{n}$, where $a = 1-\delta$ and $b = \beta$. One of the main contributions of this paper making this argument rigorous. Indeed, given the nonlinear coupled dynamics of the SIS model, it is typically intractable to argue rigorously about asymptotic state characteristics. But for star graphs, we are able to show that the SIS model exhibits phase transition behavior, and moreover that this threshold is both necessary and sufficient. Thus, below this threshold the virus dies out, and above the system converges to a non-trivial steady state *independent* of the initial state (provided only that the initial state is non-trivial). One consequence of this is that even if a single spoke node is infected initially, so long as the model parameters lie beyond the phase transition point, the infection will not die out (i.e., the node infection probabilities will not converge to the trivial point). We prove our results through a novel two-step argument, by first reducing the problem to one with a smaller graph size, and then applying the intermediate value theorem to the dynamics over the reduced graph.

1.2. **Problem Setup.** Y. Wang, C. Deepayan, C. Wang and C. Faloutsos [WDWF] proposed the following propagation model. Denote by β, the probability at any time step that an infected node infects its neighbors, and by δ, the probability at any time step that an infected node is cured.

If $p_{i,t}$ is the probability a node i is infected at time t, the SIS model is governed by the following equation:

$$1 - p_{i,t} \ = \ (1 - p_{i,t-1})\, \zeta_{i,t} + \delta p_{i,t} \zeta_{i,t}, \tag{1.1}$$

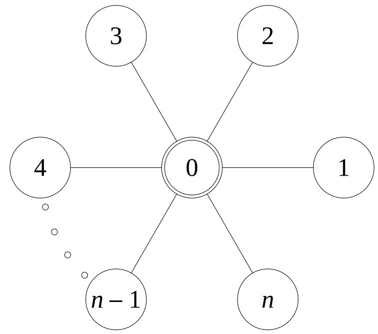

FIGURE 1. Star graph with 1 central hub and n spokes.

where $\zeta_{i,t}$ is the probability that a node i is not infected by its neighbors at time t. We can express $\zeta_{i,t}$ as follows:

$$\zeta_{i,t} \;=\; \prod_{j\sim i} p_{j,t-1}\,(1-\beta) + (1-p_{j,t-1}) \;=\; \prod_{j\sim i}(1-\beta p_{j,t-1}) \tag{1.2}$$

(where $j \sim i$ means i and j are neighbors — i.e., are connected by an edge of the graph). Given the non-linear coupled form of this system, a closed form expression for $p_{i,t}$ for the general topology case seems infeasible.

We therefore consider a specific graph topology, that of a star graph (see Figure 1). This is a graph in which there is a single "hub" node which is connected to all the other nodes, the "spokes." Suppose the graph has $n+1$ nodes: the hub is numbered 0 and the spokes are numbered 1 through n.

Proposition 1.1. *For any initial configuration, as time evolves all the spokes converge to a common behavior.*

Proof. (1.1) becomes

$$\begin{aligned} p_{0,t} \;&=\; 1-(1-p_{0,t-1})\prod_{j=1}^{n}(1-\beta_{j,t-1}) - \delta p_{0,t}\prod_{j=1}^{n}(1-\beta p_{j,t-1}) \\ p_{i,t} \;&=\; 1-(1-p_{i,t-1})\,(1-\beta p_{0,t-1}) - \delta p_{i,t}\,(1-\beta p_{0,t-1})\,, \quad 1 \;\le\; n \;\le\; n+1. \end{aligned} \tag{1.3}$$

We can immediately observe that all the spokes assume identical values quite rapidly. We prove this below by showing that for $i, j \neq 0$, $|p_{i,t} - p_{j,t}| \to 0$ as $t \to \infty$. We have

$$\begin{aligned} p_{i,t} - p_{j,t} \;&=\; (p_{i,t-1} - p_{j,t-1})\,(1-\beta p_{0,t-1}) - \delta\,(p_{i,t} - p_{j,t})\,(1-\beta p_{0,t-1}) \\ &=\; \left(\frac{1-\beta p_{0,t-1}}{1+\delta\,(1-\beta p_{0,t-1})}\right) p_{i,t-1} - p_{j,t-1}. \end{aligned} \tag{1.4}$$

Thus we have

$$|p_{i,t} - p_{j,t}| \;=\; \left(\frac{1-\beta p_{0,t-1}}{1+\delta\,(1-\beta p_{0,t-1})}\right)^{t} |p_{i,0} - p_{j,0}|\,. \tag{1.5}$$

Since the quantity to the t^{th} power cannot stabilize at 1 as the denominator is at least $1+\delta$ and the numerator is at most 1, the right-hand side in (1.5) decays to 0 as $t \to \infty$. □

An important consequence of this observation is that it allows us to simplify our model to a model in terms of x_t, the probability that the hub is infected, and y_t, the probability that a spoke is infected. These then evolve according to

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} \;=\; F\begin{pmatrix} x_t \\ y_t \end{pmatrix}, \tag{1.6}$$

where

$$F(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} 1-(1-x)(1-\beta y)^n - \delta x(1-\beta y)^n \\ 1-(1-y)(1-\beta x) - \delta y(1-\beta x) \end{pmatrix}$$

$$= \begin{pmatrix} 1-(1-ax)(1-by)^n \\ 1-(1-ay)(1-bx) \end{pmatrix}; \tag{1.7}$$

recall that we have defined $a := 1-\delta$ and $b := \beta$ to simplify the algebra.

1.3. **Main Results and Consequences.** Our main result is the following.

