ON SUMMAND MINIMALITY OF GENERALIZED ZECKENDORF DECOMPOSITIONS

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ABSTRACT. Zeckendorf proved that every positive integer can be uniquely represented as a sum of non-consecutive Fibonacci numbers. This has been extended in many ways, including to linear recurrences $H_n = c_1 H_{n-1} + \cdots + c_t H_{n-t}$ where the c_i are non-negative integers and c_1, $c_t \geq 1$. Every integer has a unique generalized Zeckendorf decomposition (gzd) – a representation composed of blocks that are lexicographically less than $(c_1, \dots, c_t)$, which we call the signature. We prove that the gzd of a natural number m uses the fewest number of summands out of all representations for m using the same recurrence sequence, for all m, if and only if the signature of the linear recurrence is weakly decreasing (i.e., $c_1 \geq \cdots \geq c_t$). Following the parallel with well-known base d representations, we develop a framework for naturally moving between representations of the same number using a linear recurrence, which we then utilize to construct an algorithm to turn any representation of an integer into the gzd. To prove sufficiency, we show that if the signature is weakly decreasing then our algorithm results in fewer summands. To prove necessity we proceed by divide and conquer, breaking the analysis into several cases. When $c_1 > 1$, we give an example of a non-gzd representation of an integer and show that it has fewer summands than the gzd by performing the same above-mentioned algorithm. When $c_1 = 1$, we non-constructively prove the existence of a counterexample by utilizing the irreducibility of a certain family of polynomials together with growth rate arguments.

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1. INTRODUCTION

A celebrated theorem of Zeckendorf [**Ze**] states that every positive integer can be uniquely represented as a sum of non-consecutive Fibonacci numbers (with $F_1 = 1, F_2 = 2$). This result has sparked significant interest since its introduction. Brown [**Br1**, **Br2**] and Keller [**Ke**] noticed that the Fibonacci numbers are distinguished as the only sequence such that every positive integer has a unique representation as the sum of non-consecutive members in the sequence. Thus, many authors have furthered Zeckendorf's theorem by specifying a rule for unique representation and deducing the structure of the sequence which allows for this representation rule (see for example [**Day**, **DDKMMV**, **CFHMN1**]).

Several authors have interpreted Zeckendorf's theorem as a way of using the Fibonacci numbers as a number system. This analogy is particularly apt because we can think of the conventional base d representation as arising from a degree one recurrence $H_n = dH_{n-1}$ (with appropriate initial conditions). In following this analogy, several authors have extended Zeckendorf's theorem for sequences arising from a larger class of linear recurrences. Namely, given a linear recurrence sequence, one may ask if there is a "natural" set of rules allowing for unique representation of each integer. Shortly following Zeckendorf's original result, many authors extended Zeckendorf's theorem to broader classes of recurrences of some specific forms (see for instance [**Ho**, **Day**, **Fr**, **HW**]). Given a linear recurrence $H_n = c_1H_{n-1} + \cdots + c_tH_{n-t}$, we call $\sigma = (c_1, \ldots, c_t)$ the signature of the recurrence. Fraenkel [**Fr**] generalized Zeckendorf's result to all recurrences with weakly decreasing signature. More recently, Miller and Wang [**MW1**, **MW2**] and independently Hamlin and Webb [**Ha**], have generalized Zeckendorf's theorem to non-negative signatures with $c_1 \geq 1$. Furthermore, various authors ([**Ha**, **CFHMN1**, **CFHMN2**, **CFHMNPX**]) provided evidence that this is the broadest class of signatures for which one can expect Zeckendorf's theorem to extend in a simple way, as negative coefficients or $c_1 = 0$ can lead to relations where infinitely many integers do not have a unique decomposition. *Thus we shall always assume $c_1 \geq 1$ and all c_i are non-negative integers below.*

For a non-negative linear recurrence, the unique representation of a positive integer using the recurrence sequence is called its **generalized Zeckendorf decomposition** (**gzd**). Classically, many authors have asked questions about the number of summands in Zeckendorf decompositions (see [**Lek**] for example). More recently, [**BBGILMT**, **BILMT**, **LM**, **Ste**] (among others) investigated the distribution of the number of summands and the gap between consecutive summands in the gzd.

Of all the decompositions of an integer as a sum of Fibonacci numbers, the Zeckendorf decomposition is minimal in that no other decomposition has fewer summands. The proof is immediate and follows by the introduction of an appropriate monovariant, and keeping track of how one moves from an arbitrary decomposition to the Zeckendorf one. Given a decomposition of m into a sum of Fibonacci numbers, consider the sum of indices of terms in the decomposition. If we have two adjacent summands F_i and F_{i+1} we do not increase the index sum by replacing them with F_{i+2}. If we have F_1 twice use F_2 instead, if we have F_2 twice use F_1 and F_3, and in general if we have F_k twice use $2F_k = F_{k-2} + F_{k-1} + F_k = F_{k-2} + F_{k+1}$, which decreases the index sum for $k \geq 3$ and yields a larger Fibonacci summand. We can only do this a bounded number of times or we would have a Fibonacci summand larger than the largest Fibonacci number less than m, thus when the process terminates there are no repeats or adjacencies. Thus we end in the Zeckendorf decomposition, and see that it cannot have more summands than the original decomposition. This minimality property holds for other decompositions. Alpert [**Al**] obtained a Zeckendorf-like result for representations of integers using Fibonacci numbers where the representation can have mixed sign summands (± 1). She showed that this far-difference representation uses the fewest number of summands among all mixed sign representations. This result is generalized for "Skipponacci" sequences in [**DDKMV**].

The goal of this paper is to extend these arguments to a larger class of recurrence relations. We call a representation of n summand minimal if no other representation of n uses fewer summands. We say that a recurrence sequence is summand minimal if the gzd is summand minimal for all n. Given that the Zeckendorf decomposition (arising from the Fibonacci numbers) is summand minimal, one may ask if the gzd is always summand minimal. We completely answer this question for positive linear recurrence relations.

Theorem 1.1. *A recurrence sequence with signature $\sigma = (c_1, \dots, c_t)$ where the c_i are non-negative integers with $c_1, c_t \geq 1$ is summand minimal if and only if $c_1 \geq c_2 \geq \cdots \geq c_t$.*

In order to establish this result, we formulate two natural rules that allow us to move from one representation to another (using the same recurrence sequence) while keeping track of the change in the total number of summands. Using these two rules, we are then able to construct an algorithm that will turn any representation of any integer into the gzd. In fact, our proof on the termination of this algorithm provides an alternative proof of the generalized Zeckendorf theorem originally proven by [**MW1**] and [**Ha**]. We then utilize this algorithm to prove that if a recurrence sequence has a weakly decreasing signature, then the recurrence sequence is summand minimal. We exploit the fact that for linear recurrences with weakly decreasing signature, anytime we perform a complete step of the algorithm, the total number of summands always weakly decreases.

The proof of the other direction of Theorem 1.1 requires significantly more work. In particular, we prove the contrapositive by splitting it into four broad cases, three of which consider signatures starting with $c_1 > 1$. When $c_1 > 1$, we provide explicit non-gzd representations with fewer summands than the gzd, which we construct by utilizing the algorithm to move from this representation to the gzd. Although the proof involves a lot of casework and bookkeeping, the technique used to prove all three cases is similar: keep track of the changes in the number of summands at each iteration of the algorithm used to find the gzd.

Lastly, the case where $c_1 = 1$ is much more interesting because we cannot directly apply our previous technique without an enormous amount of casework. Here, we make use of the key piece of information on the signature – the specific value of $c_1 = 1$, and show that there exists an integer of the form $2H_n$ for which the gzd has at least 3 summands. The proof involves determining the form of the summand minimal gzd of $2H_n$, if it exists, given the growth rate of the sequence and then establishing multiple properties, such as irreducibility of families of polynomials, in order to show that such form cannot hold for all $n \in \mathbb{N}$.

In §2 we establish the terminology that we use throughout the paper, and also details on the two aforementioned rules that allow us to move from one representation to another. Section 3 gives the algorithm that turns any initial representation into the gzd, and proves that this algorithm terminates. The forward direction of Theorem 1.1 is proved in §4, and the reverse direction in §5.

In particular, subsections §5.1–§5.4 handle linear recurrences with signature starting with $c_1 > 1$, while subsection §5.5 deals with linear recurrences with signature starting with $c_1 = 1$.

2. Preliminaries

Following the parallel between the gzd and conventional base d proves to be extremely advantageous in our case: String-like representations allow us to depict and distinguish different representations easily, faciliating the algorithm of moving from one representation to the next. In particular, we define a representation of an integer using a recurrence sequence in the following way.

Definition 2.1. *Let $H = \{H_n\}_{n\in\mathbb{N}_0}$ be a recurrence sequence. A sequence of non-negative integers $R = [r_s, r_{s-1}, \ldots, r_0]$ such that $n = \sum_i r_i H_i$ is called a **representation** of n using H.*

Hamlin and Webb [**Ha**] have studied the gzd in a very similar framework; as such, we adopt some of his terminologies but also define several others that are crucial given our question on summand minimality. In this section, we detail these new definitions together with some examples to showcase their importance.

First, in order to reinterpret the generalized Zeckendorf theorem in our framework, we need the following definitions.

Definition 2.2. *Suppose a linear recurrence is defined by $H_n = c_1 H_{n-1} + \cdots + c_t H_{n-t}$. Then $\sigma = (c_1, \ldots, c_t)$ is called the **signature** of the recurrence.*

Definition 2.3. *A linear recurrence with $\sigma = (c_1, \ldots, c_t)$ is called **positive** if the c_i are non-negative integers and $c_1, c_t \geq 1$.*

Since we shall only be concerned with positive linear recurrences, hereafter we shall simply use the word *recurrence* in place of positive linear recurrence.

Definition 2.4. *Suppose $\sigma = (c_1, \ldots, c_t)$. Then $[b_1, \ldots, b_k]$ is called an **allowable block** (or **valid block**) if $k \leq t$, $b_i = c_i$ for $i < k$, and $0 \leq b_k < c_k$.*

Example 2.5. *If $\sigma = (4, 3, 2)$ then the set of allowable blocks is*

$$\{[0], [1], [2], [3], [4, 0], [4, 1], [4, 2], [4, 3, 0], [4, 3, 1]\},$$

while if $\sigma = (2, 0, 0, 3)$ then the allowable blocks are

$$\{[0], [1], [2, 0, 0, 0], [2, 0, 0, 1], [2, 0, 0, 2]\}.$$

Definition 2.6. *Suppose $\sigma = (c_1, \ldots, c_t)$. Then $H_{-(t-1)} = H_{-(t-2)} = \cdots = H_{-1} = 0$ and $H_0 = 1$ are called the **ideal initial conditions**.*

Henceforth, when referring to a sequence arising from a recurrence, we shall assume without explicit mention that the sequence is obtained from the aforementioned recurrence with ideal initial conditions. The symbol H_σ shall refer to the recurrence sequence obtained from signature σ and ideal initial conditions.

We are now ready to reinterpret the generalized Zeckendorf theorem in our language.

Theorem 2.7. *Let $H = \{H_n\}_{n\in\mathbb{N}_0}$ be a recurrence sequence with signature $\sigma = (c_1, \ldots, c_t)$. Then every non-negative integer N has a unique representation composed of allowable blocks. This representation is called the **generalized Zeckendorf decomposition (gzd)**.*

Example 2.8. *Suppose $\sigma = (1,1)$, the signature for the Fibonacci numbers. The allowable blocks are $\{[0],[1,0]\}$. Therefore Theorem 2.7 implies that every integer has a unique representation composed of $[0]$ and $[1,0]$, which is Zeckendorf's theorem.*

Example 2.9. *Suppose $\sigma = (d)$. Then the allowable blocks are $\{[0],[1],\dots,[d-1]\}$. Theorem 2.7 implies that every integer can be uniquely expressed as a sum of powers of d such that each coefficient in the sum is between 0 and $d-1$, which is the base d representation.*

If our recurrence is depth one (as is the case for base d representations), then our recurrence sequence contains no zeros because we have only one initial condition, $H_0 = 1$. However, if our recurrence is of depth $t \geq 2$, and we use the ideal initial conditions, then we have $(t-1)$ zeros whose indices are negative (see Definition 2.6). In Definition 2.1 of the representation, we were only concerned with the coefficients in the representation that have non-negative indices. While we could have defined what it means for a representation to include coefficients whose indices go all the way to $-(t-1)$, there is no need. For example, suppose $\sigma = (1,1)$. Then $H_n = F_n$ (the Fibonacci numbers). Suppose we change our definition of representation to include the -1 index coefficient. Then, for example, we could represent 3 as $[\dots,0,1,0,1,0]$ $(1\times2+0\times1+1\times1+0\times0)$ or as $[\dots,0,1,0,1,100]$ $(1\times2+0\times1+1\times1+100\times0)$. These representations are not different in a meaningful way. The -1^{st} coefficient could be arbitrarily large without fundamentally changing the representation. Thus, so that we don't distinguish representations which are "really the same", we should think of any representation as using "as many zeros as we want." Therefore, given a representation, we implicitly assume that this representation has $(t-1)$ negative index entries, all of which are ∞ (so, for example, using $\sigma = (1,1)$, our two "distinct representations" for 3 from above would now be the single representation $[\dots,0,1,0,1,\infty]$). When our recurrence is of depth t, we shall use the shorthand ∞_{t-1} to denote $\underbrace{\infty,\dots,\infty}_{t-1}$, or when it's clear how many infinities there are, we shall simply use $\infty,\dots$ (in some cases we shall omit including the infinities, implying that the rightmost index is 0). The justification for using infinities becomes even clearer below.

The signature provides a way of moving between representations of the same number. Thus for example if our signature is (10), one representation for 312 is $[3,1,2]$ (indeed, this is the gzd). However, we may also represent it as $[2,11,2]$ (by "borrowing" from the 100's place). Analogously, say we have the representation $[6,23,4]$, that is $6\times100+23\times10+4\times1 = 834$. However, since the 10's place currently has $23 \geq 10$, we can "carry" over to the 100's place to get the representation $[7,13,4]$, and then carry again to get $[8,3,4]$.

