# Linear Recurrences of Order at Most Two in Nontrivial Small Divisors and Large Divisors

Hùng Việt Chu[1], Kevin Huu Le[2] Steven J. Miller[3], Yuan Qiu[4], and Liyang Shen[5]

[1] Department of Mathematics, Texas A&M University, College Station, TX 77843, USA,
hungchu@tamu.edu; hung.viet.chu@gmail.com
[2] Department of Mathematics, Texas A&M University, College Station, TX 77843, USA,
kevinhle@tamu.edu
[3] Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267, USA,
sjm1@williams.edu; Steven.Miller.MC.96@aya.yale.edu
[4] Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267, USA,
yq1@williams.edu
[5] Courant Institute of Mathematical Sciences, New York University, New York, NY 10012, USA,
Liyang.Shen@nyu.edu

**Abstract.** For each positive integer $N$, define

$$S'_N = \{1 < d < \sqrt{N} : d|N\} \text{ and } L'_N = \{\sqrt{N} < d < N : d|N\}.$$

Recently, Chentouf characterized all positive integers $N$ such that the set of small divisors $\{d \leq \sqrt{N} : d|N\}$ satisfies a linear recurrence of order at most two. We nontrivially extend the result by excluding the trivial divisor 1 from consideration, which dramatically increases the analysis complexity. Our first result characterizes all positive integers $N$ such that $S'_N$ satisfies a linear recurrence of order at most two. Moreover, our second result characterizes all positive $N$ such that $L'_N$ satisfies a linear recurrence of order at most two, thus extending considerably a recent result that characterizes $N$ with $L'_N$ being in an arithmetic progression.

**Keywords:** divisors, linear recurrence, order two

## 1 Introduction

For a positive integer $N$, the set of small divisors of $N$ is

$$S_N := \{d : 1 \leq d \leq \sqrt{N}, d \text{ divides } N\}.$$

Since the case $N = 1$ is trivial, we assume throughout the paper that $N > 1$. In 2018, Iannucci characterized all positive integers $N$ whose $S_N$ forms an arithmetic progression (or AP, for short). Iannucci's key idea was to show that if $S_N$ forms an AP, then the size $|S_N|$ cannot exceed 6. Observing that the trivial divisor 1 plays an important role in Iannucci's proofs (see [4, Lemma 3 and Theorem 4]), Chu [3] excluded both 1 and $\sqrt{N}$ from the definition of $S_N$ to obtain a more general theorem that characterizes all $N$ whose

$$S'_N \ := \ \{d \,:\, 1 < d < \sqrt{N}, d \text{ divides } N\}$$

is in an AP. Interestingly, with the trivial divisor 1 excluded, [2, Theorem 1.1] still gives that $|S'_N| \leq 5$. Recently, Chentouf generalized Iannucci's result from a different perspective by characterizing all $N$ whose $S_N$ satisfies a linear recurrence of order at most two. In particular, for each tuple $(u, v, a, b) \in \mathbb{Z}^4$, there is an integral linear recurrence, denoted by $U(u, v, a, b)$, of order at most two, given by

$$n_i \ = \ \begin{cases} u & \text{if } i = 1, \\ v & \text{if } i = 2, \\ an_{i-1} + bn_{i-2} & \text{if } i \geq 3. \end{cases}$$

Noting that the appearance of the trivial divisor 1 contributes nontrivially to the proof of [1, Theorem 3, Lemma 8, Theorem 10], we generalize Chentouf's result in the same manner as [2, Theorem 1.1] generalizes [4, Theorem 4]: we characterize all positive integers $N$ whose $S'_N$ satisfies a linear recurrence of order at most two.

**Definition 1.** A positive integer $N$ is said to be small recurrent if $S'_N$ satisfies a linear recurrence of order at most two. When $|S'_N| \leq 2$, $N$ is vacuously small recurrent.

**Theorem 1.** *Let $p, q, r$ denote prime numbers such that $p < q < r$ and $k$ be some positive integer. A positive integer $N > 1$ is small recurrent if and only if $N$ belongs to one of the following forms.*

1. *$N = p^k$ for some $k \geq 1$. In this case, $S'_N = \{p, p^2, \ldots, p^{\lfloor (k-1)/2 \rfloor}\}$ satisfies $U(p, p^2, p, 0)$.*
2. *$N = p^k q$ or $N = pq^k$ for some $1 \leq k \leq 3$. A restriction for $N = p^3 q$ is that either $p < q < p^2$ or $p^3 < q$.*
3. *$N = p^k q$ for some $k \geq 4$ and $q > p^k$. In this case, $S'_N = \{p, p^2, p^3, \ldots, p^k\}$ satisfies $U(p, p^2, p, 0)$.*
4. *$N = p^k q$ for some $k \geq 4$ and $\sqrt{q} < p < q$. In this case, $S'_N = \{p, q, p^2, pq, \ldots\}$ satisfies $U(p, q, 0, p)$.*
5. *$N = pq^k$ for some $k \geq 4$ and $p < q$. In this case, $S'_N = \{p, q, pq, q^2, \ldots\}$ satisfies $U(p, q, 0, q)$.*
6. *$N = pq^k r$ for some $k \geq 2$, $p < q$, and $r > pq^k$. In this case, $S'_N = \{p, q, pq, q^2, \ldots, pq^{k-1}, q^k, pq^k\}$ satisfies $U(p, q, 0, q)$.*
7. *$N = p^2 q^2$ for some $p < q < p^2$. In this case, $S'_N = \{p, q, p^2\}$.*
8. *$N = pqr$ for some $p < q < r$. If $r < pq$, then $S'_N = \{p, q, r\}$. If $r > pq$, then $S'_N = \{p, q, pq\}$.*