Theorem 1.2. *Let $a, b \in (0,1)$ and F as in* (1.7) *describes the limiting behavior of the spoke and star network.*

I. If $b \le (1-a)/\sqrt{n}$, then

- *(a) the unique fixed point of F is $(0,0)$, and*
- *(b) the system converges to this fixed point, that is, the virus dies out.*

II. If $b > (1-a)/\sqrt{n}$ then, so long as the initial configuration is not the trivial point $(0,0)$,

- *(a) F has a unique, non-trivial fixed point (x_f, y_f), where x_f and y_f are functions of a, b and n, and*
- *(b) the system evolves to this non-trivial fixed point.*

Remark 1.3. *In the notation of* [WDWF]*, the critical threshold for the epidemic is $\beta/\delta < 1/\lambda_{1,A}$, where $\lambda_{1,A}$ is the largest eigenvalue of the adjacency matrix A of the network. For a star graph with n spokes connected to the central hub, $\lambda_{1,A} = \sqrt{n}$. Recalling our $a = 1-\delta$ and $b = \beta$, their condition is equivalent to $b = (1-a)/\sqrt{n}$, exactly the condition we have.*

While previous work suggested the veracity of the above claim, it was through heuristic arguments and numerical simulations. We opted for a theoretical investigation, so as to lend additional plausibility to the general conjecture and to develop some techniques potentially useful for eventually resolving it.

The proof of this theorem is distributed over the next few sections. In §2, we prove parts I(a) and II(a) by determining the fixed points of F. Using convexity arguments, we show that the trivial fixed point is the only fixed point if $b \le (1-a)/\sqrt{n}$, but there is a unique, additional fixed point for larger b. We prove I(b) in §3, namely that for $b \le (1-a)/\sqrt{n}$ (so b is at or below the critical threshold) all initial configurations evolve to the trivial fixed point. The proof involves linearly approximating the map F near the trivial fixed point and controlling the resulting eigenvalues. Finally, we show II(b) in §4, where we prove that all non-trivial initial configurations converge to the unique non-trivial fixed point when $b > (1-a)/\sqrt{n}$. This last case is handled by noting that there is a natural partition of the domain $[0,1]^2$ of F into four regions (see Figure 3), where the partitions are induced from functions related to determining the location of F's fixed points. The analysis of F on all of $[0,1]^2$ is complicated, but the restrictions of each region lead to F having simple behavior in each region. We end with a discussion of the rate of convergence and the restriction of F to these regions in §5, and discuss some generalizations to other graph topologies.

2. Determination of Fixed Points of F

In this section we determine the behavior of the fixed points of the system as a function of the parameters a, b and n, proving Theorem 1.2, I(a) and II(a). The proof relies on some auxiliary lemmas, which we first show. Specifically, the proofs look for partial fixed points, namely points where either the x or y-coordinate is unchanged. We prove that the set of partial fixed points can be defined by continuous functions ϕ_1 and ϕ_2, whose intersections are the fixed points of the system (see Figure 2).

We begin with the following lemma characterizing these curves.

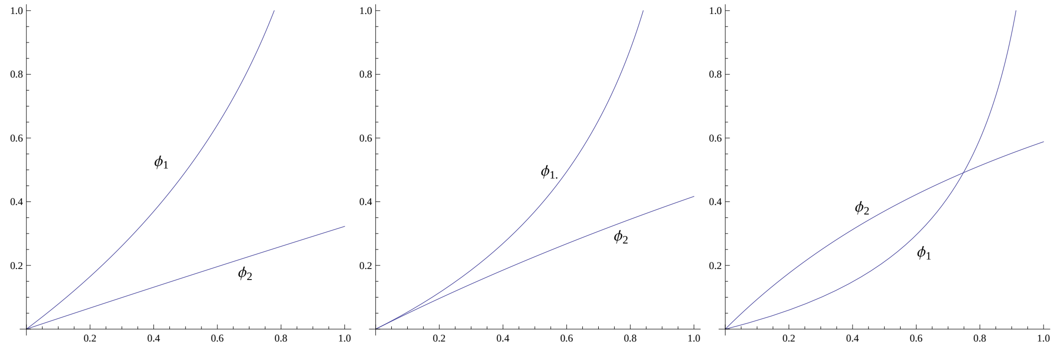

FIGURE 2. Partial fixed points from ϕ_1 and ϕ_2 when (from left to right) $b < (1-a)/\sqrt{n}$, $b = (1-a)/\sqrt{n}$, $b > (1-a)/\sqrt{n}$ ($b = 3, n = 4, a = .1, .4, .7$).

Lemma 2.1. *Consider the map F given by* (1.7).

(1) *There exists a continuous, twice differentiable convex function $\phi_1 : [0,1] \to [0,1]$ such that, for each $y \in [0,1]$, there is a $y' \in [0,1]$ with $F(\phi_1(y), y) = (\phi_1(y), y')$.*
(2) *There exists a continuous, twice differentiable concave function $\phi_2 : [0,1] \to [0,1]$ such that, for each $x \in [0,1]$, there is an $x' \in [0,1]$ with $F(x, \phi_2(x)) = (x', \phi_2(x))$.*

Proof. We define

$$g_1(x,y) \;=\; (1-(1-ax)(1-by)^n) - x \tag{2.1}$$

and

$$g_2(x,y) \;=\; (1-(1-ay)(1-bx)) - y. \tag{2.2}$$

We first analyze the set of pairs $(x,y) \in [0,1]^2$ where $g_1(x,y) = 0$. We immediately see that $g_1(0,0) = 0$, $g_1(0,y) > 0$ for $y \in (0,1]$, and $g_1(1,y) < 0$ for $y \in [0,1]$. Thus by the Intermediate Value Theorem, for each $y \in (0,1]$ there is a number (which we denote by $\phi_1(y)$) such that $g_1(\phi_1(y), y) = 0$ and $\phi_1(y) \in [0,1]$. It is easy to see that $\phi_1(y)$ is a continuous and differentiable function of y; in fact,

$$\begin{aligned} \phi_1(y) &= \frac{1-(1-by)^n}{1-a(1-by)^n} \\ \phi_1'(y) &= \frac{nb(1-a)(1-by)^{n-1}}{(1-a(1-by)^n)^2}. \end{aligned} \tag{2.3}$$

Note $\phi_1(y) \in [0,1]$: it is clearly positive, and $\frac{1-c}{1-ac} > 1$ for $c > 0$ only when $a > 1$. As $a, b \in (0,1)$, $\phi_1'(y) > 0$. Thus $\phi_1(y)$ is strictly increasing.