The ideas of "borrowing" and "carrying" from base d arithmetic extend to all recurrences. For example, suppose $\sigma = (2,1)$. The terms in our recurrence sequence would be $[0,1,2,5,12,\dots]$. If we have the representation $[3,0,0,\infty]$ (which represents 15 since $15 = 3\times5$, we can "borrow" at index 2 (remember, we start indexing from 0) to get the representation $[2,2,1,\infty]$ $(15 = 2\times5+2\times2+1\times1)$. Now suppose we borrow at index 2. We then get the representation $[2,1,3,\infty+1]$. If we extend our arithmetic to include ∞ such that $\infty \pm n = \infty$ for any $n < \infty$, and $\infty \times 0 = 0$, then we can still "borrow" even when it results in terms accumulating in the "infinities places." Suppose instead that we have the representation $[3,4,\infty]$ (which represents 10 since $10 = 3\times2+4\times1$). Since $3 \geq 2$ and $4 \geq 1$, we "carry" to index 2 to get the representation $[1,1,3,\infty]$. If we think of ∞ as a number which is larger than any finite number, then we can carry again to index 1 to get the representation $[1,2,1,\infty-1] = [1,2,1,\infty]$. Thus, using ∞ also allows us to "carry" even when it involves the infinities places.

The above discussion motivates the following formal definitions.

Definition 2.10. *Given a positive linear recurrence with signature $\sigma = (c_1, \dots, c_t)$, we have associated* ***borrow*** *and* ***carry rules****. Consider a representation $R = [\dots, 0, r_n, r_{n-1}, \dots, r_0, \infty_{t-1}]$. Let $B(R, i)$, $C(R, i)$ be defined as*

$$B : \mathbb{N}_0^\infty \times \mathbb{N} \to \mathbb{N}_0^\infty$$
$$([\dots, r_n, \dots, r_0, \infty_{t-1}], i) \mapsto [\dots, r_n, \dots, r_i - 1, r_{i-1} + c_1, \dots, r_{i-t} + c_t, r_{i-(t+1)}, \dots, r_0, \infty_{t-1}]$$
$$C : \mathbb{N}_0^\infty \times \mathbb{N} \to \mathbb{N}_0^\infty$$
$$([\dots, r_n, \dots, r_0, \infty_{t-1}], i) \mapsto [\dots, r_n, \dots, r_i + 1, r_{i-1} - c_1, \dots, r_{i-t} - c_t, r_{i-(t+1)}, \dots, r_0, \infty_{t-1}].$$

We call the application of $B(R, i)$ ***borrowing from*** *i and the application of $C(R, i)$* ***carrying to*** *i.*

Remark 2.11. *Borrowing and carrying are best visualized by the following tables:*

		$\mathbf{n}$	$\dots$	$\mathbf{i}$	$\mathbf{i-1}$	$\dots$	$\mathbf{i-t}$	$\mathbf{i-(t+1)}$	$\dots$	$\mathbf{0}$	$\mathbf{-1}$	$\dots$	$\mathbf{-(t-1)}$
$\mathbf{R}$	$\dots$	r_n	$\dots$	r_i	r_{i-1}	$\dots$	r_{i-t}	$r_{i-(t+1)}$	$\dots$	r_0	∞	$\dots$	∞
Borrow from i				-1	c_1	$\dots$	c_t						
$\mathbf{B(R, i)}$	$\dots$	r_n	$\dots$	$r_i - 1$	$r_{i-1} + c_1$	$\dots$	$r_{i-t} + c_t$	$r_{i-(t+1)}$	$\dots$	r_0	∞	$\dots$	∞

TABLE 1. Borrow from i.

		$\mathbf{n}$	$\dots$	$\mathbf{i}$	$\mathbf{i-1}$	$\dots$	$\mathbf{i-t}$	$\mathbf{i-(t+1)}$	$\dots$	$\mathbf{0}$	$\mathbf{-1}$	$\dots$	$\mathbf{-(t-1)}$
$\mathbf{R}$	$\dots$	r_n	$\dots$	r_i	r_{i-1}	$\dots$	r_{i-t}	$r_{i-(t+1)}$	$\dots$	r_0	∞	$\dots$	∞
Carry to i				1	$-c_1$	$\dots$	$-c_t$						
$\mathbf{C(R, i)}$	$\dots$	r_n	$\dots$	$r_i + 1$	$r_{i-1} - c_1$	$\dots$	$r_{i-t} - c_t$	$r_{i-(t+1)}$	$\dots$	r_0	∞	$\dots$	∞

TABLE 2. Carry to i.

Remark 2.12. *When we borrow from/carry to an index $i \geq t$, the change in the number of summands is $\pm(-1 + c_1 + \cdots + c_t)$. In such a case, we call these actions* ***pure borrow*** *and* ***pure carry****. Note that after one pure borrow and one pure carry (independent of the position of index), the change in the number of summands is zero. In the case where we borrow from/carry to an index $i < t$, the change in the number of summands is $\pm(-1 + c_1 + \cdots + c_i)$. We call these* ***impure borrow*** *and* ***impure carry****.*

Definition 2.13. *Let $R = [\dots, 0, r_m, r_{m-1}, \dots, r_0, \infty_{t-1}]$ be a representation using the sequence H with signature $\sigma = (c_1, \dots, c_t)$. We say the representation R is* ***legal up to*** s *if $[\dots, 0, r_m, \dots, r_s]$ can be expressed in the form $[\dots, [0], [B_1], \dots, [B_j]]$ where for $1 \leq i \leq j$, each $[B_i]$ is an allowable block and $[B_1] \neq [0]$.*

Definition 2.14. *The* ***minimum legal index (m.l.i.)*** *of a representation R is the smallest index s such that R is legal up to s.*

Notice that if $\sigma = (c_1, \dots, c_t)$ and R is a representation using H_σ whose m.l.i. is s, then $r_{s-1} \geq c_1$. If $r_{s-1} = c_1$, then $r_{s-2} \geq c_2$. At some point, we must either have that $r_{s-j} > c_j$ or for all $1 \leq j < t$, $r_{s-j} = c_j$ and $r_{s-t} \geq c_t$. This motivates the following definition.

Definition 2.15. *Suppose a representation R has m.l.i. equal to s. Let j be the smallest index such that $r_{s-j} > c_j$, or if $r_{s-i} = c_i$ for all $1 \leq i \leq t$, then let $j = t$. The **violation index** is $s - j$ and we call r_{s-j} the **violation**.*

Definition 2.16. *Suppose a representation R has m.l.i. equal to s and violation index equal to j. Then $[r_{s-1}, \ldots, r_{s-(j-1)}] = [c_1, \ldots, c_{j-1}]$ is called the **prefix of the violation**. The prefix and the violation together comprise the **violation block**.*

Definition 2.17. *Let R be a representation with m.l.i. s and violation index j. We say that R is **semi-legal** up to q if $q = s - j + 1$. We call q the **semi-legal index (s.l.i.)**.*

There are two key remarks following the above definitions.

Remark 2.18. *We note that the s.l.i. is the index to the left of the violation index, i.e., equal to the violation index plus one. Furthermore, the difference between the m.l.i. and the s.l.i. is exactly the length of the violation prefix, which is 0 in the case that the violation prefix is empty.*

Remark 2.19. *The m.l.i. and s.l.i. essentially give us a way to think about how "close" a representation is to its gzd. Indeed, with Definitions 2.13, 2.14 and 2.17, we have that the m.l.i. and the s.l.i. of any gzd are both equal to 0, implying the gzd is legal up to 0. For any representation, we can now say that it is legal up to x and semi-legal up to y; as such, the more positive x and y are, the "further" away a representation is from the gzd.*

Definition 2.20. *Let R be some representation. We say that we are **able to carry to** j if for all $1 \leq i \leq t$ we have $r_{j-i} \geq c_i$.*

We now illustrate the above definitions via the following example.

Example 2.21. *Let $\sigma = (c_1, c_2, c_3) = (3, 2, 4)$ and consider the representation $R = [3, 2, 1, 1, 3, 0, 3, 3, 5]$. We have*

$$R = \left[\boxed{3,2,1}, \boxed{1}, \boxed{3,0}, \boxed{3,3\ \,}, 5\right],$$

where each closed box represents a valid block, while the right-opened box ($\boxed{3,3\ \,}$) represents the violation block. The violation index is 1, the s.l.i. is 2 and the m.l.i is 3. We are able to carry to 3 (the m.l.i.) because $r_2 = 3 = c_1$, $r_1 = 3 > 2 = c_2$ and $r_0 = 5 > 4 = c_3$. After carrying, we get the following representation

$$\left[\boxed{3,2,1}, \boxed{1}, \boxed{3,1}, \boxed{0}, \boxed{1}, \boxed{1}\right],$$

which is the gzd. We note that the m.l.i. and s.l.i equal 0.

However, consider the same signature but the representation $R' = [3, 2, 1, 1, 3, 0, 3, 3, 1]$. The violation index, m.l.i., and s.l.i. are still the same but we cannot carry because $r_0 = 1 < 4 = c_3$. This motivates the following definitions.

Definition 2.22. *Let R be a representation with m.l.i. equal to s. We call $s - \ell$ the **carry obstruction index (c.o.i.)** if for all $1 \leq i < \ell$, $r_{s-i} \geq c_i$ and $r_{s-\ell} < c_\ell$.*

Definition 2.23. *Let R be a representation whose m.l.i. is s and c.o.i. is $s - \ell$. Then $s - e$ is called the **rightmost excess index (r.e.i.)** if for all $e < i < \ell$, $r_{s-i} = c_i$ and $r_{s-e} > c_e$.*

Example 2.24. *Consider the aforementioned example with $\sigma = (3,2,4)$ and $R' = [3,2,1,1,3,0,$ $3,3,1]$. Here, m.l.i. $= 3$, s.l.i. $= 2$ and the violation index is 1. The c.o.i. is 0 and the r.e.i. is 1 because $r_1 = 3 > c_2 = 2$. The idea is that we can borrow from the r.e.i. to make our c.o.i. large enough so we are able to carry, hence the name. We demonstrate this in Table 3.*

	8	**7**	**6**	**5**	**4**	**3**	**2**	**1**	**0**	**−1**	**−2**
$\mathbf{R}'$	3	2	1	1	3	0	3	3	1	∞	∞
Borrow from 1								-1	3	2	4
	3	2	1	1	3	0	3	2	4	∞	∞
Carry to 3						1	-3	-2	-4		
	3	2	1	1	3	1	0	0	0	∞	∞

TABLE 3. Sequence of borrows and carries to move to the gzd.

3. ALGORITHM: FROM ANY REPRESENTATION TO THE GZD

Recall from Remark 2.19 that a representation can be thought of as being "far" from the gzd if its m.l.i. and s.l.i. are large. As such, a natural way to turn any representation into the gzd is to try decreasing the m.l.i. and s.l.i. of the representation to zero. To do so, one can trace the representation from left to right, find the first violation and attempt to "fix" it. Notice that because any valid block is lexicographically less than the signature of the linear recurrence (see Definition 2.4), the entry at the violation index is always "too large", which suggests that we can carry in order to fix it, as in Example 2.21. In the case where we cannot carry, which means the c.o.i. exists, we would borrow from the r.e.i. in order to carry, as in Example 2.24. As one goes along and performs these borrows and carries to fix all possible violations, one would expect to decrease the m.l.i. and the s.l.i. to zero to reach the gzd. Though there is more subtlety in the actual algorithm and why it terminates, this is the key idea of the process.

We now present our algorithm formally.

Algorithm 3.1.
Input:*a representation, R, of n*
Output:*the gzd of n*
while *the m.l.i. is not 0* **do**
  if *able to carry to m.l.i.* **then**
    carry to m.l.i.
    while *left neighbor block is $[c_1, \dots, c_t]$* **do**
      carry to next block
    end
  else
    borrow from the r.e.i.
  end
end

Remark 3.2. *Recall that given a signature $(c_1, \dots, c_t)$ we can decompose any representation R as*

$$[\dots, [0], [B_1], \dots, [B_j], [V], r_m, r_{m-1}, \dots, r_0, \infty_{t-1}],$$

where each $[B_i]$, $1 \leq i \leq j$, represents a valid block with $[B_1] \neq [0]$ and $[V]$ is the violation block. In this sense, the left neighbor block of $[V]$ is $[B_j]$, the left neighbor block of any $[B_i]$ with $2 \leq i \leq j$ is $[B_{i-1}]$, and the left neighbor block of $[B_1]$ is $[0]$.

We demonstrate this algorithm in the following example.

Example 3.3. *Let $\sigma = (5, 3, 1)$. We apply the algorithm to turn the representation $R = [1, 5, 3, 0, 5, 4, 0, 6]$ into the gzd.*

Index	$\dots$	**7**	**6**	**5**	**4**	**3**	**2**	**1**	**0**	$\mathbf{-1}$	$\dots$
m.l.i.$= 4$*, s.l.i.*$= 3$*, c.o.i.*$= 1$*, r.e.i.*$= 2$		1	5	3	0	5	4	0	6	∞	$\dots$
borrow at 2							-1	5	3	1	$\dots$
able to carry		1	5	3	0	5	3	5	9	∞	$\dots$
carry to 4					1	-5	-3	-1			$\dots$
left neighbor block in form to carry		1	**5**	**3**	**1**	0	0	4	9	∞	$\dots$
carry to 7		1	-5	-3	-1						
m.l.i=s.l.i.$= 1$		2	0	0	0	0	0	4	9	∞	$\dots$
carry to 2								1	-5	-3	
m.l.i=2, s.l.i$= 1$		2	0	0	0	0	0	5	4	∞	$\dots$
carry to 3							1	-5	-3	∞	
m.l.i.=s.l.i.$= 0$*, gzd*		2	0	0	0	0	1	0	1	∞	$\dots$

Note from the above example that the m.l.i may increase at some step of the algorithm, but the s.l.i. can never increase.

In what follows, we give the proof of why Algorithm 3.1 terminates in the gzd. We first show that the s.l.i. weakly decreases in Lemma 3.4, then we show that the s.l.i. strictly decreases after finitely many steps in Lemma 3.5. Finally, using those two lemmas, we prove that the m.l.i. decreases to 0 in finitely many steps.

