# RATIONAL IRRATIONALITY PROOFS

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Proving the irrationality of $\sqrt{2}$ is a rite of passage for mathematicians. The purpose of this note is to spread the word of a remarkable geometric proof, and to generalize it. The proof was discovered by Stanley Tennenbaum [Te] in the 1950's, and first appeared in print in John H. Conway's article in Power [Co]. In the interest of space, we often leave out the algebra justifications for the lengths of the sides in our figures. The reader is encouraged to prove these expressions for themselves, or see the arxiv post [MM] for complete details.

## 1. Tennenbaum's proof

We now describe Tennenbaum's wonderful geometric proof of the irrationality of $\sqrt{2}$. Suppose that $\sqrt{2} = a/b$ for some positive integers $a$ and $b$; then $a^2 = 2b^2$. We may assume that $a$ is the smallest positive integer for which this is possible. We interpret this geometrically by constructing a square of side $a$ and, within it, two squares of side $b$ (see Figure 1). Since the combined areas of the squares of side $b$ equals the area of the square of side $a$, the pink, doubly-counted square must have the same area as the two white squares. We have therefore found a smaller pair of integers $u$ and $v$ with $u^2 = 2v^2$, which is a contradiction. Thus $\sqrt{2}$ is irrational.

## 2. The square-root of 3 is irrational

We generalize Tennenbaum's geometric proof to show $\sqrt{3}$ is irrational. Suppose not, so $\sqrt{3} = a/b$, and again we may assume that $a$ and $b$ are the smallest positive integers

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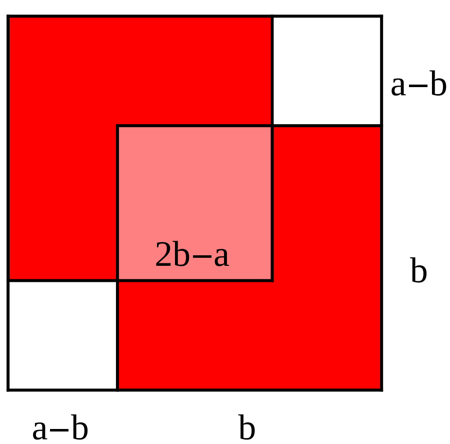


FIGURE 1. Geometric proof of the irrationality of $\sqrt{2}$.

satisfying $a^2 = 3b^2$. As the area of an equilateral triangle is proportional to the square of its side $s$ (the area is $s^2 \cdot \sqrt{3}/4$), we may interpret $a^2 = 3b^2$ as the area of one equilateral triangle of side length $a$ equals the area of three equilateral triangles of side length $b$. We represent this in Figure 2 , which consists of three equilateral triangles of

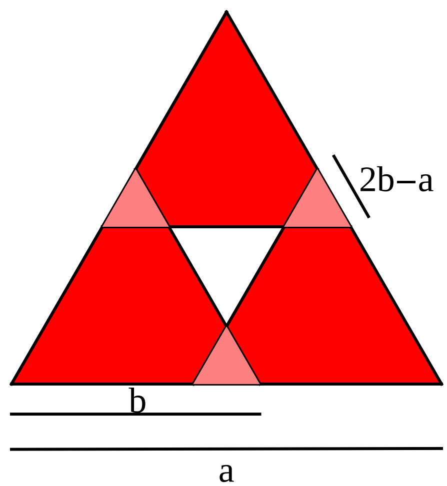


FIGURE 2. Geometric proof of the irrationality of $\sqrt{3}$. The white equilateral triangle in the middle has sides of length $2a - 3b$.

side length $b$ placed at the corners of an equilateral triangle of side length $a$. Note that the area of the three doubly covered, pink triangles (which have side length $2b - a$) is therefore equal to that of the uncovered, equilateral triangle in the middle (which has integral sides of length $2a - 3b$). This is clearly a smaller solution, contradiction!

## 3. THE SQUARE-ROOT OF 5 IS IRRATIONAL

For the irrationality of $\sqrt{5}$, we have to slightly modify our approach as the overlapping regions are not so nicely shaped. As the proof is similar, we omit many of the details. Similar to the case of $\sqrt{3}$ and triangles, there are proportionality constants relating the area to the square of the side lengths of regular $n$-gons; however, as these constants appear on both sides of the equations, we may ignore them.

Suppose $a^2 = 5b^2$ with, as always, $a$ and $b$ minimal. We place five regular pentagons of side length $b$ at the corners of a regular pentagon of side length $a$ (see Figure 3). Note that this gives five small triangles on the edge of the larger pentagon which are uncovered, one uncovered regular pentagon in the middle of the larger pentagon, and five kite-shaped doubly covered regions. As before, the doubly covered region must have the same area as the uncovered region.