We analyze $g_2(x,y) = 0$ similarly. We find

$$g_2(x,y) \;=\; (1-(1-ay)(1-bx)) - y \;=\; 0. \tag{2.4}$$

Note $g_2(0,0) = 0$, $g_2(x,0) > 0$ for $x \in (0,1]$, and $g_2(x,1) < 0$ for $x \in [0,1]$. Solving yields

$$y \;=\; \phi_2(x) \;=\; \frac{bx}{1-a+abx}. \tag{2.5}$$

We can rewrite this as a function of y as follows:

$$x \;=\; \phi_2(y) \;=\; \frac{(1-a)\,y}{b\,(1-ay)}. \tag{2.6}$$

This is clearly continuously differentiable, and

$$\phi_2'(y) \;=\; \frac{1-a}{b(1-ay)^2} \;>\; 0. \tag{2.7}$$

Thus $\phi_2(y)$ is an increasing function of y.

We now prove that $\phi_1(y)$ is convex and $\phi_2(y)$ is concave. Straightforward differentiation and some algebra gives

$$\phi_1''(y) = -\frac{b^2 n(1-a)(1-by)^{n-2} \cdot (n-1+a(1-by)^n + a(n+1)(1-by)^n)}{(1-a(1-by)^n)^3} < 0$$

$$\phi_2''(y) = \frac{2a(1-a)}{b(1-ay)^3} > 0. \tag{2.8}$$

Thus $\phi_1(y)$ is convex while $\phi_2(y)$ is concave. Direct inspection shows each function is twice continuously differentiable. □

The next lemma is useful in determining the number and location of fixed points of our map F.

Lemma 2.2. *Let h_1, h_2 be twice continuously differentiable functions such that $h_1(x)$ is convex and $h_2(x)$ is concave. If there exists some p such that $h_1'(p) \leq h_2'(p)$ and $h_1(p) = h_2(p)$, then $h_1(x) \neq h_2(x)$ for all $x > p$.*

Proof. As $h_1(x)$ is convex and $h_2(x)$ is concave, $h_1'(x)$ is decreasing and $h_2'(x)$ is increasing. Thus, since $h_1'(p) \leq h_2'(p)$, $h_1'(x) < h_2'(x)$ for all $x > p$. As $h_1(p) = h_2(p)$, this implies that $h_1(x) < h_2(x)$ for all $x > p$. □

We now determine the location of the fixed points.

Proof of Theorem 1.2, I(a). Note that

$$\phi_1'(0) = \frac{bn}{1-a}, \quad \phi_2'(0) = \frac{1-a}{b}. \tag{2.9}$$

From these equations, we can see that $\phi_2'(0) \geq \phi_1'(0)$ when $b \leq (1-a)/\sqrt{n}$. Thus by Lemma 2.2, when $b \leq (1-a)/\sqrt{n}$, there is no $y > 0$ such that $\phi_1(y) = \phi_2(y)$. The trivial fixed point is thus the unique fixed point in $[0,1]^2$. □

We next prove that for $b > (1-a)/\sqrt{n}$, there exists a unique non-trivial fixed point. The key ingredient is the following lemma.

Lemma 2.3. *Let $h_1, h_2 : [0,1] \to [0,1]$ be twice continuously differentiable functions such that $h_1(x)$ is convex, $h_2(x)$ is concave, $h_1(0) = h_2(0) = 0$ and $h_1(x) \neq h_2(x)$ for $x > 0$ sufficiently small. Then there exists at most one other $x > 0$ for which $h_1(x) = h_2(x)$.*

Proof. The claim is trivial if there is only one point of intersection, so assume there are at least two. Without loss of generality we may assume $p > 0$ is the first point above zero where h_1 and h_2 agree. Such a smallest point exists by continuity, as we have assumed $h_1(x) \neq h_2(x)$ for $x > 0$ sufficiently small; if there are infinitely many points x_n where they are equal, let $p = \liminf_n x_n > 0$.

Because $h_1(x)$ is convex, $h_1'(x)$ is increasing. By the Mean Value Theorem there is a point $c_1 \in (0,p)$ such that

$$h_1'(c_1) = \frac{h_1(p) - h_1(0)}{p - 0} = \frac{h_1(p)}{p}. \tag{2.10}$$

As h_1' is increasing, we have $h_1'(p) > h_1(c_1)$; further, $h_1'(x) > h_1(c_1)$ for all $x \geq p$. As $h_2(x)$ is concave, $h_2'(x)$ is decreasing. Again by the Mean Value Theorem there is a point $c_2 \in (0,p)$ such that

$$h_2'(c_2) = \frac{h_2(p) - h_2(0)}{p - 0} = \frac{h_2(p)}{p}, \tag{2.11}$$

$h_2'(p) < h_2'(c_2)$, and $h_2'(x) < h_2'(c_2)$ for all $x \geq p$. But since $h_1(p) = h_2(p)$, $h_1'(c_1) = h_2'(c_2)$, so $h_1'(x) > h_2'(x)$ for all $x \geq p$. Thus we know from Lemma 2.2 that there cannot be another point of intersection after p. □

We are now ready to complete the analysis.

Theorem 1.2, II(a). We first prove existence and then uniqueness. When $b > (1-a)/\sqrt{n}$, we know from the proof of Theorem 1.2, I(a) (see (2.9)) that $\phi_1(y)$ is above $\phi_2(y)$ near the origin since $\phi_1'(0) > \phi_2'(0)$. The existence of the non-trivial point of intersection follows from the Intermediate Value Theorem. We recall that $y = \phi_2(x)$ is defined in $[0,1]$ for all $x \in [0,1]$, and $x = \phi_1(y)$ is defined in $[0,1]$ for all $y \in [0,1]$. As $x \to 1$ we have $\phi_2(x)$ tends to a number strictly less than 1. Thus the curve $y = \phi_2(x)$ hits the line $x = 1$ below $(1,1)$. Similarly the curve $x = \phi_1(y)$ hits the line $y = 1$ to the left of $(1,1)$. Thus the two curves flip as to which is above the other, implying that there must be one point where the two curves are equal.

We now have two fixed points, the trivial fixed point and the non-trivial fixed point from the second intersection of the two curves. By Lemmas 2.1 and 2.3 there are no other fixed points, and thus there is a unique, non-trivial fixed point. □

3. Dynamical Behavior: $b \le (1-a)/\sqrt{n}$

In this section we show how an eigenvalue perspective can completely determine the dynamics if $b \le (1-a)/\sqrt{n}$, proving Theorem 1.2, I(b). As these methods fail for larger b, we adopt a different perspective in §4.