Lemma 3.4. *The semi-legal index (s.l.i.) monotonically decreases.*

Proof. Notice that if we are not able to carry, then the semi-legal index either stays the same or decreases. Thus, without loss of generality, suppose that we are able to carry.

Firstly, suppose that the block before the violation block is $[c_1, \dots, c_{\ell-1}, d_\ell]$ with $d_\ell < c_\ell$. If $d_\ell < c_\ell - 1$, then we still have a valid block after carrying. The entries between the old m.l.i. and the old s.l.i. are zeros after carrying, which are valid blocks. Therefore the s.l.i. will have either stayed the same or decreased.

Secondly, suppose that $d_\ell = c_\ell - 1$ and $\ell < t$. After carrying, we have something of the form

$$[c_1, \dots, c_{\ell-1}, c_\ell, 0, \dots, 0, v - c_{k+1}, \dots],$$

where v is the original violation and the length of $[0, \dots, 0]$ is the length of the violation prefix before carrying.

We cannot have a violation before $v - c_{k+1}$ because a violation requires that an entry is greater than its corresponding entry in the signature; however, the first ℓ terms agree with the signature and the remaining terms are all zero, so they are either equal to or less than the corresponding terms in the signature. Therefore the earliest possible violation is at $v - c_{k+1}$, so the s.l.i. either stays the same or decreases.

Finally, suppose $\ell = t$ and $d_\ell = c_t - 1$. After we carry, the left neighbor block[1] of the violation block is now $[c_1, \ldots, c_t]$, so we immediately carry again by the algorithm. If the next left neighbor block is not $[c_1, \ldots, c_t - 1]$, then after the carry, at worst, we have $[c_1, \ldots, c_m]$ followed by at least t zeros, where $m < t$. This must be made up of allowable blocks because we have $[c_1, \ldots, c_m, \underbrace{0, \ldots, 0}_{t-m}]$ and $c_t \geq 1$. So in this case the s.l.i. has not increased. If the left neighbor block is $[c_1, \ldots, c_t - 1]$, then after the carry, we carry again by the algorithm. We know that at some point the left neighbor block is not $[c_1, \ldots, c_t - 1]$ since there are finitely many non-zero blocks. As such, at some point this process terminates without increasing the s.l.i. $\square$

Lemma 3.5. *Suppose s.l.i. ≥ 1. Then, after finitely many steps of the algorithm, the s.l.i. decreases.*

Proof. Let us consider the signature $(c_1, \ldots, c_t)$ and the representation

$$R = [\ldots, [0], [B_1], \ldots, [B_j], c_1, \ldots, c_m, v, \ldots],$$

where $0 \leq m < t$, v is the first violation reading from left to right, and each B_i is a valid block. Without loss of generality, let $B_j = [c_1, \ldots, c_{\ell-1}, d_\ell]$, with $d_\ell < c_\ell$.

We now show that after finitely many steps, v decreases in size. First, when we perform the algorithm, every time we borrow, the value in the rightmost excess index (r.e.i.) decreases by 1. If we keep borrowing and are never able to carry, then at some point, we must have decreased the value at the original r.e.i. to the point where it is no longer "excess". At this point, the r.e.i. increases, and so after finite time the r.e.i. becomes equal to the violation index and thus when we borrow from there, v decreases.

Otherwise, after some borrows, we are able to carry. If we carry a positive number from v, v decreases; however, it is possible that we carry 0 from v, in which case v does not decrease. More specifically, if $m \neq 0$ and $c_{m+1} = 0$, then after carrying, we obtain:

$$[\ldots, [0], [B_1], \ldots, [B_{j-1}], c_1, \ldots, c_{\ell-1}, d_\ell + 1, \underbrace{0, \ldots, 0}_{m}, v, \ldots].$$

Notice that v has not decreased; however, there are now $m \geq 1$ zeros to the left of v. By Lemma 3.4, after performing any possible carries in the left neighbor blocks, the s.l.i. will at worst stay the same, in which case v is still the violation and the algorithm repeats. Again, repeating the above arguments, we have that any time we borrow from v, v will decrease. Otherwise, the only case where we can carry from v without decreasing it is when the violation block is of the form $[c_1, c_2, \ldots, c_q, v]$ and $c_{q+1} = 0$. After the first carry, because the first m entries to the left of v are all zeros, we must have that $c_q = \cdots = c_{q-m+1} = 0$, hence q must be larger than m. As such, after we perform this second carry, the number of zeros to the left of v increases by $q - m \geq 1$.

Therefore, when we repeat the algorithm, either v decreases or we carry and v does not decrease but the number of zeros immediately to the left of v increases. If v never decreases, then there must be a point where there are t zeros in front of v. By Lemma 3.4, at worse the s.l.i. stays the same and v remains the violation index. Since the difference between the m.l.i. and the s.l.i. is either 0 or equal to the length of the violation prefix (which is bounded above by $t - 1$), and because there are now t zeros in front of v, the m.l.i. must equal the s.l.i. Repeating the algorithm, either we borrow repeatedly until we borrow from v or at some point we carry to the m.l.i. and v decreases by at least $c_1 \geq 1$.

[1] see Remark 3.2 for more information on left neighbor block

As a result, in all cases we must have that v decreases. Therefore, after finitely many steps, v must be small enough that the s.l.i. decreases. $\square$

Theorem 3.6. *The algorithm terminates in the gzd.*

Proof. By Lemma 3.5, the s.l.i. decreases to zero. Theorem 3.6 holds if and only if the m.l.i. decreases to zero after finitely many steps. Therefore, we need only show that when the s.l.i. is zero, the m.l.i. goes to zero.

Consider the signature $(c_1, \dots, c_t)$. Suppose the s.l.i. is zero. If the m.l.i. is not equal to 0, then our representation is of the form $R = [\dots, [B_1], \dots, [B_j], c_1, c_2, \dots, c_m, \infty_{t-1}]$, where each B_i is an allowable block and $0 < m \leq t$. Without loss of generality, suppose that $[B_j] = [c_1, \dots, c_{\ell-1}, d_\ell]$ with $d_\ell < c_\ell$ and $0 < \ell \leq t$. By the algorithm, using the ∞ places, we can immediately carry to get

$$[\dots, [B_1], \dots, [B_{j-1}], c_1, \dots, c_{\ell-1}, d_\ell + 1, \underbrace{0, \dots, 0}_{m}, \infty_{t-1}].$$

If $d_\ell + 1 < c_\ell$, then the m.l.i. is 0.

Otherwise, we have $d_\ell + 1 = c_\ell$. If $\ell = t$, then by the algorithm, we perform a carry to the next neighbor block. By the proof of Lemma 3.4, this process of carrying to the next neighbor block terminates without changing the s.l.i. In this case in particular, we have that the first $m + t > t$ values in our representation (from right to left) must all be 0. As such, we cannot have a violation prefix placed right to the left of the ∞ places because the length of any violation prefix is less than t. Therefore, the m.l.i. must be equal to 0 in this case.

This implies that the only subcase in which the m.l.i. is not 0 is when $c_{\ell+1} = \dots = c_{\ell+m} = 0 \leq 1 = c_t$. As such, we have $(\ell + m) < t$ and the violation block $[c_1, \dots, c_\ell, 0, \dots, 0]$. Notice that the length of the violation prefix has now increased to $m + \ell - 1$. We can repeat the algorithm and all the previous arguments to have that anytime the m.l.i. does not go to 0, the length of the violation prefix increases. However, the length of the violation prefix is bounded above by $t - 1$. As such, at some point the m.l.i. must become zero. $\square$

Remark 3.7. *We note that the proof of Theorem 3.6 on the termination of our algorithm gives an alternative proof of the generalized Zeckendorf theorem for positive linear recurrences (Theorem 2.7).*

4. Weakly Decreasing Signature Implies Summand Minimality

In this section, we prove one direction of Theorem 1.1, namely if a recurrence sequence has a weakly decreasing signature, then the recurrence sequence is summand minimal. In particular, the idea is to show that for linear recurrences with weakly decreasing signatures, whenever we perform one "complete" step of the algorithm (where a "complete" step refers to one iteration of the first "while" statement in Algorithm 3.1), the number of summands monotonically decreases. As such, the gzd is summand minimal.

Notice that by Remark 2.12, carrying decreases the number of summands while borrowing increases it. In our algorithm, the only time we need to borrow is when the c.o.i. exists, and by the proof of Lemma 3.5, we need to keep borrowing until we are able to carry or until we borrow from the violation enough times that it is no longer a violation. The key observation in the case of weakly decreasing signatures is that anytime we need to borrow, we are able to carry immediately, which ensures that the total number of summands is weakly decreasing. This is well illustrated by the following example.

Example 4.1. *Consider the signature $\sigma = (c_1, c_2, c_3)$ with $c_1 \geq c_2 \geq c_3 \geq 1$ and the representation $[c_1 + 1, 0, 0, \infty]$. Here, both the m.l.i. and s.l.i. are equal to 3, the violation index and the r.e.i. are equal to 2 and the c.o.i. equals 1. We borrow from the r.e.i. to get*

$$[c_1, c_1, c_2, \infty].$$

Notice that we can carry now since $c_1 \geq c_2$ and $c_2 \geq c_3$ because the signature is weakly decreasing. Hence we carry right away to get $[1, 0, c_1 - c_2, c_2 - c_3, \infty]$, which is a valid gzd with the total number of summands equal to $c_1 + 1 - c_3 < c_1 + 1$, which is the total number of summands we had at the start.

We formalize the above discussion in the following proposition and its proof.

Proposition 4.2. *If σ is weakly decreasing, then the gzd is summand minimal for all n.*

Proof. Let us consider the signature $(c_1, \dots, c_t)$ with $c_1 \geq c_2 \geq \cdots \geq c_t \geq 1$ and a representation of the form

$$r = [\dots, [0], [B_1], \dots, [B_j], c_1, \dots, c_m, v, r_{s-m-2}, r_{s-m-3}, \dots, r_0, \infty_{t-1}],$$

where v is the violation and each $[B_i]$ is a valid block with $[B_1] \neq [0]$.

First, recall that every time we need to carry in the algorithm, the number of summands decreases. As such, it suffices to consider the cases where we need to borrow. The only time this is required in the algorithm is when the c.o.i. exists. Without loss of generality, suppose that $s - \ell$ is the c.o.i. and $s - e$ is the r.e.i. Then the string of length t to the right of the m.l.i. in our representation looks like:

$$[c_1, \dots, c_m, v, r_{s-m-2}, \dots, r_{s-e}, c_{e+1}, \dots, c_{\ell-1}, r_{s-\ell}, \dots, r_{s-t}].$$

After borrowing from the r.e.i., we obtain

$$[c_1, \dots, c_m, v, r_{s-m-2}, \dots, r_{s-e} - 1, c_{e+1} + c_1, \dots, c_{\ell-1} + c_{\ell-e+1}, r_{s-\ell} + c_{\ell-e}, \dots, r_{s-t} + c_{s-t-e}].$$

Notice that every entry with index at least $s - \ell$ either stays the same or increases with the exception of the entry at the r.e.i. However, we have that $r_{s-e} - 1 \geq c_e$ and hence for all $1 \leq i \leq \ell$, we have that $r_{s-i} \geq c_i$. Furthermore, for all $e < j \leq t$, we now have that the entry in the index $s - j$ is equal to $r_{s-i} + c_{s-i-e} \geq c_{s-i-e} \geq c_{s-i}$ because our signature is weakly decreasing. As such, we can carry right away.

Therefore, every time we have a violation, we need to borrow at most once before we can carry. Furthermore, by the algorithm, we always borrow at the r.e.i., which is at most the violation index, and carry to the m.l.i., which is strictly larger than the violation index. As such, by Remark 2.12, any borrow, pure or impure, can be matched with another carry in the same step that results in a non-positive net change of summands. Therefore, in moving from any representation to the gzd, after one complete step of the algorithm, the number of summands never increases. This is equivalent to the statement that gzd is summand minimal. □

Note that this proposition proves the forward direction of Theorem 1.1. We now turn our attention to the other direction.

5. SUMMAND MINIMALITY IMPLIES WEAKLY DECREASING SIGNATURE

In order to prove the other direction of Theorem 1.1 we will prove the contrapositive. That is, for every non-weakly decreasing signature, we prove the existence of a non-gzd representation with fewer summands than the gzd. Some case work is required. Specifically, there are four broad categories of cases, three of which handle linear recurrences with signatures starting with $c_1 > 1$ and the last of which deals with linear recurrences whose signatures start with $c_1 = 1$.

In particular, the case where $c_1 > 1$ is not the largest term in the signature (i.e., there exists $c_i > c_1$) can be proven very cleanly and is presented in subsection §5.1. However, the technique used in this case does not generalize well and is only helpful in certain subcases; as such, we motivate a new, different approach to prove the other two cases where $c_1 > 1$ in subsection 5.2 and then proceed to prove those two cases in subsections §5.3 and §5.4. Lastly, the case where $c_1 = 1$ requires a completely different method of proof and is detailed in subsection §5.4.

5.1. Case 1: $\exists\ \mathbf{c_i > c_1 \geq 2}$.

We first deal with the case where c_1 is not the largest term in the signature. Here, we find a form of representations that can be easily shown to always use fewer summands than the gzd. We state this formally.

Proposition 5.1. *Suppose $\sigma = (c_1, \dots, c_t)$. Suppose there exists an i such that $c_i > c_1 \geq 2$. Then the representation $[c_1, \dots, c_{i-2}, c_{i-1} + 1, 0]$ has fewer summands than the gzd.*

Proof. First notice that this representation is not the gzd because we have a violation at index 1. We borrow at index 1:

	...	**i**	...	**2**	**1**	**0**	**−1**	...
	...	c_1	...	c_{i-2}	$c_{i-1}+1$	0	∞	...
Borrow from 1					-1	c_1	c_2	...
	...	c_1	...	c_{i-2}	c_{i-1}	c_1	∞	...