We now take the uncovered triangles from the edge and match them with the doubly covered part at the “bottom” of the kite, and regard each as covered once instead of one covered twice and one uncovered (see Figure 4). This leaves five doubly covered pentagons, and one larger pentagon uncovered.

A straightforward analysis shows that the five doubly covered pentagons are all regular, with side length $a - 2b$, and the middle pentagon is also regular, with side length

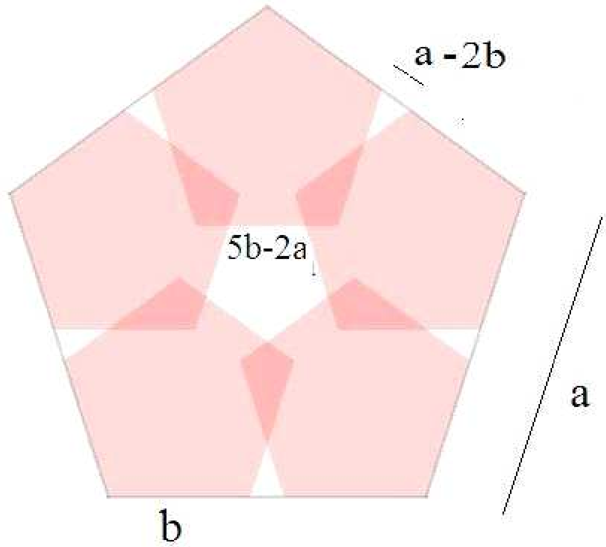


FIGURE 3. Geometric proof of the irrationality of $\sqrt{5}$.

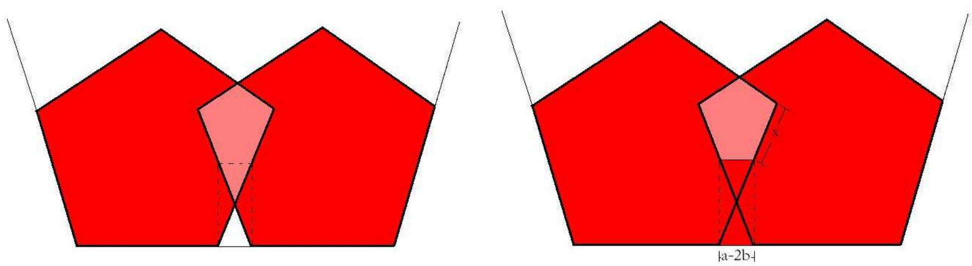


FIGURE 4. Geometric proof of the irrationality of $\sqrt{5}$: the kites, triangles and the small pentagons.

$b - 2(a - 2b) = 5b - 2a$ (see [MM] for the full calculations). We now have a smaller solution, with the five doubly counted regular pentagons having the same area as the omitted pentagon in the middle. Specifically, we have $5(a-2b)^2 = (5b-2a)^2$; as $a = b\sqrt{5}$ and $2 < \sqrt{5} < 3$, note that $a - 2b < b$ and thus we have our contradiction.

## 4. HOW FAR CAN WE GENERALIZE: TO $\sqrt{6}$ AND BEYOND

We conclude with a discussion of one generalization of our method that allows us to consider certain triangular numbers, though other generalizations are possible and yield similar results. We hope the reader will explore these constructions further.

Figure 5 shows the construction for the irrationality of $\sqrt{6}$. Assume $\sqrt{6} = a/b$ so $a^2 = 6b^2$; as always, we assume $a$ and $b$ are the smallest positive integers satisfying this relation. The large equilateral triangle has side length $a$ and the six medium equilateral triangles have side length $b$. The 7 smallest equilateral triangles (6 double counted, one in the center triple counted) have side length $t = (3b - a)/2$. It's a little work, but not too bad, to show the triple counted one is the same size. For the three omitted triangles,

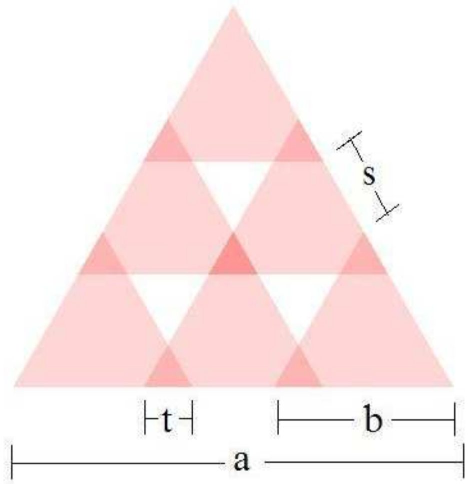


FIGURE 5. Geometric proof of the irrationality of $\sqrt{6}$.

they are all equilateral (angles equal) and of side length $s = b - 2(3b - a)/2 = a - 2b$. As the area of the smaller equilateral triangles is proportional to $t^2$ and for the larger it is proportional to $s^2$, we find $8t^2 = 3s^2$ or $16t^2 = 6s^2$ so $(4t/s)^2 = 6$. Note that although $t$ itself may not be an integer, $4t = 2(3b - a)$ is an integer, and we obtain our contradiction as we have found a smaller solution.