3.1. Technical Preliminaries.

Our analysis of the dynamical behavior relies on the following lemma.

Lemma 3.1. *Let $a, b \in (0,1)$ with $b < (1-a)/\sqrt{n}$, and let $\lambda_1 \ge \lambda_2$ denote the eigenvalues of the matrix $\begin{pmatrix} a\alpha & nb\beta \\ b\gamma & a\delta \end{pmatrix}$, where $\alpha, \beta, \gamma, \delta \in [0,1]$. Then $-1 < \lambda_1, \lambda_2 < 1$.*

Proof. The sum of the eigenvalues is the trace of the matrix (which is $a(\alpha+\delta)$), and the product of the eigenvalues is the determinant (which is $a^2\alpha\delta - nb^2\beta\gamma$). Thus the eigenvalues satisfy the characteristic equation

$$\lambda^2 - a(\alpha+\delta)\lambda + (a^2\alpha\delta - nb^2\beta\gamma). \tag{3.1}$$

The eigenvalues are therefore

$$\frac{a(\alpha+\delta) \pm \sqrt{a^2(\alpha+\delta)^2 - 4(a^2\alpha\delta - nb^2\beta\gamma)}}{2} = \frac{a(\alpha+\delta) \pm \sqrt{a^2(\alpha-\delta)^2 + 4nb^2\beta\gamma}}{2}. \tag{3.2}$$

As the discriminant is positive, the eigenvalues are real. Since $a(\alpha+\delta) \ge 0$, we have $|\lambda_2| \le \lambda_1$, where

$$0 \le \lambda_1 = \frac{a(\alpha+\delta) + \sqrt{a^2(\alpha-\delta)^2 + 4nb^2\beta\gamma}}{2}. \tag{3.3}$$

As $\beta\gamma \le 1$, $nb^2 < (1-a)^2$ and $\sqrt{u+v} \le \sqrt{u} + \sqrt{v}$ for $u, v \ge 0$ we find

$$\begin{aligned}
\lambda_1 &< \frac{a(\alpha+\delta) + \sqrt{a^2(\alpha-\delta)^2} + \sqrt{4(1-a)^2}}{2} \\
&= \frac{a(\alpha+\delta) + a|\alpha-\delta| + 2(1-a)}{2} \\
&= \frac{2a\max(\alpha,\delta) + 2(1-a)}{2} \\
&= 1 - (1-\max(\alpha,\delta))\,a \le 1,
\end{aligned} \tag{3.4}$$

where the last claim follows from $a, \alpha, \delta \in [0,1]$. □

3.2. **Proofs.** Armed with the following, we now prove the first half of our main result, the dynamical behavior at or below the critical threshold.

We prove the claim by using the Mean Value Theorem and an eigenvalue analysis of the resulting matrix. From Theorem 1.2, I(a) we know $(0,0)$ is the unique fixed point. We have

$$f\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) \ = \ \begin{pmatrix} 1-(1-au)(1-bv)^n \\ 1-(1-av)(1-bu) \end{pmatrix}. \tag{3.5}$$

Let

$$c(t) \ = \ (1-t)\begin{pmatrix} 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} x \\ y \end{pmatrix}, \quad c'(t) \ = \ \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.6}$$

Thus $c(t)$ is the line connecting the trivial fixed point to $\begin{pmatrix} x \\ y \end{pmatrix}$, with $c(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $c(1) = \begin{pmatrix} x \\ y \end{pmatrix}$. Let

$$\mathcal{F}(t) \ = \ f(c(t)) \ = \ \begin{pmatrix} 1-(1-atx)(1-bty)^n \\ 1-(1-aty)(1-btx) \end{pmatrix}. \tag{3.7}$$

Then simple algebra (or the chain rule) yields

$$\mathcal{F}'(t) \ = \ \begin{pmatrix} a(1-bty)^n & nb(1-atx)(1-bty)^{n-1} \\ b(1-aty) & a(1-btxu) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.8}$$

We now apply the one-dimensional chain rule twice, once to the x-coordinate function and once to the y-coordinate function. We find there are values t_1 and t_2 such that

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) - f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \ = \ \begin{pmatrix} a(1-bt_1y)^n & nb(1-at_1x)(1-bt_1y)^{n-1} \\ b(1-at_2y) & a(1-bt_2x) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.9}$$

To see this, look at the x-coordinate of $\mathcal{F}(t)$: $h(t) = 1-(1-atx)(1-bty)^n$. We have $h(1)-h(0) = h(1) = h'(t_1)(1-0)$ for some t_1. As

$$\begin{aligned} h'(t_1) \ &= \ ax(1-bt_1y)^n + nby(1-at_1x)(1-bt_1y)^{n-1} \\ &= \ \left(a(1-bt_1y)^n,\ nb(1-at_1x)(1-bt_1y)^{n-1}\right)\begin{pmatrix} x \\ y \end{pmatrix}, \end{aligned} \tag{3.10}$$

the claim follows; a similar argument yields the claim for the y-coordinate (though we might have to use a different value of t, and thus denote the value arising from applying the Mean Value Theorem here by t_2). We therefore have

$$\begin{aligned} f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) \ &= \ \begin{pmatrix} a(1-bt_1y)^n & nb(1-at_1x)(1-bt_1y)^{n-1} \\ b(1-at_2y) & a(1-bt_2x) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \ A(a,b,x,y,t_1,t_2)\begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned} \tag{3.11}$$

To show that f is a contraction mapping, it is enough to show that, for all a, b with $b < (1-a)/\sqrt{n}$ and all $x, y \in [0,1]$ that the eigenvalues of $A(a,b,x,y,t_1,t_2)$ are less than 1 in absolute value; however, this is exactly what Lemma 3.1 gives (note our assumptions imply that $\alpha = (1-bt_1y)^n$ through $\delta = (1-bt_2x)$ are all in $(0,1)$). Let us denote $\lambda_{\max}(a,b)$ the maximum value of λ_1 for fixed a and b as we vary $t_1, t_2, x, y \in [0,1]$. As we have a continuous function on a compact set, it attains its maximum and minimum. As λ_1 is always less than 1, so is the maximum. Here it is important that we allow ourselves to have $t_1, t_2 \in [0,1]$, so that we have a closed and bounded set; it is immaterial (from a compactness point of view) that $a, b \in (0,1)$ as they are fixed. As