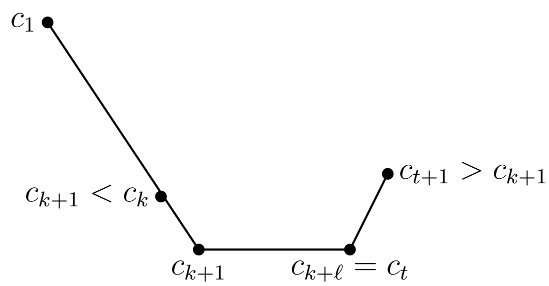
Notice that we have reached the gzd because $c_1 < c_i$. The change in the number of summands is $\Delta S = -1 + c_1 \geq 1$. Therefore, the gzd has more summands than the starting representation. □

There are several items to note about this proof. First off, notice that this technique does not work when $c_1 = 1$. In fact, as we have mentioned before, $c_1 = 1$ will be a case of its own, and will require a completely different approach (see section 5.5). Furthermore, notice that if $c_i \leq c_1$, then after the borrow, we are in a position to carry, in which case the number of summands will decrease (though we may not immediately have the gzd). Thus, we must seek a different technique to handle $c_i \leq c_1$ for all i. However, as we shall see in subsection §5.4, an adaptation of the above technique will be useful in handling some of these cases.

5.2. The "cutting" technique.

In sections 5.2-5.4, we suppose that $c_1 \geq 2$ and $c_i \leq c_1$ for all i. We now develop some new terminology, the **cutting** technique, which will be relevant to handling these cases, and detail the motivation of our approach.

First, since we know that the signature is not weakly decreasing, there exists a first point (from left to right) of increase. Let t be this position, i.e., the smallest index such that $c_t < c_{t+1}$. Notice that this implies that $c_1 \geq c_2 \geq \cdots \geq c_t$. Let $k < t$ be the largest index such that $c_k > c_t$. Let $\ell = t - k$. Notice that this implies that $c_1 \geq c_2 \geq \cdots \geq c_k > c_{k+1} = c_{k+2} = \cdots = c_{k+\ell}$ and $c_{k+\ell} = c_t < c_{t+1}$. With this terminology, we have the following visualization of the first part of our signature (up to the first increasing index):



We now describe the technique we call **cutting**. Heuristically, cutting at i refers to "placing the infinity places" at the position i places to the right of some fixed position. This is best demonstrated by example. Suppose that we have the representation $[c_1+1, \underbrace{0,\dots,0}_{k+\ell-1}, \infty_{t-1}]$. Then, if we borrow from the $k+\ell-1$ index and carry to the $k+\ell$ index, we have

	$\mathbf{k+\ell}$	$\mathbf{k+\ell-1}$	$\mathbf{k+\ell-2}$	$\mathbf{k+\ell-3}$	$\dots$	$\mathbf{0}$	$\mathbf{-1}$	$\dots$
		c_1+1					∞	$\dots$
Borrow from $k+\ell-1$		-1	c_1	c_2	$\dots$	$c_{k+\ell-1}$	$c_{k+\ell}$	
Carry to $k+\ell$	1	$-c_1$	$-c_2$	$-c_3$	$\dots$	$-c_{k+\ell}$	$-c_{t+1}$	
	1	0	c_1-c_2	c_2-c_3	$\dots$	$c_{k+\ell-1}-c_{k+\ell}=0$	∞	$\dots$

Notice that we have now reached the gzd after one borrow and one carry, similar to Example 4.1. On the other hand, if we "shift" our representation out by 1, i.e., consider the representation $[c_1+1, \underbrace{0,\dots,0}_{k+\ell}, \infty]$ and apply the shifted borrow and carry, i.e., borrow from $k+\ell$ and carry to $k+\ell+1$, we then achieve

	$\mathbf{k+\ell+1}$	$\mathbf{k+\ell}$	$\mathbf{k+\ell-1}$	$\mathbf{k+\ell-2}$	$\dots$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{-1}$	$\dots$
		c_1+1						∞	$\dots$
Borrow from $k+\ell$		-1	c_1	c_2	$\dots$	$c_{k+\ell-1}$	$c_{k+\ell}$	c_{t+1}	
Carry to $k+\ell+1$	1	$-c_1$	$-c_2$	$-c_3$	$\dots$	$-c_{k+\ell}$	$-c_{t+1}$	$-c_{t+2}$	
	1	0	c_1-c_2	c_2-c_3	$\dots$	$c_{k+\ell-1}-c_{k+\ell}=0$	$c_{k+\ell}-c_{t+1}$	∞	$\dots$

Note that the two representations are the same all the way down to index 1 but differ at index 0. Furthermore, in this shifted case, we have not found a valid gzd since $c_{k+\ell} - c_{t+1} < 0$ (because $t+1$ is the point of increase in the signature). We thus make the following observations.

Given a finite string of numbers, we can create "shifted" representations of the same form but with different length (note that each "shifted" representation also corresponds to a different integer) by positioning the string relative to the ∞ places, possibly with added zeros between the string and the ∞ places or with the ∞ places absorbing a part of the string. If we align any two "shifted" representations of the same form on the left and apply the same borrow or carry action (after shifting appropriately), reading from left to right, the two resulting representations will be the same up to the first ∞ place. In this sense, one "shifted" representation may reach the gzd more quickly than another.

This is of interest to us as we want to keep track of the number of borrows and carries required to move from a representation to the gzd in order to compute the net change in the number of summands. Furthermore, it suggests that the position of our string of numbers in relation to the ∞ places in the representation is important, and thus induces the following approach to complete the proof of Theorem 1.1: in order to construct an example where the number of summands of the gzd is not minimal, one can choose a string of numbers and place them appropriately in relation to the ∞ places so that, in moving towards the gzd, the number of summands accumulated from the borrows is strictly larger than the number of summands absorbed by the carries. However, since any two "shifted" representations are the same up to some point under the same action of borrowing and carrying, a better way to think about this strategy is to extend the string of numbers bi-infinitely by adding zeros on both sides. We then perform the necessary borrows and carries to move towards the gzd and choose an appropriate position to put ∞ places. This is the heart of the "cutting" technique.

The above examples illustrate another important observation: the ∞ places must be placed in a way such that we can detect the non-weakly decreasing property of the signature. In the first example, c_{t+1}, which is the first entry (reading from left to right) exhibiting the non-weakly decreasing property of the signature, is absorbed into the ∞ places, and as such, with only one borrow, we can carry right away and arrive at the gzd (similar to the case of weakly decreasing signature). However, in the second case, this index is not absorbed in the ∞ places when we perform the carry, and hence we see that we have a negative number in the resulting representation, which we must fix by at least one more borrow. Lastly, this also suggests that $c_1 + 1$ is a good string to use because it exhibits the need for more than one borrow before a carry can be made in order for the resulting representation to be non-negative.

As it turns out in the proof that follows, $c_1 + 1$ is the right string to use, and, in most cases, putting the ∞ places $k + \ell$ places away from it (so we "detect" c_{t+1}, as in the second example) is the right choice. However, there are some edge cases that require further placement, which we detail in appendices.

Per the above discussion, in certain cases it makes sense to deviate from the algorithm and to carry right after one borrow, even if this introduces negative entries. We can then consider the number of borrows needed to "fix" the negative entries because borrowing will only increase the number of summands. We use this idea repeatedly in the proofs presented in sections 5.3 and 5.4, hence we state it formally in the following lemma.

Lemma 5.2. *Consider a representation that is semi-legal up to q, with some non-zero value at an index at or beyond q and a negative value at index $q-1$ with absolute value no more than c_1. Then after a sequence of borrows, the m.l.i is at most $q-1$.*

Proof. This proof is very similar to the proof of Lemma 3.5. Essentially, we can always borrow from the positive entry closest to the left of the negative entry (analogous to borrowing from the r.e.i. to fix the c.o.i.) until the negative entry in question becomes non-negative. For details, see Appendix A. □

Lastly, to facilitate the flow of the proof, we formally define "cutting" and the change in number of summands.

Definition 5.3. *Define the action of cutting at the i^{th} column to be putting the first term in the representation at the i^{th} position to the left of the $\infty, \ldots, \infty$. In other words, the changes in the $(i+1)^{st}$ column onward when borrowing and carrying are ignored.*

Definition 5.4. *Let ΔS be the difference in the number of summands between the starting representation and the resulting representation after a series of borrows and carries. Then ΔS is called the* ***net change of summands****.*

5.3. Case 2: $c_{t+1} < c_1$ and $c_1 \geq 2$.

Proposition 5.5. *Given a non-weakly decreasing signature with $c_{t+1} < c_1$ and $c_1 \geq 2$, there exists a shifted representation of the string $c_1 + 1$ such that the gzd is not summand minimal.*

To prove this, we use the "cutting" technique outlined in the previous subsection. First, we present Table 4, which illustrates the two borrows and one carry needed to turn any representation of the form $[c_1 + 1, 0, \ldots, 0]$ into the gzd. In fact, the two borrows are required by the algorithm; meanwhile, we perform the carry regardless of whether the algorithm calls for it and try to "fix" any negative entries that arise by using Lemma 5.2. As discussed in the last subsection on the cutting technique, we index the columns of the table from left to right, so that the position of $c_1 + 1$ is 0, and try to find an appropriate column to the right of the $c_1 + 1$ to place the ∞'s (i.e., "cut").

$\mathbf{-1}$	$\mathbf{0}$	$\mathbf{1}$	$\ldots$	$\mathbf{k-1}$	$\mathbf{k}$	$\mathbf{k+1}$	$\ldots$	$\mathbf{k+\ell-1}$	$\mathbf{k+\ell}$	$\mathbf{t+1}$	$\mathbf{t+2}$	$\mathbf{t+3}$	$\ldots$
	c_1+1												
	-1	c_1	$\ldots$	c_{k-1}	c_k	c_{k+1}	$\ldots$	$c_{k+\ell-1}$	$c_{k+\ell}$	c_{t+1}	c_{t+2}	c_{t+3}	$\ldots$
					-1	c_1	$\ldots$	$c_{\ell-1}$	c_ℓ	$c_{\ell+1}$	$c_{\ell+2}$	$c_{\ell+3}$	$\ldots$
1	$-c_1$	$-c_2$	$\ldots$	$-c_k$	$-c_{k+1}$	$-c_{k+2}$	$\ldots$	$-c_{k+\ell}$	$-c_{t+1}$	$-c_{t+2}$	$-c_{t+3}$	$-c_{t+4}$	$\ldots$
1	0	c_1-c_2	$\ldots$	$c_{k-1}-c_k$	$c_k-c_{k+1}-1$	c_1	$\ldots$	$c_{\ell-1}$	$c_{k+\ell}+c_\ell-c_{t+1}$	$-$	$-$	$-$	$\ldots$

TABLE 4. Base case table

If we cut at the $(k+\ell)^{\text{th}}$ column, then the net change in the number of summands is

$$\Delta S \;=\; -c_{t+1} - 1 + c_1 + \cdots + c_\ell.$$

Subcase (i): $c_{t+1} < c_1 - 1$.

Since $c_{t+1} < c_1 - 1$, we have $\Delta S \geq 1$. If we have $c_{k+\ell} + c_\ell - c_{t+1} \geq 0$, then we have reached the gzd with at least 1 more summand since $c_{k+\ell} + c_\ell - c_{t+1} \leq c_\ell$ (as $c_{k+\ell} < c_{t+1}$). Otherwise, the column is negative and we do the process of borrowings as detailed in Lemma 5.2 and arrive at the gzd with even more summands (since borrowing only increases the number of summands).

Subcase (ii): $c_{t+1} = c_1 - 1$**.**

Now suppose $c_{t+1} = c_1 - 1$. Cutting at the $(k+\ell)^{\text{th}}$ column gives the net summand change $\Delta S = c_2 + \cdots + c_\ell$.

Suppose $\ell \geq 2$. If $c_2 \geq 1$, then $\Delta S \geq 1$. Arguing as in the case above, whether or not the $(k+\ell)^{\text{th}}$ column is negative, we can still cut there and get that the gzd is not summand minimal.

If $c_2 = 0$, then $k = 2$ and so we have $c_3, \ldots, c_{k+\ell} = 0$. Hence the value at the $(k+\ell)^{\text{th}}$ column is exactly $-c_{t+1} = -c_1 + 1 < 0$ since $c_1 \geq 2$. Again, by Lemma 5.2, we can borrow repeatedly to arrive at the gzd, with the least possible increase in summands from borrowing being $-1 + c_1 \geq 1$.

We are now left with the case where $c_{t+1} = c_1 - 1$ and $\ell = 1$. This particular case requires cutting further right and we now look at the $(t+1)^{\text{st}}$ column as well. Notice that everything up to the k^{th} column is valid. As such, for brevity, we concentrate on the $(k+\ell) = (k+1)^{\text{st}}$ column and $(t+1)^{\text{st}}$ column from Table 4.

$\mathbf{k+1}$	$\mathbf{t+1}$
c_{k+1}	$c_1 - 1$
c_1	c_2
$-(c_1 - 1)$	$-c_{t+2}$
$c_{k+1} + 1$	$c_1 - 1 + c_2 - c_{t+2}$

If $c_1 - 1 \geq c_1 - 1 + c_2 - c_{t+2} > 0$, then cutting at the $(t+1)^{\text{st}}$ column gives us a valid gzd with $\Delta S = c_1 - 1 + c_2 - c_{t+2} > 0$.

If $c_1-1+c_2-c_{t+2} \geq c_1$, then we carry once to the $(k+1)^{\text{st}}$ column and $c_{k+1}+2 \leq c_{t+1}+1 = c_1$ in the $(k+1)^{\text{st}}$ column and $c_2 - c_{t+2} - 1 < c_2$ in the $(t+1)^{\text{st}}$ column. Hence, we have reached a valid gzd with

$$\Delta S = c_1 - 1 + c_2 - c_{t+2} + 1 - c_1 = c_2 - c_{t+2} \geq 1.$$

Lastly, the cases when $c_1 - 1 + c_2 - c_{t+2} < 0$ and $c_1 - 1 + c_2 - c_{t+2} = 0$ are dealt with in Appendix B and C, respectively. The proofs follow similarly, but there are a few more edge cases which require tedious casework.