Can we continue this argument? We may interpret the argument here as adding three more triangles to the argument for the irrationality of $\sqrt{3}$; thus the next step would be adding four more triangles to this to prove the irrationality of $\sqrt{10}$. Proceeding along these lines leads us to study the square-roots of triangular numbers. Triangular numbers are of the form $n(n + 1)/2$ for some positive integer $n$, and thus the first few are $1, 3, 6, 10, 15, \dots$. We continue more generally by producing images like Figure 5 with $n$ equally spaced rows of side length $b$ triangles. This causes us to start with $a^2 = \frac{n(n+1)}{2}\, b^2$, so we can attempt to show that $\sqrt{n(n + 1)/2}$ is irrational.

By similar reasoning to the above, we see that the smaller multiply covered equilateral triangles all have the same side length $t$, and that the uncovered triangles also all have the same side length $s$. Further $t$ equals $(nb - a)/(n - 1)$, and we have that $s = b-2t$, so $s = b-2(nb-a)/(n-1) = (2a-(n+1)b)/(n-1)$. To count the number of side length $t$ triangles, we note that there will be $(n - 2)(n - 1)/2$ triply covered triangles (as there is a triangle-shaped configuration of them with $n - 2$ rows), and that there will be $3(n-1)$ doubly covered triangles around the edge of the figure, for a grand total of $2(n-2)(n-1)/2+3(n-1) = (n-1)(n+1)$ coverings of the smaller triangle. Further, note that in general there will be $(n-1)n/2$ smaller, uncovered triangles, so we have that $(n - 1)(n + 1)t^2 = ((n - 1)n/2)s^2$. Writing out the formula for $s, t$ (to verify that our final smaller solution is integral), we have $(n-1)(n+1)((nb-a)/(n-1))^2 = ((n - 1)n/2)((2a - (n + 1)b)/(n - 1))^2$. We now multiply both sides of the equation by $n - 1$ to ensure integrality, giving $(n + 1)(nb - a)^2 = (n/2)(2a - (n + 1)b)^2$. We multiply both sides by $n/2$ to achieve a smaller solution to $a^2 = (n(n+1)/2)b^2$, giving us $(n(n+1)/2)(nb-a)^2 = (n(2a-(n+1)b)/2)^2$. Note that this solution is integral, as $n$ odd implies that $2a-(n+1)b$ is even. Finally, to show that this solution is smaller, we just need that $nb-a<b$. This is equivalent to $n-\sqrt{n(n+1)/2}<1$.

We see that this inequality holds for $n \le 4$, but not for $n > 4$. So, we have shown that the method used above to prove that $\sqrt{6}$ is irrational can also be used to show that $\sqrt{10}$ (the square root of the fourth triangular number) is irrational, but that this method will not work for any further triangular numbers. It is good (perhaps it is better to say, 'it is not unexpected') to have such a problem, as some triangular numbers are perfect squares. For example, when $n = 49$ then we have $49 \cdot 50/2 = 7^2 \cdot 5^2$, and thus we should not be able to prove that this has an irrational square-root!

## 5. Final Remarks

There are many proofs of the irrationality of $\sqrt{2}$; see for example [Ap, Bo, HW]. One particularly nice one can be interpreted as an origami construction (see proof 7 of [Bo] and the references there, and pages 183–185 of [CG] for the origami interpretation). Cwikel [Cw] has generalized these origami arguments to yield the irrationality of other numbers as well.

The purpose of this note is to describe a geometric method which can be pushed further than one might initially expect. The examples studied are in no sense meant to be exhaustive, but rather should be viewed as a representative sample of what can be done. Our hope is that the reader will find and communicate many more. For example, after reading an earlier draft Walter Stromquist [St] found a similar proof of the irrationality of $\sqrt{6}$. Instead of using triangles he uses the squares of Tennenbaum's original proof of $\sqrt{2}$. Taking $a$ and $b$ as in that proof, he considers a square of side length $a+b$, and finds two doubly counted squares of length $2b-a$ must equal the area of three uncounted squares of length $a-b$. Doubling, we find that four doubly counted squares must equal the area of six uncounted squares; as four is a perfect square, this would imply that six is a perfect square as well.

## Acknowledgements

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