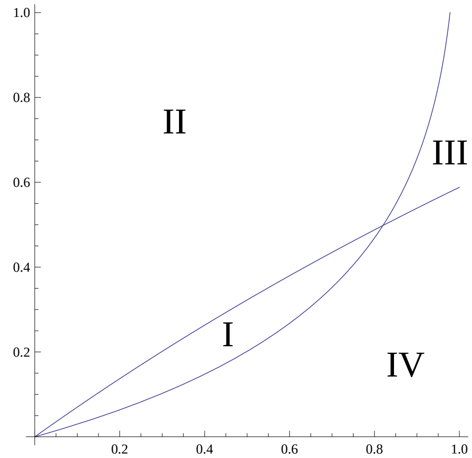

FIGURE 3. The four regions determined by ϕ_1 and ϕ_2 when $b > (1-a)/\sqrt{n}$.

$0 < a, b < 1$, we have $\alpha, \beta, \gamma, \delta < 1$ and thus the inequalities claimed in Lemma 3.1 hold. For any matrix M we have $||Mv|| \leq |\lambda_{\max}|||v||$; thus

$$\left|\left|f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)\right|\right| \;\leq\; \lambda_{\max}(a,b)\left|\left|\begin{pmatrix} x \\ y \end{pmatrix}\right|\right|; \tag{3.12}$$

as $\lambda_{\max}(a,b) < 1$ we have a contraction map. Therefore any non-zero $\begin{pmatrix} x \\ y \end{pmatrix}$ iterates to the trivial fixed point if $b < (1-a)/\sqrt{n}$ and $n \geq 2$. In particular, the trivial fixed point is the only fixed point (if not, $A(a,b,x,y,t_1,t_2)v = v$ for v a fixed point, but we know $||A(a,b,x,y,t_1,t_2)v|| < ||v||$ if v is not the zero vector).

Remark 3.2. *Unfortunately this eigenvalue approach does not work in a simple, closed form manner for general $b > (1-a)/\sqrt{n}$. We include details of such an attempted analysis in Appendix B of the arxiv version of this paper,* [BGKMRS].

4. DYNAMICAL BEHAVIOR: $b > (1-a)/\sqrt{n}$

In this section we prove Theorem 1.2, II(b), establishing convergence to the non-trivial fixed point.

4.1. **Properties of the Four Regions.** Unfortunately, the method of eigenvalues does not seem to naturally generalize to large b. While it is possible to compute the eigenvalues of the associated matrix, it does not appear feasible to obtain a workable expression that can be understood as the parameters vary; however, breaking the analysis of F into regions induced from the maps ϕ_1 and ϕ_2 of §2 turns out to be very fruitful. This is because these curves determine partial fixed points. See Figure 3 for the four regions.

We first study the effect of F in Regions I and III. Our first lemma provides some general information about the image of these regions under F, which we then use to show in the next lemma that F maps each of these Regions I and III to themselves.

Lemma 4.1. *Let $b > (1-a)/\sqrt{n}$. Points in Region* I *strictly increase in x and y on iteration by F, and points in Region* III *strictly decrease in x and y on iteration.*

Proof. A point (x,y) in Region I satisfies the inequalities

$$x \;<\; \frac{1-(1-by)^n}{1-a(1-by)^n} \tag{4.1}$$

and

$$y < \frac{bx}{1-a+abx}. \tag{4.2}$$

By multiplying by the denominator on both sides for both inequalities, we find that

$$\begin{aligned} x - ax\left(1-by\right)^n &< 1-\left(1-by\right)^n \\ y - ay + abxy &< bx. \end{aligned} \tag{4.3}$$

Rearranging these terms gives

$$x < 1-(1-by)^n + ax(1-by)^n = 1-(1-ax)(1-by)^n = f_1\left(x,y\right) \tag{4.4}$$

and

$$y < ay + bx - abxy = 1-(1-ay)(1-bx) = f_2\left(x,y\right). \tag{4.5}$$

Thus, the x and y coordinates of the iterate of a point in Region I are strictly greater than the x and y coordinates of the initial point.

The proof for points in Region III is exactly analogous except with the inequalities flipped. Thus

$$x > \frac{1-(1-by)^n}{1-a(1-by)^n} \tag{4.6}$$

and

$$y > \frac{bx}{1-a+abx} \tag{4.7}$$

imply that

$$x > 1-(1-ax)(1-by)^n = f_1\left(x,y\right) \tag{4.8}$$

and

$$y > 1-(1-ay)(1-bx) = f_2\left(x,y\right), \tag{4.9}$$

i.e., the x and y coordinates of the iterate of a point in Region III are strictly less than the x and y coordinates of the initial point. ☐

Lemma 4.2. *Let $b > (1-a)/\sqrt{n}$. The image of Region* I *under F is contained in* I*, and the image of Region* III *under F is contained in Region* III*.*

Proof. We prove that for a point (x,y) in Region I, its iterated x-coordinate satisfies (4.1) and its iterated y-coordinate satisfies (4.2).

x-Coordinate Iteration:

We must show that

$$1-(1-ax)(1-by)^n < \frac{1-(1-b(1-(1-ay)(1-bx)))^n}{1-a(1-b(1-(1-ay)(1-bx)))^n}. \tag{4.10}$$

We'll do this by first showing the left hand side is less than $\frac{1-(1-by)^n}{1-a(1-by)^n} > 1-(1-ax)(1-by)^n$, which we then show is less than the right hand side.