5.4. Case 3: $c_{t+1} = c_1$ and $c_1 \geq 2$.

Proposition 5.6. *Given a non-weakly decreasing signature with $c_{t+1} = c_1$ and $c_1 \geq 2$, then there exists a number for which the gzd is non summand minimal.*

Notice this is the last case to cover for signatures starting with $c_1 \geq 2$. Again, the general approach is to use the cutting technique that is detailed in §5.2. We first handle the case where $\ell \geq 3$. The case where $\ell = 2$ is handled in Appendix D (for this case, no new ideas are required, and the analysis ultimately comes down to casework). The majority of the $\ell = 1$ case is handled here, except for one tedious subcase which is relegated to Appendix E. We note that in the case where $\ell = 1$, we employ multiple arguments that are inspired by the technique that was used to handle the straightforward case of §5.1 (where there exists $c_i > c_1$).

Subcase (i): $\ell \geq 3$

Our table looks like

$\mathbf{-1}$	$\mathbf{0}$	$\mathbf{1}$	$\ldots$	$\mathbf{k-1}$	$\mathbf{k}$	$\mathbf{k+1}$	$\ldots$	$\mathbf{k+\ell-1}$	$\mathbf{k+\ell}$	$\mathbf{t+1}$	$\mathbf{t+2}$	$\mathbf{t+3}$	$\ldots$
	c_1+1												
	-1	c_1	$\ldots$	c_{k-1}	c_k	c_{k+1}	$\ldots$	$c_{k+\ell-1}$	$c_{k+\ell}$	c_1	c_{t+2}	c_{t+3}	$\ldots$
					-1	c_1	$\ldots$	$c_{\ell-1}$	c_ℓ	$c_{\ell+1}$	$c_{\ell+2}$	$c_{\ell+3}$	$\ldots$
1	$-c_1$	$-c_2$	$\ldots$	$-c_k$	$-c_{k+1}$	$-c_{k+2}$	$\ldots$	$-c_{k+\ell}$	$-c_1$	$-c_{t+2}$	$-c_{t+3}$	$-c_{t+4}$	$\ldots$
1	0	c_1-c_2	$\ldots$	$c_{k-1}-c_k$	$c_k-c_{k+1}-1$	c_1	$\ldots$	$c_{\ell-1}$					

Suppose we cut at the $(k+\ell)^{\text{th}}$ column. Notice that columns -1 to k are allowable blocks of length 1. In columns $k+1$ to $k+\ell-1$ we get the prefix $[c_1, \ldots, c_{\ell-1}]$. We know that $c_{k+1} < c_1$, so $c_{k+\ell}+c_\ell-c_1 < c_\ell$, implying that the $(k+\ell)^{\text{th}}$ column is less than c_ℓ. If this column is non-negative, then cutting at the $(k+\ell)^{\text{th}}$ column results in the gzd. If this column is negative, then by Lemma 5.2, we know that we can borrow some number of times such that the resulting representation is the gzd.

Assuming we did not borrow any more (if we did borrow more, ΔS would be even larger), the net change in the number of summands is

$$\Delta S \ = \ -1 - c_1 + c_1 + c_2 + c_3 + \cdots + c_\ell \ = \ c_2 + \cdots + c_\ell - 1.$$

Therefore, if $-1 + c_2 + \cdots + c_\ell \geq 1$, or equivalently, $c_2 + \cdots + c_\ell \geq 2$, then the net change in the number of summands is positive. Suppose instead that $c_2 + \cdots + c_\ell \leq 1$. Since $c_2 \geq c_3 \geq \cdots \geq c_\ell$, if $c_3 \geq 1$, then $c_2 + \cdots + c_\ell \geq 2$. Therefore, $c_3 = 0$ (implying that $c_3 = c_4 = \cdots = c_{k+\ell} = 0$). Thus, the only way that $c_2 + \cdots + c_\ell \leq 1$ is if $c_2 = 0$ or $c_2 = 1$.

Suppose $c_2 = 1$. The $k+\ell$ column will then contain $c_{k+1} + c_\ell - c_1 = 0 + 0 - c_1 = -c_1$, so we will need to borrow at least once more, in which case we get

$$\Delta S \ \geq \ c_2 + \cdots + c_\ell - 1 - 1 + c_1.$$

Since $c_1 - 1 \geq 1$ and $c_2 + \cdots + c_\ell - 1 = 0$, ΔS is positive. Thus the case $c_2 = 1$ is handled.

Now suppose $c_2 = 0$. This implies that $k = 1$, which in turn implies that columns $k+2$ through $k + \ell - 1$ (of which there are at least 1) are 0. Therefore, in order to make the $(k+\ell)^{\text{th}}$ column positive, we need to borrow at least twice. Thus,

$$\Delta S \geq (c_2 + \cdots + c_\ell - 1) + (-1 + c_1) + (-1 + c_1) = 2c_1 - 3 \geq 1.$$

Again, in this case, $\Delta S > 0$.

Subcase (ii): $\ell = 2$: The method of proof follows similarly as above for this case but requires tedious casework, so we refer the interested reader to Appendix D.

Subcase (iii): $\ell = 1$:

Now suppose that $\ell = 1$ and again $c_{t+1} = c_1$. Suppose that $c_2 < c_{t+2}$. Then we use a trick that is similar to the one used in section 5.1:

$\mathbf{1}$	$\mathbf{2}$	$\dots$	$\mathbf{k}$	$\mathbf{k+1}$	$\mathbf{t+1}$	$\mathbf{t+2}$	$\dots$
c_1	c_2	$\dots$	c_k	$c_{k+1}+1$ -1	c_1	c_2	$\dots$
c_1	c_2	$\dots$	c_k	c_{k+1}	c_1	c_2	$\dots$

Since $c_2 < c_{t+2}$, $[c_1, c_2, \dots, c_{k+1}, c_1, c_2]$ is an allowable block, so we can cut at column $t+2$ in which case the number of summands has changed by $-1 + c_1 + c_2 \geq 1$.

Now suppose that $c_2 > c_{t+2}$. This implies that $c_2 - c_{t+2} \geq 1$. Our table looks like

$\mathbf{-1}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{2}$	$\dots$	$\mathbf{k-1}$	$\mathbf{k}$	$\mathbf{k+1}$	$\mathbf{t+1}$	$\mathbf{t+2}$	$\mathbf{t+3}$	$\dots$
	c_1+1										
	-1	c_1	c_2	$\dots$	c_{k-1}	c_k	c_{k+1}	c_1	c_{t+2}	c_{t+3}	$\dots$
						-1	c_1	c_2	c_3	c_4	$\dots$
1	$-c_1$	$-c_2$	$-c_3$	$\dots$	$-c_k$	$-c_{k+1}$	$-c_1$	$-c_{t+2}$	$-c_{t+3}$	$-c_{t+4}$	$\dots$
1	0	$c_1 - c_2$	$c_2 - c_3$	$\dots$	$c_{k-1} - c_k$	$c_k - c_{k+1} - 1$	c_{k+1}	$c_1 + (c_2 - c_{t+2})$			

Suppose we cut at the $(t+1)^{\text{st}}$ column. Since $c_2 - c_{t+2} > 0$, we would need to carry. In which case, we get

$\mathbf{-1}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{2}$	$\dots$	$\mathbf{k-1}$	$\mathbf{k}$	$\mathbf{k+1}$	$\mathbf{t+1}$	$\mathbf{t+2}$	$\mathbf{t+3}$	$\dots$
1	0	$c_1 - c_2$	$c_2 - c_3$	$\dots$	$c_{k-1} - c_k$	$c_k - c_{k+1} - 1$	c_{k+1}	$c_1 + (c_2 - c_{t+2})$			
							1	$-c_1$	$-c_2$	$-c_3$	$\dots$
1	0	$c_1 - c_2$	$c_2 - c_3$	$\dots$	$c_{k-1} - c_k$	$c_k - c_{k+1} - 1$	$c_{k+1} + 1$	$c_2 - c_{t+2}$			

The net change in summands is $\Delta S = c_2 - c_{t+2} \geq 1$. Therefore, if cutting at column $t+1$ results in a valid gzd, we are done. We handle the case when we have not yet reached the gzd, which requires some tedious edge cases, in Appendix E.

We have now handled all cases when $\ell = 1$ and $c_2 \neq c_{t+2}$. Let us now consider when $c_2 = c_{t+2}$. When $c_3 < c_{t+3}$, we do a similar trick as before:

$\mathbf{1}$	$\mathbf{2}$	$\dots$	$\mathbf{k}$	$\mathbf{k+1}$	$\mathbf{t+1}$	$\mathbf{t+2}$	$\mathbf{t+3}$	$\dots$
c_1	c_2	$\dots$	c_k	$c_{k+1}+1$ -1	c_1	c_2	c_3	$\dots$
c_1	c_2	$\dots$	c_k	c_{k+1}	c_1	c_2	c_3	$\dots$

If we cut at the $(t+3)^{\text{rd}}$ column, we have a valid gzd and the number of summands has increased by $-1 + c_1 + c_2 + c_3 \geq c_1 - 1 \geq 1$.

Now suppose that $c_{t+3} < c_3$, so that $-c_{t+3} + c_3 \geq 1$. We need to carry at the $k+1$ column:

-1	0	1	2	$\dots$	$k-1$	k	$k+1$	$t+1$	$t+2$	$t+3$	$\dots$
1	0	$c_1 - c_2$	$c_2 - c_3$	$\dots$	$c_{k-1} - c_k$	$c_k - c_{k+1} - 1$	c_{k+1}	$c_1 + (c_2 - c_{t+2})$			
							1	$-c_1$	$-c_2$	$-c_3$	$\dots$
1	0	$c_1 - c_2$	$c_2 - c_3$	$\dots$	$c_{k-1} - c_k$	$c_k - c_{k+1} - 1$	$c_{k+1} + 1$	0	$c_3 - c_{t+3}$		

The net change in summands is $\Delta S = -c_{t+3} + c_3 \geq 1$, so if the resulting representation is a valid gzd, the net change in summands is positive. Suppose $c_{k+1} + 1 \leq c_1 - 1$. If $c_3 - c_{t+3} < c_1$, then we have a valid gzd. Else, suppose $c_3 - c_{t+3} = c_1$ meaning that $c_3 = c_1$ and $c_{t+3} = 0$. We then carry, in which case we get that the last 3 columns are $[c_{k+1} + 1, 1, 0]$ which must be a valid gzd. The net change in summands is $-c_{t+3} + c_3 + 1 - c_1 = 1$, so we're done.

Now suppose that $c_{k+1} + 1 = c_1$. If $c_3 - c_{t+3} \neq c_1$, then we have a valid gzd since $c_2 \geq c_{k+1} = c_1 - 1 \geq 1$, meaning that columns $k+1$ and $t+1$ form an allowable block. If $c_3 - c_{t+3} = c_1$, then $c_3 = c_1$, implying that either $c_{k+1} = c_2 = c_1 - 1$, in which case, after the carry, columns $k+1$, $t+1$, and $t+2$ must form an allowable block, or else $c_2 = c_1$, in which case columns $k+1$ and $t+1$ form an allowable block and column $t+2$ is 0. Thus in all cases, in moving to the gzd, the number of summands increases.

Thus, the only unhandled case is $c_3 = c_{t+3}$. If $c_4 < c_{t+4}$, we use the same trick as before. If $c_4 > c_{t+4}$, then after the carry, the net change in summands is $\Delta S = -c_{t+4} + c_4 \geq 1$, so we have more summands. If $c_4 - c_{t+4} \neq c_1$, then we must have a valid gzd because either $c_{k+1} + 1 < c_1$, in which case we are clearly done, or $c_{k+1} + 1 = c_1$ implying that $c_{k+1} = c_1 - 1 \geq 1$ so $c_2 \geq c_{k+1} = 1$. There will be a 0 in the $(t+1)^{\text{st}}$ column, so columns $k+1$ and $t+1$ will form an allowable block. If we carry ($c_4 = c_1$, $c_{t+4} = 0$), then we will be left with a valid gzd. The change in the number of summands is

$$\Delta S = -c_{t+4} + c_4 + 1 - c_1 = 1$$

and we're done. Thus the only remaining case is $c_4 = c_{t+4}$.

From the above arguments it is clear that if for any m, $c_m \neq c_{t+m}$, we can find a representation with fewer summands than the gzd. Since the length of the signature is finite, and the last element of the signature is non-zero, there must be some point where $c_m \neq c_{t+m}$. Therefore, the case of $\ell = 1$ is complete.

5.5. **Case 4: $c_1 = 1$.** Lastly, we give the proof of the $\Longleftarrow$ direction of Theorem 1.1 for non-weakly decreasing signatures $(c_1, \dots, c_t)$ where $c_1 = 1$ and $c_t \neq 0$. In particular, we note that this is comprised of all non-negative signatures starting with 1 except for the ones of the form $(1, 1, \dots, 1)$. The techniques we've developed so far do not handle $c_1 = 1$ since we previously made repeated use of the inequality $c_1 - 1 \geq 1$.

Formally, we will prove the following proposition.

Proposition 5.7. *Given a non-weakly decreasing signature with $c_1 = 1$, then there exists n large enough such that the gzd of $2H_n$ has at least 3 summands.*

The proof of the above proposition is intricate and uses properties of the characteristic polynomial and the growth rate of a recurrence sequence. We start by giving the outline and key ideas. Using the fact that $c_1 = 1$, we know that if the gzd of $2H_n$ is summand minimal, it must have one of the forms $[1, 0, \dots, 0, 1, 0, \dots, 0]$ or $[1, 0, \dots, 0]$, where we do not yet know the lengths of the strings of zeros $[0, \dots, 0]$ in either form. By analyzing the growth rate of the sequence, we are able to establish that for large n, there are only three possible choices for the gzd: we can only have $2H_n = H_{n+r-s} + H_{n-s}$, $H_{n+r-s} + H_{n-s+1}$, or $2H_n = H_{n+r-s}$ (with fixed r, s), where the first two choices are of the form $[1, 0, \dots, 0, 1, 0, \dots, 0]$ and the last one is of the form $[1, 0, \dots, 0]$. Notice that each of the three previous possible relations corresponds to a characteristic polynomial: $x^r - 2x^s + 1$, $x^{r-1} - 2x^{s-1} + 1$ and $x^{r-s} - 2$, respectively. We use another growth rate argument in conjunction with some results on the factorization of these polynomials to show that for large n, if the gzd of $2H_n$ has at most two summands, it must always be of the same form.