Since (x,y) is in Region I, we know that

$$x < 1-(1-ax)(1-by)^n, \tag{4.11}$$

which implies that

$$\frac{x}{1-(1-ax)(1-by)^n} < 1. \tag{4.12}$$

Since $0 < a,b,y < 1$, we know that $a(1-by)^n > 0$. Thus,

$$1 - \frac{ax(1-by)^n}{1-(1-ax)(1-by)^n} > 1-a(1-by)^n. \tag{4.13}$$

We simplify the left side of the inequality:

$$\begin{aligned}
\frac{1-(1-ax)(1-by)^n}{1-(1-ax)(1-by)^n} - \frac{ax(1-by)^n}{1-(1-ax)(1-by)^n} &> 1-a(1-by)^n \\
\frac{1-(1-by)^n + ax(1-by)^n}{1-(1-ax)(1-by)^n} - \frac{ax(1-by)^n}{1-(1-ax)(1-by)^n} &> 1-a(1-by)^n \\
\frac{1-(1-by)^n}{1-(1-ax)(1-by)^n} &> 1-a(1-by)^n. \qquad (4.14)
\end{aligned}$$

Finally, we rearrange the inequality, and obtain our intermediate step:

$$\frac{1-(1-by)^n}{1-a(1-by)^n} > 1-(1-ax)(1-by)^n. \qquad (4.15)$$

For the second part of the proof, recall that

$$y < 1-(1-ay)(1-bx), \qquad (4.16)$$

which implies

$$(1-b(1-(1-ay)(1-bx)))^n < (1-by)^n. \qquad (4.17)$$

Now we let $(1-b(1-(1-ay)(1-bx)))^n = c$ and $(1-by)^n = c+\delta$ where $0 < c < 1$ and $\delta > 0$ such that $c < c+\delta < 1$. Then we can write

$$\begin{aligned}
-\delta &< -a\delta \\
1-c-\delta-ac+ac^2+ac\delta &< 1-c-ac+ac^2-a\delta+a\delta c \\
(1-ac)(1-c-\delta) &< (1-ac-a\delta)(1-c) \\
\frac{1-(c+\delta)}{1-a(c+\delta)} &< \frac{1-c}{1-ac}. \qquad (4.18)
\end{aligned}$$

Thus

$$\frac{1-(1-b(1-(1-ay)(1-bx)))^n}{1-a(1-b(1-(1-ay)(1-bx)))^n} > \frac{1-(1-by)^n}{1-a(1-by)^n}. \qquad (4.19)$$

The desired result follows from (4.15) and (4.19).

y-Coordinate Iteration:

We must show that

$$1-(1-ay)(1-bx) < \frac{b(1-(1-ax)(1-by)^n)}{1-a+ab(1-(1-ax)(1-by)^n)}. \qquad (4.20)$$

We argue similarly as before, first showing the left hand side is less than $\frac{bx}{1-(1-ay)(1-bx)}$, which we then show is less than the right hand side. Since (x,y) is in Region I, we know that

$$y < 1-(1-ay)(1-bx), \qquad (4.21)$$

which implies that

$$\frac{y}{1-(1-ay)(1-bx)} < 1. \qquad (4.22)$$

Since $0 < a, b, x < 1$, we know that $abx - a < 0$. Thus,

$$1 + \frac{y(abx-a)}{1-(1-ay)(1-bx)} > 1-a+abx. \qquad (4.23)$$

We simplify the left side of the inequality:

$$\begin{aligned}
\frac{1-(1-ay)(1-bx)}{1-(1-ay)(1-bx)} + \frac{y(abx-a)}{1-(1-ay)(1-bx)} &> 1-a+abx \\
\frac{ay+bx-abxy}{1-(1-ay)(1-bx)} + \frac{abxy-ay}{1-(1-ay)(1-bx)} &> 1-a+abx \\
\frac{bx}{1-(1-ay)(1-bx)} &> 1-a+abx. \qquad (4.24)
\end{aligned}$$

Rearranging the inequality yields our intermediate step:

$$\frac{bx}{1-a+abx} > 1-(1-ay)(1-bx). \qquad (4.25)$$

For the second part of the proof, recall that for a point in Region I

$$x < 1-(1-ax)(1-by)^n. \qquad (4.26)$$

This allows us to write $1-(1-ax)(1-by)^n = x+c$ for some $c > 0$ such that $x < x+c < 1$. Since $c > 0$ and $a, b < 1$ we see that

$$\begin{aligned}
bc - abc &> 0 \\
bx + bc - abx - abc + ab^2x^2 + ab^2xc &> bx - abx + ab^2x^2 + ab^2xc \\
b(x+c)(1-a+abx) &> bx(1-a+ab(x+c)). \qquad (4.27)
\end{aligned}$$

Thus

$$\frac{b(x+c)}{1-a+ab(x+c)} > \frac{bx}{1-a+abx}, \qquad (4.28)$$

that is,

$$\frac{b(1-(1-ax)(1-by)^n)}{1-a+ab(1-(1-ax)(1-by)^n)} > \frac{bx}{1-a+abx}. \qquad (4.29)$$

The desired result follows from (4.25) and (4.29).

The proof showing that all points in Region III iterate inside Region III under F is essentially the same, now taking (4.8) and (4.9) as the initial inequalities. Thus given a point in Region III, we find that its iterated x-coordinate satisfies (4.6) and its iterated y-coordinate satisfies (4.7). □

4.2. **Limiting Behavior.** Before proving Theorem 1.2, II(b) in general, we concentrate on the special case when the initial state is in Region I or III.

Lemma 4.3. *Let* $b > (1-a)/\sqrt{n}$*. All non-trivial points in Regions* I *and* III *iterate to the non-trivial fixed point under* F*.*

Proof. Consider any non-trivial point $z_0 = (x_0, y_0)$ in Region I. Define a sequence by setting $z_{t+1} = F(z_t)$. By Lemma 4.1, we know that z_t is monotonically increasing in each component, and is always in Region I. Furthermore, we know that z_t is bounded by (x_f, y_f) (the unique, non-trivial fixed point). Thus, z_t must converge. Suppose it converges to z', i.e., $\lim_{t\to\infty} z_t = z'$. We consider the iterate of z'. Since F is continuous, we have

$$F(z') = F\left(\lim_{t\to\infty} z_t\right) = \lim_{t\to\infty} F(z_t) = \lim_{t\to\infty} z_{t+1} = \lim_{t\to\infty} z_t = z'. \qquad (4.30)$$