The above implies that if the gzd is summand minimal, there must exist a truncated sequence of our original sequence whose minimal polynomial[2] divides one of the three polynomials $x^r - 2x^s + 1$, $x^{r-1} - 2x^{s-1} + 1$ or $x^{r-s} - 2$. We then establish that, given the ideal initial conditions, the minimal polynomial of any truncated sequence arising from our original recurrence sequence must be the characteristic polynomial of the linear recurrence. Lastly, we show that the characteristic polynomial associated to a linear recurrence with a non-weakly decreasing signature and $c_1 = 1$ does not divide $x^r - 2x^s + 1$, $x^{r-1} - 2x^{s-1} + 1$ or $x^{r-s} - 2$; the proof of this relies on the special form of these characteristic polynomials, the factorization forms of $x^r - 2x^s + 1$ and $x^{r-1} - 2x^{s-1} + 1$, and the irreducibility of $x^{r-s} - 2$.

The above arguments show that there must exist an n such that the gzd of $2H_n$ has at least 3 summands, completing our proof.

Remark 5.8. *Notice that this approach cannot be applied easily to the previous cases where $c_1 > 1$ because there are too many possible valid forms for the summand minimal gzd of $(c_1 + 1)H_n$ to take into consideration. Here, it is possible because we have exploited the key information on the specific value of $c_1 = 1$.*

We now proceed to fill in the details of the above proof sketch. First, we establish a few propositions showing how the growth rate of the sequences determines the form of the gzd.

Proposition 5.9. *Consider a linear recurrence with signature $(c_1, \dots, c_t)$ where $c_1 = 1$ and $c_t \neq 0$. Let β be the largest real root of the corresponding characteristic polynomial. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, if the gzd of $2H_n$ uses 2 or fewer summands, it must always be of one of the following forms: $H_{n+\ell_1}$, $H_{n+\ell_1} + H_{n-\ell_2}$ or $H_{n+\ell_1} + H_{n-\ell_2-1}$, where $\ell_1 = \lfloor \log_\beta 2 \rfloor$ and $\ell_2 = \lfloor \log_\beta (2 - \beta^{\ell_1})^{-1} \rfloor$.*

Proof. If the gzd of a linear recurrence with signature starting with $c_1 = 1$ is summand minimal, then the gzd of $2H_n$ must be of the form $H_{n+\ell_1(n)} + H_{n+\ell_2(n)}$ or $H_{n+\ell_3(n)}$, where ℓ_1, ℓ_2 and ℓ_3 are functions of n. We show in Lemma F.1 that as n approaches infinity, H_n is on the order of β^n, where β is the largest root of the characteristic polynomial corresponding to our sequence. From this, we are able to deduce that for large n, $\ell_3(n) = \ell_1(n) = \lfloor \log_\beta 2 \rfloor$ and $\ell_2(n) = \lfloor \log_\beta (2 - \beta^{\ell_1})^{-1} \rfloor$ or $\lceil \log_\beta (2 - \beta^{\ell_1})^{-1} \rceil$. For details, see Appendix F. □

For our purpose, Proposition 5.9 essentially says that for large n, the only three possible forms of the gzd of $2H_n$ with two or fewer summands are $H_{n+r-s} + H_{n-s}$, $H_{n+r-s} + H_{n-s+1}$, and H_{n+r-s}, for some fixed r, s.

Next, we develop the following proposition, which establishes a condition to rule out possible valid representations of $2H_n$ for large n.

Proposition 5.10. *Suppose the characteristic polynomial corresponding to a signature $(c_1, \dots, c_t)$ does not share a root with a polynomial of the form $x^r - 2x^s + 1$ (resp. $x^r - 2$). Then, for all $n \geq N$ where N is sufficiently large, $2H_n$ cannot be written as $H_{n+r-s} + H_{n-s}$ (resp. H_{n+r}).*

[2] We have a more in-depth, formal discussion on the distinction between characteristic polynomials and minimal polynomials of linear recurrences later in this section, after Remark 5.12.

Proof. Suppose that β is not a root of a polynomial of the forms $x^r - 2x^s + 1$ or $x^r - 2$. Since the argument for both is the same, we will only deal with $x^r - 2x^s + 1$.

We now show that for sufficiently large k, $2H_{k+s} \neq H_{k+r} + H_k$. By Lemma F.1, we have that $\lim_{n\to\infty} H_n = C\beta^n$ for some constant C. As such, for each n, we can write $H_n = C\beta^n(1+\varepsilon(n))$ where ε is a function from $\mathbb{N}$ to $\mathbb{R}$ such that as $N \to \infty$, $\varepsilon(N) \to 0$. We then have

$$H_{k+r} - 2H_{k+s} + H_k \;=\; C\beta^k\left[\beta^r(1+\varepsilon(k+r)) - 2\beta^s(1+\varepsilon(k+s)) + 1 + \varepsilon(k)\right]. \tag{5.1}$$

Since $\varepsilon(k+r)\beta^r - 2\varepsilon(k+s)\beta^s + \varepsilon(k)$ goes to 0 as $k \to \infty$, there exists K such that for all $k \geq K$, $|\varepsilon(k+r)\beta^r - \varepsilon(k+s)\beta^s + \varepsilon(k)| < |\beta^r - 2\beta^s + 1|$. Therefore, for all $k \geq K$, $2H_{k+s} \neq H_{k+r} + H_k$. $\square$

Given the above proposition, if we can prove that the three polynomials $x^r - 2x^s + 1$, $x^{r-1} - 2x^{s-1} + 1$ and $x^{r-s} - 2$ do not share any positive real root, then our characteristic polynomial can only share a root with at most one of those three polynomials. As such, Proposition 5.10 implies that if the gzd is summand minimal, then for large n, the form of the gzd of $2H_n$ must always be the same. Note that $x^{r-s} - 2$ is clearly irreducible for all r, and its only positive real root is $\sqrt[r-s]{2}$. The factorization of each of the other aforementioned polynomials is more complicated. We utilize a result from [**Schin**] (which can also be found in English in [**FS**]) on the factorization of such polynomials.

Theorem 5.11. *Let $g(r,s) = x^r - 2x^s + 1$ with $r, s \in \mathbb{N}$ and $r > s$. The polynomial*

$$h(r,s) \;=\; \frac{g(r,s)}{x^{\gcd(r,s)} - 1} \tag{5.2}$$

is irreducible for all r, s except for $(r,s) = (7k, 2k)$ or $(7k, 5k)$, in which case $h(r,s)$ factors into irreducible pieces

$$h(r,s) \;=\; (x^{3k} + x^{2k} - 1)(x^{3k} + x^k + 1) \quad \text{and} \quad (x^{3k} + x^{2k} + 1)(x^{3k} - x^k - 1), \tag{5.3}$$

respectively.

Note that in the case where $(r,s) \neq (7k,2k), (7k,5k)$, for $\gcd(r,s) = d$, we have

$$h(r,s) \;=\; \frac{g(r,s)}{x^d - 1} = x^{r-d} + x^{r-2d} + \cdots + x^s - x^{s-d} - \cdots - 1. \tag{5.4}$$

Theorem 5.11 then tells us that $g(r,s)$ factors into a cyclotomic piece $C(r,s) = x^{\gcd(r,s)} - 1$ and either one or two irreducible pieces. By the same theorem, one can easily see that the non-cyclotomic irreducible pieces of $g(r,s)$, $g(r-1,s-1)$ and $x^{r-s} - 2$ are distinct, so they cannot share any root. As such, $g(r,s)$, $g(r-1,s-1)$ and $x^{r-s} - 2$ cannot share any positive real root. Therefore, the characteristic polynomial of our linear recurrence can share a positive real root with at most one of the three polynomials.

Remark 5.12. *Given the special form of factorization of $g(r,s)$, one may suspect that our characteristic polynomial cannot share its largest positive real root with a polynomial of this form. However, this is not true as the following example demonstrates:*

$$h(6,5)(x^2+1) \;=\; \frac{g(6,5)}{x-1}(x^2+1) \;=\; x^7 - x^6 - 2x^4 - 2x^3 - 2x^2 - x - 1,$$

which shows that $g(6,5)/(x-1)$ divides the characteristic polynomial of a linear recurrence with signature $(1,0,2,2,2,2,1,1)$.

This, together with the discussion above on Proposition 5.10, implies that if the gzd of our sequence is summand minimal, then there exists a truncated sequence $\{H_n\}_{n\geq q}$ of the original sequence H_σ that always satisfies exactly one of the following relations: $2H_n = H_{n+r-s} + H_{n-s}$, $2H_n = H_{n+r-s} + H_{n-s+1}$ or $2H_n = H_{n+r-s}$. This motivates the following discussion on minimal polynomials of sequences.

Let $H = \{H_n\}_{n\in\mathbb{N}_0}$ be some sequence with terms H_i which satisfies some linear recurrence over $\mathbb{Z}$. Let $J(H)$ be the set of all polynomials which arise as characteristic polynomials of linear recurrences H satisfies. It is easy to see that $J(H)$ is closed under addition and mulitiplication by elements in $\mathbb{Q}[x]$, and therefore forms an ideal in $\mathbb{Q}[x]$. Since $\mathbb{Q}[x]$ is a principal ideal domain, there must be a unique monic minimal polynomial $s(x)$ generating $J(H)$ which corresponds to a minimal depth linear recurrence generating H. We call this polynomial the minimal polynomial of H. Now, let $\overline{H}_\ell = \{H_n\}_{n\geq\ell}$ be the sequence consisting of all terms in H with index at least ℓ. Together with the above discussion, this means that if our linear recurrence is summand minimal, then there exists an ℓ such that the minimal polynomial of $\overline{H}_\ell$ divides one of the polynomials $x^r - 2x^s + 1$, $x^{r-1} - 2x^{s-1} + 1$ and $x^{r-s} - 2$.

Suppose we fix a polynomial $p(x)$. Note that it is possible for the associated recurrence relation from p to satisfy two different sequences H, H' but for $p(x)$ to be the minimal polynomial for H but not for H'. We necessarily ask the following question: suppose we have a sequence H and suppose $s(x)$ is the minimal polynomial for H. Is $s(x)$ necessarily the minimal polynomial for $\overline{H}_\ell$ (where $\overline{H}_\ell$ is defined previously as a truncated sequence of S)?

Theorem 5.13. *Suppose $s(x)$ is the minimal polynomial for the sequence H. Then $s(x)$ is the minimal polynomial for all truncations of H.*

Before proving this theorem, we need the following definition and lemma.

Definition 5.14. *Let H be a sequence satisfying some linear recurrence. Let $H_{n,k}$ be*

$$H_{n,k} \;=\; \begin{bmatrix} H_n & H_{n+1} & \dots & H_{n+k} \\ H_{n+1} & H_{n+2} & \dots & H_{n+k+1} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n+k} & H_{n+k+1} & \dots & H_{n+2k} \end{bmatrix}. \tag{5.5}$$

We need the following result (Lemma 3 of [**Sal**]).

Lemma 5.15. *A sequence satisfying a linear recurrence satisfies some degree k linear recurrence if and only if $\det(H_{n,k}) = 0$ for all n.*

Proof of Theorem 5.13. Let H be some recurrence and let $s(x)$ be the minimal polynomial for H. Let $s(x) = x^t + c_1x^{t-1} + \cdots + c_t$. Note that $c_t \neq 0$. We shall show that $\det(H_{n,t-1}) \neq 0$ for all n. Let $D = \det(H_{1,t-1})$. In particular we will show that $|\det(H_{n,t-1})| = |c_t^n D|$.

We proceed by induction. Suppose we have

$$H_{n,t-1} \;=\; \begin{bmatrix} H_n & H_{n+1} & \dots & H_{n+t-1} \\ H_{n+1} & H_{n+2} & \dots & H_{n+t} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n+t-1} & H_{n+t} & \dots & H_{n+2t-2} \end{bmatrix}. \tag{5.6}$$

We index our columns starting from zero. Notice that the first through $(t-1)^{\text{th}}$ columns will appear as columns in $H_{n+1,t-1}$. Furthermore, notice that if we multiply the zeroth column by c_t

and add to it c_{t-1} times the first column plus c_{t-2} times the second column plus $\dots$ plus c_1 times the $t-1$ column, then the columns of the resulting matrix agree with the columns of $H_{n+1,t-1}$. The determinant has gone up by a factor of c_t. In order to move the resulting matrix to the form of $H_{n+1,t-1}$, we need to permute some columns, which may change the sign of the determinant, but not the magnitude.

We know that $\overline{H}_n$ is satisfied by $s(x)$, so $s(x)$ is in $J(\overline{H}_n)$. However, if $s(x)$ did not generate $J(\overline{H_n})$, then there must be some polynomial of lower degree in $J(\overline{H}_n)$. In particular, there must be some polynomial of degree $t-1$ in $J(\overline{H}_n)$. If this were so, then $\det(H_{m,t-1})$ would be zero for all $m \geq n$, which is a contradiction. Therefore, we must have that $s(x)$ is the lowest degree polynomial in $J(\overline{H}_n)$ for all n, and thus is the minimal polynomial for all $\overline{H}_n$. $\square$

Corollary 5.16. *Let H_σ be a sequence arising from a non-negative linear recurrence with characteristic polynomial $f(x)$ of degree t and ideal initial conditions $H_{-1} = \cdots = H_{-(t-1)} = 0$ and $H_0 = 1$. Then $f(x)$ is the minimal polynomial for all $\overline{H}_n$.*

Proof. By Theorem 5.13, it suffices to show that $\det H_{-(t-1),t-1} \neq 0$. It is immediate from writing out the matrix $H_{-(t-1),t-1}$ that by switching columns, we can make it lower triangular with diagonal entries all equal to 1, hence $\det H_{-(t-1),t-1} = 1$. $\square$

Corollary 5.16 implies that if H is a sequence whose minimal polynomial is $f(x)$, and $p(x)$ is some other polynomial, then, if $f \nmid p$, there does not exist a point n such that the recurrence relation arising from p is valid for all elements in $\overline{H}_n$.

With all that precedes, in order for us to prove that the gzd for positive linear recurrences with $c_1 = 1$ is not summand minimal it suffices to show that its characteristic polynomial f does not divide any polynomial of the form $x^r - 2x^s + 1$ or $x^r - 2$ for any r, s. Clearly, since $x^r - 2$ is irreducible and f is of the form $x^m - x^{m-1} - \cdots$ (because $c_1 = 1$), we must have $f \nmid (x^r - 2)$. It suffices then to show the following.

Proposition 5.17. *Let f be the characteristic polynomial for some non-negative linear recurrence. Then, $f \nmid g(r,s)$ for any r, s except for $r = s+1$, in which case $f = x^{r-1} - x^{r-2} - \cdots - 1$.*

Proof. Recall from Theorem 5.11 for $g(r,s) = x^r - 2x^s + 1$ that $g(r,s) = C(r,s)h(r,s)$ where $C(r,s) = x^{\gcd(r,s)-1}$ is cyclotomic and for $\gcd(r,s) = d$ and $(r,s) \neq (7k, 2k)$ or $(7k, 5k)$, we have

$$h(r,s) \;=\; \frac{g(r,s)}{x^d - 1} \;=\; x^{r-d} + x^{r-2d} + \cdots + x^s - x^{s-d} - \cdots - 1. \tag{5.7}$$

We thus know that any divisor of $g(r,s)$ must be made up either entirely of cyclotomic pieces, of the irreducible piece, or of some combination of the two. We shall show that all divisors of $g(r,s)$ either contain some positive coefficient other than the leading coefficient, or else have a zero as the coefficient of the second highest degree monomial. That is to say, no divisor of $g(r,s)$ is a valid characteristic polynomial for a non-negative linear recurrence with leading coefficient equal 1.

We know that 1 is not a root of $f(x)$, so when considering divisors of $g(r,s)$, we need only consider those not having $x-1$ as a divisor. All cyclotomic polynomials, $c(x)$, other than $x-1$, are self-reciprocal polynomials, which means $c(x) = x^n c(1/x)$. Furthermore, the product of any collection of such polynomials is of this form. Because of this, $f(x)$ cannot be a cyclotomic polynomial because that would imply that $c_t = 1$, which is a contradiction.

We now consider two mutually exclusive cases. In the first case, $\gcd(r,s) = 1$. In this case, the only cyclotomic part is $C(r,s) = (x-1)$. Thus, the only divisor of $g(r,s)$ we're interested in is

$I(r,s)$. If $r \neq s+1$, then this polynomial has a 1 as the x^{r-2} coefficient, so it is not equal to any valid f. Since it's irreducible, f cannot divide it.

Now suppose $\gcd(r,s) > 1$. Then the x^{r-d-1} term of $I(r,s)$ is 0, the coefficient of the x term in $I(r,s)$ is 0 and the constant coefficient is -1. Let $c(x)$ be some cyclotomic part of $g(r,s)$. Suppose $c(x) = x^\ell + c_1 x^{\ell-1} + \cdots + c_1 x + 1$. Therefore, since the coefficients of $c(x)$ are symmetric, in $c(x)I(r,s)$, the coefficient of $x^{r-d+\ell-1}$ is c_1. The coefficient of x is $-c_1$. If $c_1 = 0$, then $c(x)I(r,s)$ is not of a valid form to be the characteristic polynomial of a non-negative linear recurrence since then the first term of the signature would be zero. Otherwise, $c_1 \neq 0$, in which case either the coefficient of $x^{r-d+\ell-1}$ or x is positive, so this divisor cannot equal f.

Lastly we need to handle the case when $(r,s) = (7k, 2k)$ or $(r,s) = (7k, 5k)$. Suppose $(r,s) = (7k, 2k)$. Let $I_1 = x^{3k} + x^{2k} - 1$ and $I_2 = x^{3k} + x^k + 1$. By the same arguments above, any divisor of g containing both I_1 and I_2 cannot equal f. Suppose instead we only have I_1 and a cyclotomic piece. This case follows from the same arguments as before. Suppose we only have I_2 and a cyclotomic piece. Then, since the constant term in the cyclotomic piece must be 1, this divisor cannot equal f.

Now suppose that $(r,s) = (7k, 5k)$. Let $I_1 = x^{3k} + x^{2k} + 1$ and $I_2 = x^{3k} - x^k - 1$. By the same arguments as above, we can't have both I_1 and I_2. If we have just I_1 and a cyclotomic piece, then the constant term in the product would be 1, so this divisor cannot be f. If we just have I_2 and a cyclotomic piece, then by the same arguments as above, we would either have that the coefficient of $x^{3k+\ell-1}$ is 0, or one of the coefficients on $x^{3k+\ell-1}$ and x is positive, which in both cases implies this divisor is not equal to f. $\square$

Remark 5.18. *Recall from Remark 5.12 that the irreducible factor of a polynomial $g(r,s)$ can divide the characteristic polynomial of some positive linear recurrence with $c_1 = 1$. However, since our characteristic polynomial remains the minimal polynomial of any truncated sequence, this becomes unimportant. Hence, it only matters whether the characteristic polynomial can divide some polynomial of the form $g(r,s)$, which we have proven to be impossible above.*

Finally, we note that all of the above arguments combined give us the proof of Proposition 5.7. Furthermore, Propositions 5.1, 5.5, 5.6 and 5.7 complete the proof of Theorem 1.1.

APPENDIX A. PROOF OF LEMMA 5.2

Proof. Denote the entry at any index j by v_j. We have $v_{q-1} < 0$ and $|v_{q-1}| \leq c_1$. Since there is a positive entry at some index at least q, there exists $i = \min\{j \geq q \mid v_j > 0\}$. Call this the rightmost positive index in relation to $q-1$ and denote it by r.p.i. As such, $v_j = 0$ for all $i < j \leq q$. We now borrow from i. If v_{q-1} becomes non-negative, then the m.l.i. is at most $q-1$ since all blocks are valid up to i and from $i-1$ to $q-1$ we have a valid block of the form $[c_1, \ldots, c_{q-i}, c_{q-i+1} - v_{q-1}]$.

Otherwise, $c_{q-i+1} < v_{q-1}$ and we still have a negative value at index $q-1$. This also means that $i \neq q$ (otherwise, we have $c_1 < v_{q_1}$, contradiction). In this case, the value at index $i-1$ is now $c_1 > 0$. Thus, the r.p.i. has now decreased to be at most $i-1$. Repeating the process of borrowing from the r.p.i., we must have that at some step, the value at $q-1$ becomes non-negative and the m.l.i. decreases to be at most $q-1$ since we know the process definitely terminates when r.p.i.$= q$. $\square$

APPENDIX B. SUBCASE 1 OF $c_{t+1} < c_1$ AND $\ell = 1$

Appendices B, C, D and E deal with the remaining bad cases mentioned in Subsections 5.3 and 5.4. In each of these cases, although the general technique is the same, many edge cases arise and are treated separately. Since we have illustrated the technique in §5.1–§5.4, for brevity, here we only summarize where the cutting is placed and what the net change in the number of summands ΔS is for each edge case. A detailed proof can be found on the arXiv preprint at https://arxiv.org/abs/1606.08110.

First, recall that we were at the step where our representation is

$\mathbf{-1}$	$\mathbf{0}$	$\mathbf{1}$	$\ldots$	$\mathbf{k-1}$	$\mathbf{k}$	$\mathbf{k+1}$	$\mathbf{t+1}$
1	0	$c_1 - c_2$	$\ldots$	$c_{k-1} - c_k$	$c_k - c_{k+1} - 1$	$c_{k+1} + 1$	$c_1 - 1 + c_2 - c_{t+2}$

The only cases we did not resolve occur when $c_1 - 1 + c_2 - c_{t+2} \leq 0$. This appendix deals with the strict inequality, i.e., when $c_1 - 1 + c_2 - c_{t+2} < 0$.

A. If any one of the conditions $c_1 = 2$, $c_2 = 0$, and $c_1 = c_{t+2}$ does not hold, then cutting at the $(t+1)^{\text{st}}$ column, we reach the valid gzd with $\Delta S = -c_{t+2} - 1 + c_1 + c_2 - 1 + c_1 > 0$.

B. If $c_1 = 2$, $c_2 = 0$, and $c_1 = c_{t+2}$, then our signature starts with $(2, 0, 1, 2)$. If $c_{t+3} = 2$, then we cut at the $(t+2)^{\text{nd}}$ column and get the gzd $[1, 0, 1, 0, 1, 1]$ with 4 summands.

C. If $c_{t+3} = 0$, then our signature starts with $(2, 0, 1, 2, 0)$. After cutting at $(t+2)^{\text{nd}}$ column, we carry twice to get the gzd $[1, 0, 1, 1, 0, 1]$ with 4 summands.

D. Otherwise, we must have $c_{t+3} = 1$ and our signature starts with $(2, 0, 1, 2, 1)$. After cutting at $(t+3)^{\text{rd}}$ column, we carry twice to get the gzd $[1, 0, 1, 1, 0, 0, 3 - c_{t+4}]$. If $c_{t+4} = 2$, we have reached the gzd with 4 summands.

E. Otherwise, we must have $c_{t+4} = 0$ or 1. We carry once more to get $[1, 0, 1, 1, 0, 1, 1 - c_{t+4}]$ which is gzd with at least 4 summands.

Appendix C. Subcase 2 of $c_{t+1} < c_1$ and $\ell = 1$

Here, we deal with the final subcase of the $c_{t+1} < c_1$ case and $\ell = 1$, i.e., when $c_1 - 1 + c_2 - c_{t+2} = 0$. This can happen only when $c_{t+2} = c_1 - 1$ and $c_2 = 0$ or when $c_{t+2} = c_1$ and $c_2 = 1$.

Subcase (i): $c_{t+2} = c_1 - 1$ **and** $c_2 = 0$. In this case, it must be that $k = 1$ and our signature starts with $(c_1, 0, c_1 - 1, c_1 - 1)$. Our table is

$\mathbf{-1}$	$\mathbf{0}$	$\mathbf{k = 1}$	$\mathbf{k+1 = 2}$	$\mathbf{t+1}$	$\mathbf{t+2}$	$\mathbf{t+3}$
0	c_1	c_1	0	$c_1 - 1$	$c_1 - 1$	c_{t+3}
		-1	c_1	0	$c_1 - 1$	$c_1 - 1$
1	$-c_1$	-0	$-(c_1 - 1)$	$-(c_1 - 1)$	$-c_{t+3}$	$-c_{t+4}$
1	0	$c_1 - 1$	1	0	$2c_1 - 2 - c_{t+3}$	$c_{t+3} + c_1 - 1 - c_{t+4}$

A. If $2c_1 - 2 - c_{t+3} \geq c_1$, then after cutting at the $(t+2)^{\text{nd}}$ column, we can carry once and obtain the gzd $[1, 0, c_1 - 1, 1, 1, c_1 - 2 - c_{t+3}]$ with $\Delta S = c_1 - 1 - c_{t+3} \geq 1$.

B. If $2c_1 - 2 - c_{t+3} < c_1$, then cutting at the $(t+2)^{\text{nd}}$ column also results in a valid gzd with $\Delta S = 2c_1 - 2 - c_{t+3} \geq 0$. Hence, we get a non summand minimal gzd except for the case when $c_{t+3} = c_1$ and $c_1 = 2$.

C. In that case, our signature starts with $(2, 0, 1, 1, 2)$. For this, if $c_{t+3} = 2$, we cut at the $(t+3)^{\text{rd}}$ column and get the gzd $[1, 0, 1, 1, 0, 0, 1]$, which has 4 summands.

D. Otherwise, $c_{t+3} = 0$ or 1; we cut at the same place but carry once more to obtain the gzd $[1, 0, 1, 1, 0, 1, 1 - c_{t+3}]$, which has at least 4 summands.

Subcase (ii): $c_{t+2} = c_1$ **and** $c_2 = 1$. In this case, our table is:

$\mathbf{-1}$	$\mathbf{0}$	$\mathbf{1}$	$\dots$	$\mathbf{k-1}$	$\mathbf{k}$	$\mathbf{k+1}$	$\mathbf{t+1}$	$\mathbf{t+2}$	$\mathbf{t+3}$
	$c_1 + 1$								
	-1	c_1	$\dots$	c_{k-1}	c_k	c_{k+1}	$c_{t+1} = c_1 - 1$	$c_{t+2} = c_1$	c_{t+3}
					-1	c_1	$c_2 = 1$	c_3	c_4
1	$-c_1$	$-c_2 = -1$	$\dots$	$-c_k$	$-c_{k+1}$	$-c_{t+1} = -(c_1 - 1)$	$-c_{t+2} = -c_1$	$-c_{t+3}$	$-c_{t+4}$
1	0	$c_1 - 1$	$\dots$	$c_{k-1} - c_k$	$c_k - c_{k+1} - 1$	$c_{k+1} + 1$	0	$c_1 + c_3 - c_{t+3}$	$c_{t+3} + c_4 - c_{t+4}$

E. If $c_1 + c_3 - c_{t+3} \geq c_1$ we cut at the $(t+2)^{\text{nd}}$ column and then carry to get a gzd with $\Delta S = -c_{t+3} - 1 + c_1 + c_2 + c_3 + 1 - c_1 = c_2 + c_3 - c_{t+3} \geq c_2 = 1$.

F. Otherwise, since $c_{t+3} \leq c_1$, we have $0 \leq c_1 + c_3 - c_{t+3} \leq c_1 - 1$. If $c_1 + c_3 - c_{t+3} > 0$, we cut at the $(t+2)^{\text{nd}}$ column and reach the gzd with $\Delta S = -c_{t+3} - 1 + c_1 + c_2 + c_3 = c_1 + c_3 - c_{t+3} > 0$.

G. If $c_1 + c_3 - c_{t+3} = 0$, we must have that $c_3 = 0$ and $c_{t+3} = c_1$. This implies our signature starts with $(c_1, 1, 0, c_1 - 1, c_1, c_1)$ and hence $k = 2$. If $2c_1 - 1 - c_{t+4} < c_1$, cutting at the $(t+3)^{\text{rd}}$ column, we will reach the gzd with $\Delta S = 2c_1 - 1 - c_{t+4} \geq c_1 - 1 \geq 1$.