Thus, z' is a fixed point. Since $z_0 > (0,0)$ and z_t is increasing, z' cannot be the trivial fixed point. Thus z' must be the unique non-trivial fixed point. For Region III, we have a monotonically decreasing and bounded sequence z_t that must thus converge to a fixed point. By Lemma 4.2, this fixed point must be in Region III and thus can only be the unique non-trivial fixed point. □

4.3. **Proofs.** The essential idea is the following. Consider any rectangle in $[0,1]^2$ whose lower left vertex is not $(0,0)$ (the trivial fixed point introduces some complications, but we can bypass these by simply taking larger and larger rectangles). Assume the lower left and upper right vertices are in Regions I and III respectively. We show that the image of this rectangle under F is strictly contained in the rectangle by showing that the image of the lower left (respectively, upper right) point has both coordinates smaller (respectively, larger) than any other iterate. As the lower left and upper right vertices iterate to the non-trivial fixed points (since they are in Regions I and III), so too do all the other points in the rectangle, as the diameters of the iterations of the rectangle tend to zero.

We make the above argument precise. Let the rectangle be all points $(x,y) \in [0,1]^2$ with $x_\ell \leq x \leq x_u$ and $y_\ell \leq y \leq y_u$. Recall $F(x,y) = (f_1(x,y), f_2(x,y))$. We choose a point (x,y) in our rectangle and let $z_{0,1}(x,y) = x$ and $z_{0,2}(x,y) = y$. We define the sequence $z_t(x,y) = (z_{t,1}(x,y), z_{t,2}(x,y))$ (t a positive integer) by $z_{t+1,1}(x,y) = f_1(z_{t,1}(x,y), z_{t,2}(x,y))$ and $z_{t+1,2}(x,y) = f_2(z_{t,1}(x,y),\ z_{t,2}(x,y))$. We show by induction that $z_{t,1}(x_\ell, y_\ell) \leq z_{t,1}(x,y) \leq z_{t,1}(x_u, y_u)$ and $z_{t,2}(x_\ell, y_\ell) \leq z_{t,2}(x,y) \leq z_{t,2}(x_u, y_u)$. In other words, the image of any of our rectangles is contained in the rectangle, and the lower left vertex iterates to the lower left vertex of the new region (and similarly for the top right vertex).

The base case is given by our choice of (x_ℓ, y_ℓ) and (x_u, y_u), so we proceed to show the inductive step. Suppose that we have $z_{t,1}(x_\ell, y_\ell) \leq z_{t,1}(x,y)$ and $z_{t,2}(x_\ell, y_\ell) \leq z_{t,2}(x,y)$. Then

$$\begin{aligned} 1 - az_{t,1}(x_\ell, y_\ell) &\geq 1 - az_{t,1}(x,y) \\ 1 - bz_{t,2}(x_\ell, y_\ell) &\geq z_{t,2}(x,y), \end{aligned} \tag{4.31}$$

which implies that

$$(1 - az_{t,1}(x_\ell, y_\ell))(1 - bz_{t,2}(x_\ell, y_\ell))^n \ \geq\ (1 - az_{t,1}(x,y))(1 - bz_{t,2}(x,y))^n \tag{4.32}$$

for any $n \geq 1$. Then

$$1 - (1 - az_{t,1}(x_\ell, y_\ell))(1 - bz_{t,2}(x_\ell, y_\ell))^n \ \leq\ 1 - (1 - az_{t,1}(x,y))(1 - bz_{t,2}(x,y))^n. \tag{4.33}$$

That is, $z_{t+1,1}(x_\ell, y_\ell) \leq z_{t+1,1}(x,y)$. Furthermore, we have that

$$\begin{aligned} 1 - az_{t,2}(x_\ell, y_\ell) &\geq 1 - az_{t,2}(x,y) \\ 1 - bz_{t,1}(x_\ell, y_\ell) &\geq 1 - bz_{t,1}(x,y), \end{aligned} \tag{4.34}$$

which implies that

$$(1 - az_{t,2}(x_\ell, y_\ell))(1 - bz_{t,1}(x_\ell, y_\ell)) \ \geq\ (1 - az_{t,2}(x,y))(1 - bz_{t,1}(x,y)). \tag{4.35}$$

Then

$$1 - (1 - az_{t,2}(x_\ell, y_\ell))(1 - bz_{t,1}(x_\ell, y_\ell)) \ \leq\ 1 - (1 - az_{t,2}(x,y))(1 - bz_{t,1}(x,y)). \tag{4.36}$$

That is, $z_{t+1,2}(x_\ell, y_\ell) \leq z_{t+1,2}(x,y)$.

By a similar argument, we see that $z_{t,1}(x,y) \leq z_{t,1}(x_u, y_u)$ and $z_{t,2}(x,y) \leq z_{t,2}(x_u, y_u)$ implies that $z_{t+1,1}(x,y) \leq z_{t+1,1}(x_u, y_u)$ and $z_{t+1,2}(x,y) \leq z_{t+1,2}(x_u, y_u)$.

Thus $z_{t,1}(x_\ell, y_\ell) \leq z_{t,1}(x,y) \leq z_{t,1}(x_u, y_u)$ and $z_{t,2}(x_\ell, y_\ell) \leq z_{t,2}(x,y) \leq z_{t,2}(x_u, y_u)$ for all $t \in \mathbb{N}$. Taking the limit, we have

$$\lim_{t\to\infty} z_{t,1}(x_\ell, y_\ell) \ \leq\ \lim_{t\to\infty} z_{t,1}(x,y) \ \leq\ \lim_{t\to\infty} z_{t,1}(x_u, y_u) \tag{4.37}$$

and

$$\lim_{t\to\infty} z_{t,2}(x_\ell, y_\ell) \ \leq\ \lim_{t\to\infty} z_{t,2}(x,y) \ \leq\ \lim_{t\to\infty} z_{t,2}(x_u, y_u) \tag{4.38}$$

Since (x_ℓ, y_ℓ) is in Region I and (x_u, y_u) is in Region III, the inequalities become

$$x_f \ \leq\ \lim_{t\to\infty} z_{t,1}(x,y) \ \leq\ x_f \tag{4.39}$$

and

$$y_f \le \lim_{t\to\infty} z_{t,2}(x,y) \le y_f. \tag{4.40}$$

Thus $\lim_{t\to\infty} z_{t,1}(x,y) = x_f$ and $\lim_{t\to\infty} z_{t,2}(x,y) = y_f$, that is, (x,y) iterates to (x_f,y_f).