H. Otherwise, $2c_1 - 1 - c_{t+4} \geq c_1$. We cut at the $(t+3)^{\text{rd}}$ column and carry to get $[1, 0, c_1 - 1, 0, 1, 0, 1, c_1 - 1 - ct + 4]$, which is gzd with $\Delta S = 2c_1 - 1 - c_{t+4} - c_1 + 1 \geq c_1 - c_1 + 1 = 1$. This completes the proof of the whole subcase.

APPENDIX D. SUBCASE $\ell = 2$ IN CASE $c_{t+1} = c_1$

This is the subcase of $c_{t+1} = c_1$ where $\ell = 2$. Here, our table looks like:

$\mathbf{-1}$	$\mathbf{0}$	$\mathbf{1}$	$\dots$	$\mathbf{k-1}$	$\mathbf{k}$	$\mathbf{k+1}$	$\mathbf{k+2}$	$\mathbf{t+1}$	$\mathbf{t+2}$	$\mathbf{t+3}$	$\dots$
	c_1+1										
	-1	c_1	$\dots$	c_{k-1}	c_k	c_{k+1}	c_{k+2}	c_1	c_{t+2}	c_{t+3}	$\dots$
					-1	c_1	c_2	c_3	c_4	c_5	$\dots$
1	$-c_1$	$-c_2$	$\dots$	$-c_k$	$-c_{k+1}$	$-c_{k+2}$	$-c_1$	$-c_{t+2}$	$-c_{t+3}$	$-c_{t+4}$	$\dots$
1	0	c_1-c_2	$\dots$	$c_{k-1}-c_k$	$c_k-c_{k+1}-1$	c_1					

A. If $c_2 \geq 2$, we cut at the $(k+2)^{\text{nd}}$ column and get $\Delta S = -c_1 - 1 + c_1 + c_2 = c_2 - 1 \geq 1$. Thus the only unhandled cases are when $c_2 = 0$ or 1.

B. Suppose $c_2 = 0$. Then $k = 1$, and $c_2 = c_3 = 0$. Then, cutting at the $(k+2)^{\text{nd}}$ column and borrowing will give us a gzd with $\Delta S = (c_2 - 1) + (-1 + c_1) = c_1 - 2$. Therefore, unless $c_1 = 2$, we have $\Delta S \geq 1$.

C. Suppose $c_1 = 2$. Then our signature looks like $(2, 0, 0, 2, c_{t+2}, \dots)$. We know that $c_{t+2} \leq 2$. Suppose $c_{t+2} = 0$, then $\sigma = (2, 0, 0, 2, 0, c_{t+3}, \dots)$. In this case, we cut at the 4^{th} column and will obtain the gzd $[1, 0, 1, 1, 1, 0]$ with 4 summands.

D. Lastly, suppose $c_{t+2} = 1$ or 2. We can employ the following trick reminiscent of Proposition 5.1. Consider the representation $[2, 0, 1, 0, 0]$ and borrow once, we get $[2, 0, 0, 2, 0]$ which is the gzd with 4 summands. This completes the analysis on $c_2 = 0$.

E. Now suppose $c_2 = 1$. We know that $c_{k+2} \leq c_2 = 1$. Therefore $c_{k+2} = 0$ or 1.

Suppose $c_{k+2} = 0$. Then we cut at the $(k+2)^{\text{nd}}$ column, borrow once and get the gzd with $\Delta S \geq (c_2 - 1) + (-1 + c_1) = c_1 - 1 \geq 1$.

F. The only bad case for $c_2 = 1$ is when $c_{k+2} = 1$. This implies that $k = 1$. If $c_1 > 2$, then cutting at the $(k+2)^{\text{nd}}$ column and borrowing once will give us the gzd with $\Delta S \geq (c_2 - 1) + (c_1 - 1) = c_1 - 1 \geq 1$.

G. Therefore, the next bad edge case is $c_1 = 2$. Here we have $\sigma = (2, 1, 1, 2, c_{t+2}, \dots)$. If $c_{t+2} = 2$, then cutting at column 4, we get $[1, 0, 0, 2, 0, 1]$ which is a valid gzd with 4 summands.

H. Now, suppose $c_{t+2} = 0$ or 1. We carry to get $[1, 0, 0, 2, 1, 1 - c_{t+2}]$. Again, we cut at column 4. If $c_{t+2} = 1$, then the resulting representation is the gzd with 4 summands.

I. Thus, the only remaining case is $c_{t+2} = 0$. Then $\sigma = (2, 1, 1, 2, 0, c_{t+3})$. If we cut at column 5, the final representation is either gzd, or the 5^{th} column is negative, so we can arrive at the gzd by some sequence of borrows by Lemma 5.2. Here, $\Delta S \geq -c_{t+3} + 1 + 2 = 3 - c_{t+3} \geq 1$ so we are done. This concludes the analysis of $c_2 = 1$, which in turn concludes the analysis of $\ell = 2$.

Appendix E. A subcase of case $c_{t+1} = c_1$ and $\ell = 1$

In this appendix, we deal with the subcase of $c_{t+1} = c_1$ and $\ell = 1$ where cutting at the $(t+1)^{\text{th}}$ column does not result in the valid gzd. Recall that our table is:

$\mathbf{-1}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{2}$	$\dots$	$\mathbf{k-1}$	$\mathbf{k}$	$\mathbf{k+1}$	$\mathbf{t+1}$	$\mathbf{t+2}$	$\mathbf{t+3}$	$\dots$
	c_1+1										
	-1	c_1	c_2	$\dots$	c_{k-1}	c_k	c_{k+1}	c_1	c_{t+2}	c_{t+3}	$\dots$
						-1	c_1	c_2	c_3	c_4	$\dots$
1	$-c_1$	$-c_2$	$-c_3$	$\dots$	$-c_k$	$-c_{k+1}$	$-c_1$	$-c_{t+2}$	$-c_{t+3}$	$-c_{t+4}$	$\dots$
							1	$-c_1$	$-c_2$	$-c_3$	$\dots$
1	0	c_1-c_2	c_2-c_3	$\dots$	$c_{k-1}-c_k$	$c_k-c_{k+1}-1$	$c_{k+1}+1$	c_2-c_{t+2}			

A. Suppose $c_{k+1}+1 \leq c_1 - 1$. If $c_2 - c_{t+2} < c_1$, then we have a valid gzd when cutting at the $(t+1)^{\text{th}}$ column.

B. The only way $c_2 - c_{t+2} \not< c_1$ is if $c_2 = c_1$ and $c_{t+2} = 0$. We are in a position to carry again. Once we do so, the resulting representation is necessarily the gzd with $\Delta S \;=\; c_2 - c_{t+2} + 1 - c_1 \;=\; c_1 - 0 + 1 - c_1 \;=\; 1$.

Therefore, as long as $c_{k+1}+1 \leq c_1 - 1$, we are done.

C. Suppose $c_{k+1}+1 = c_1$. Then if $c_{t+2} > 0$, columns $k+1$ and $t+1$ will form an allowable block, so we will get a gzd with $\Delta S > 0$. Therefore, the bad case is when $c_{t+2} = 0$.

D. We distinguish a few possible cases. First note that since $c_{k+1} = c_1 - 1$, we must have that $c_1 = c_2 = \cdots = c_k$. Suppose that $k = 1$. Then $\sigma = (c_1, c_1 - 1, c_1, 0, c_{t+3}, \dots)$. If $c_{t+3} = 0$ or 1, then by cutting at the 4^{th} column, we get $[1, 0, 0, c_1, c_1 - 1, 1 - c_{t+3}]$, which is the gzd with $\Delta S \;=\; c_1 - c_{t+3} \;\geq\; c_1 - 1 \;\geq\; 1$.

E. Suppose instead that $c_{t+3} \geq 2$. Then we borrow to get the gzd with $\Delta S \;\geq\; c_1 - c_{t+3} - 1 + c_1 \;\geq\; c_1 - 1 \;\geq\; 1$. Thus, in all cases, the number of summands increases.

F. Now suppose $k = 2$; that is $\sigma = (c_1, c_1, c_1 - 1, c_1, 0, c_{t+3}, \dots)$. If $c_{t+3} \leq c_1 - 1$, then cutting at the 5^{th} column, we will get the gzd with $\Delta S \;=\; (-c_{t+3}) + (c_1 - 1) + (-1 + c_1) \;\geq\; c_1 - 1 \;\geq\; 1$.

G. Suppose instead that $c_{t+3} = c_1$. Then, we borrow again to get the gzd with $\Delta S \;=\; -c_{t+3} + (c_1 - 1) + (-1 + c_1) + (-1 + c_1) \;=\; 2c_1 - 3 \;\geq\; 1$.

H. Now suppose that $k \geq 3$, that is, our signature looks like $\sigma = (c_1, c_1, c_1, \dots)$. Then cutting at the $(t+2)^{\text{nd}}$ column, we have a valid gzd with $\Delta S \;=\; -c_{t+3} + c_3 \;=\; c_1 \;>\; 1$.

I. Suppose instead that $c_{t+3} > 0$. Then cutting at the $(t+2)^{\text{nd}}$ column and borrowing, we get the gzd with $\Delta S \;\geq\; -c_{t+3} + c_3 + c_1 - 1 \;\geq\; c_1 - 1 \;\geq\; 1$. Therefore, in this case we are also done.

APPENDIX F. PROOF OF PROPOSITION 5.9

In order to prove this proposition, we first need the following lemma.

Lemma F.1. *Given a non-negative signature* $c_1, \dots, c_t$, *let* $\alpha_1, \dots, \alpha_t$ *be the roots of the corresponding characteristic polynomial. Then* $\lim_{n\to\infty} H_n = C\beta^n$ *where* $\beta = \max\{|\alpha_1||, \dots, |\alpha_t|\}$.

Proof. This is a straightforward implication of Generalized Binet's Formula (see for example Theorem A.1 in [**BBGILMT**]). □

Proof of Proposition 5.9. Consider a non-weakly-decreasing signature of the type $c_1, \dots, c_t$ where $c_1 = 1$ and $c_t \neq 0$. Suppose that for $2H_n$, a summand minimal gzd exists, which means that in the gzd, we can only have one or two summands.

If the gzd has two summands, it must be of the form $[1, 0, \dots, 0, 1, 0, \dots]$, which corresponds to $2H_n = H_{\ell'_1(n)} + H_{\ell'_2(n)}$, where $\ell'_1(n) > \ell'_2(n)$. Now, since the signature is nonnegative, there exists N_1 such that for all $n \geq N_1$, $H_n > H_{n-1}$. Hence, for all $n \geq N_1$, we cannot have $\ell'_1(n) \leq n$ or $\ell'_2(n) \geq n$. As such, in the case where the gzd has two summands, for $n \geq N_1$ we can only have $2H_n = H_{n+\ell_1(n)} + H_{n-\ell_2(n)}$ where $\ell_1(n), \ell_2(n) > 0$. If the gzd has one summand, it must be that $2H_n = H_{n+\ell_3(n)}$ where $\ell_3(n) > 0$ for all $n > N_1$ because the sequence is strictly increasing after H_{N_1}.

Now we will determine the possible choices for $\ell_1(n)$, $\ell_2(n)$ and $\ell_3(n)$.

Since the greedy algorithm to construct a gzd established by [**MW1**] and [**Ha**] always includes the largest term in the sequence that is smaller than $2H_n$, we know that $H_{n+\ell_1(n)}$ and $H_{n+\ell_3(n)}$ are such that $H_{n+\ell_i(n)} \leq 2H_n < H_{n+\ell_i(n)+1}$, where $i = 1, 3$. By Lemma F.1, we can write $H_n = C\beta^n(1+\varepsilon(n))$ where ε is a function from $\mathbb{N}$ to $\mathbb{R}$ such that as $N \to \infty$, $\varepsilon(N) \to 0$. We then have that

$$C\beta^{n+\ell_i(n)}(1+\varepsilon(n+\ell_i(n))) \ \leq \ 2C\beta^n(1+\varepsilon(n)) \ < \ C\beta^{n+\ell_i(n)+1}(1+\varepsilon(n+\ell_i(n)+1)). \tag{F.1}$$

Canceling $C\beta^n$ from all 3 sides and taking the $\log$ base β, we then obtain

$$\ell_i(n) \ = \ \left\lfloor \log_\beta \frac{2(1+\varepsilon(n))}{1+\varepsilon(n+\ell_i(n))} \right\rfloor, \tag{F.2}$$

which goes to $\lfloor \log_\beta 2 \rfloor$ as $N \to \infty$. Hence, there exists N_2 such that for all $n \geq N_2$, $\ell_1(n) = \ell_3(n) = \lfloor \log_\beta 2 \rfloor$. Note that these are now constant.

Lastly, we determine $\ell_2(n)$ in the case that the gzd has two summands, i.e., $2H_n = H_{n+\ell_1(n)} + H_{n-\ell_2(n)}$. Rewriting using Lemma F.1, we have that

$$C\beta^{n-\ell_2(n)}(1+\varepsilon(n+\ell_2(n))) \ = \ 2C\beta^n(1+\varepsilon(n)) - C\beta^{n+\ell_1(n)}(1+\varepsilon(n+\ell_1(n))), \tag{F.3}$$

Again, cancelling $C\beta^n$ on both sides and taking the $\log$ base β, we get

$$\ell_2(n) \ = \ \log_\beta \left[\left[\left(2(1+\varepsilon(n)) - \beta^{\ell_1(n)}(1+\varepsilon(n+\ell_1(n)))\right) \right]^{-1} (1+\varepsilon(n+\ell_2(n))) \right]. \tag{F.4}$$

Since $\varepsilon(n+\ell_2(n)) \to 0$ as $n \to \infty$, there exists N_3 such that for all $n > N_3$, either $\ell_2(n) = \lfloor \log_\beta (2-\beta^{\ell_1})^{-1} \rfloor$ or $\lceil \log_\beta (2-\beta^{\ell_1})^{-1} \rceil$. This concludes the proof. □

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