We can isolate from the proof Theorem 1.2, II(b) information about the rapidity of convergence.

Corollary 4.4. *Assume $b > (1-a)/\sqrt{n}$. Given a point $(x,y) \in (0,1)^2$, consider a rectangle with (x,y) on the boundary and vertices $(x_{\mathrm{I}}, y_{\mathrm{I}})$ in Region* I *and $(x_{\mathrm{III}}, y_{\mathrm{III}})$ in Region* III. *Then the amount of time it takes for (x,y) to converge to the unique, non-trivial fixed point is the maximum of the time it takes $(x_{\mathrm{I}}, y_{\mathrm{I}})$ and $(x_{\mathrm{III}}, y_{\mathrm{III}})$ to converge.*

5. Future Research

While we are able to determine the limiting behavior of any configuration, a fascinating question is to understand the path iterates take when converging to the fixed point. Based on some numerical computations and some partial theoretical results, we make the following conjecture.

Conjecture 5.1. *Let $b > (1-a)/\sqrt{n}$. Points in Regions* II *and* IV *exhibit one of two behaviors, depending on a, b, n. Either:*

(1) *All points in Region* II *iterate outside Region* II *and all points in Region* IV *iterate outside Region* IV *("flipping behavior"), or*
(2) *All points in Region* II *iterate outside Region* IV *and all points in Region* IV *iterate outside Region* II *("non-flipping behavior").*

It would be interesting to find simple conditions involving a, b and n for each of the two possibilities.

Another topic for future research is to apply the methods of this paper to more general models. We present some partial results to a system which quickly follow from our arguments. We may consider star graphs with more than two levels, i.e., graphs whose spokes are themselves surrounded by additional spokes, which might themselves be surrounded by additional spokes, et cetera. We recall that (1.1) and (1.2) give us the following general system:

$$\begin{aligned} p_{i,t} &= (1-p_{i,t-1})\prod_{j\sim i}(1-\beta p_{j,t-1}) + \delta p_{i,t}\prod_{j\sim i}(1-\beta p_{j,t-1}) \\ &= 1-(1-ap_{i,t-1})\prod_{j\sim i}(1-bp_{j,t-1}). \end{aligned} \tag{5.1}$$

We keep the simplifying assumption that at each level, the number of spokes is the same. In the 3-level case, this means that we consider a graph with n_1 spoke nodes around a hub node, and n_2 spoke nodes around each of the n_1 spokes. Generalizing our result in the 2-dimensional case that in the limit all spokes have the same behavior, we can argue by induction that all nodes on the same 'level' approach a common, limiting value. Thus, in the ℓ-dimensional case, we are reduced to a system in ℓ unknowns.

We first consider the 3-dimensional case. If we let x_t be the probability that the hub is infected (the level 1 node), y_t be the probability that a spoke of the hub is infected (the level 2 nodes), and z_t be the probability a spoke of a spoke is infected (the level 3 nodes), (5.1) gives us the following system:

$$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-(1-ax)(1-by)^{n_1} \\ 1-(1-ay)(1-bx)(1-bz)^{n_2} \\ 1-(1-az)(1-by) \end{pmatrix}. \tag{5.2}$$

We again look for partial fixed points by solving

$$\begin{aligned} x &= f_1(x,y,z) \\ y &= f_2(x,y,z) \\ z &= f_3(x,y,z), \end{aligned} \tag{5.3}$$

which gives the following surfaces:

$$\begin{aligned} \phi_1(y,z) &= x = \frac{1-(1-by)^{n_1}}{1-a(1-by)^{n_1}} \\ \phi_2(x,z) &= y = \frac{1-(1-bx)(1-bz)^{n_2}}{1-ay(1-bx)(1-bz)^{n_2}} \\ \phi_3(x,y) &= z = \frac{by}{1-a+aby}. \end{aligned} \tag{5.4}$$

If we take the intersection of ϕ_1 with the plane defined by ϕ_3 and ϕ_2 with the plane defined by ϕ_3, we get two curves that look a lot like our curves from the original (2-dimensional) case. We can express these curves in terms of x and y. The first curve is already done. For the second, we can write

$$x = \frac{y-1}{b(1-ay)(1-bz)^{n_2}} + \frac{1}{b}. \tag{5.5}$$

Since we know that $z = by/(1-a+aby)$ we can write this as

$$x = \frac{y-1}{b(1-ay)\left(1-\frac{b^2y}{1-a+aby}\right)^{n_2}} + \frac{1}{b}. \tag{5.6}$$

We now have two curves, $\phi_1(y)$ and $\phi_2(y)$. If we take their derivatives at 0, we obtain

$$\begin{aligned} \phi_1'(0) &= \frac{bn_1}{1-a} \\ \phi_2'(0) &= \frac{(1-a)^2 - b^2 n_2}{b(1-a)}. \end{aligned} \tag{5.7}$$

Doing some analysis on their second derivatives shows that $\phi_1''(y) < 0$ and $\phi_2''(y) > 0$ for all $y \in [0,1]$. Thus $\phi_1(y)$ is convex and $\phi_2(y)$ is concave. All the pieces are now in place to argue as in the proof of Theorem 1.2, I(a) and II(a). We find that there exists a unique nontrivial fixed point if and only if

$$\phi_1'(0) > \phi_2'(0), \tag{5.8}$$

i.e.,

$$b > \frac{1-a}{\sqrt{n_1+n_2}}. \tag{5.9}$$

.

This leads to the following conjecture (which is known for $\ell = 2$ or 3).

Conjecture 5.2. *Consider a generalized spoke and star graph with ℓ levels. Level one consists of one node (the hub), level two consists of n_1 spokes connected to the central hub, and for each node of level k there are n_k nodes connected to it (and these are the level $k+1$ nodes). There is a unique, non-trivial fixed point if and only if $b > (1-a)/\sqrt{n_1+\cdots+n_{\ell-1}}$.*

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