9. *$N = p^3q^2$ for some $p^{3/2} < q < p^2$. In this case, $S'_N = \{p, q, p^2, pq, p^3\}$ satisfies $U(p, q, 0, p)$.*
10. *$N = p^2qr$, where $p < q < p^2 < r < pq$, $(q^2 - p^3)|(pq - r)$, $(q^2 - p^3)|(rq - p^4)$, and $r = pq - \sqrt{(q^2 - p^3)(p^2 - q)}$. In this case, $S'_N = \{p, q, p^2, r, pq\}$ satisfies $U\left(p, q, \frac{p(pq-r)}{q^2-p^3}, \frac{rq-p^4}{q^2-p^3}\right)$.*

Next, consider the set of large divisors of $N$

$$L_N := \{d : d \geq \sqrt{N}, d \text{ divides } N\},$$
$$L'_N := \{d : \sqrt{N} < d < N, d \text{ divides } N\}.$$

The second result of this paper is the characterization of all positive integers $N$ whose $L'_N$ satisfies a linear recurrence of order at most two. This considerably extends [2, Theorem 1].

**Definition 2.** A positive integer $N$ is said to be large recurrent if $L'_N$ satisfies a linear recurrence of order at most two. When $|L'_N| \leq 2$, $N$ is vacuously large recurrent.

**Theorem 2.** *Let $p, q, r$ denote prime numbers such that $p < q < r$ and $k$ be some positive integer. A positive integer $N > 1$ is large recurrent if and only if $N$ belongs to one of the following forms.*

1. *$N = p^k$ for some $k \geq 1$. Then $L'_N = \{p^{\lceil (k-1)/2\rceil+1}, p^{\lceil (k-1)/2\rceil+2}, \ldots, p^{k-1}\}$ satisfies $U(p^{\lceil (k-1)/2\rceil+1}, p^{\lceil (k-1)/2\rceil+2}, p, 0)$.*
2. *$N = p^kq$ for some $k \geq 1$ and $q > p^k$. Then $L'_N = \{q, pq, p^2q, \ldots, p^{k-1}q\}$ satisfies $U(q, pq, p, 0)$.*
3. *$N = p^kq$ for some $k \geq 2$ and $p^{k-1} < q < p^k$. Then*
$$L'_N = \{p^k, pq, p^2q, \ldots, p^{k-1}q\}$$
*satisfies $U(p^k, pq, p, 0)$.*
4. *$N = p^kq$ some for $k \geq 3$ and $p < q < p^2$. In this case,*
$$L'_N = \begin{cases} \{p^{k/2+1}, p^{k/2}q, p^{k/2+2}, \ldots, p^{k-1}q\} & \text{if } 2|k, \\ \{p^{(k-1)/2}q, p^{(k+3)/2}, p^{(k+1)/2}q, \ldots, p^{k-1}q\} & \text{if } 2 \nmid k. \end{cases}$$
*Note that $L'_N$ satisfies $U(p^{k/2+1}, p^{k/2}q, 0, p)$ and $U(p^{(k-1)/2}q, p^{(k+3)/2}, 0, p)$ for even and odd $k$, respectively.*
5. *$N = p^4q$ with $p^2 < q < p^3$, $(p^5 - q^2)|(p^2 - q)$, and $(p^5 - q^2)|(p^3 - q)$. In this case, $L'_N = \{pq, p^4, p^2q, p^3q\}$.*
6. *$N = p^3q^2$ for $p < q < p^2$. In this case, $L'_N = \{q^2, p^2q, pq^2, p^3q, p^2q^2\}$ satisfies $U(q^2, p^2q, 0, p)$.*
7. *$N = p^2q^2$ for some $p < q$. If $p < q < p^2$, then $L'_N = \{q^2, p^2q, pq^2\}$.*
8. *$N = pq^k$ for some $k \geq 2$ and $p < q$. In this case,*
$$L'_N = \begin{cases} \{pq^{k/2}, q^{k/2+1}, \ldots, q^k\} & \text{if } 2|k, \\ \{q^{(k+1)/2}, pq^{(k+1)/2}, \ldots, q^k\} & \text{if } 2 \nmid k. \end{cases}$$

*Observe that* $L'_N$ *satisfies* $U(pq^{k/2}, q^{k/2+1}, 0, q)$ *and* $U(q^{(k+1)/2}, pq^{(k+1)/2}, 0, q)$ *for even and odd* $k$*, respectively.*

9. $N = pq^k r$ *for some* $k \geq 1$ *and* $p < q < pq^k < r$*. In this case,* $L'_N = \{r, pr, qr, pqr, q^2r, \ldots, q^k r\}$ *satisfies* $U(r, pr, 0, q)$*.*

The paper is structured as follows: Section 2 studies the case when $N$ has a small number of divisors and establish some preliminary results; Section 3 characterizes small recurrent numbers, while Section 4 characterizes large recurrent numbers.

## 2 Preliminaries

For each $N \in \mathbb{N}$ with the prime factorization $\prod_{i=1}^{\ell} p_i^{a_i}$, the divisor-counting function is

$$\tau(N) \ := \ \sum_{d|N} 1 \ = \ \prod_{i=1}^{\ell}(a_i + 1). \tag{1}$$

It is easy to verify that for $N > 1$,

$$\tau(N) \ := \ \begin{cases} 2|S'_N| + 3 \ = \ 2|L'_N| + 3 & \text{if } N \text{ is a square,} \\ 2|S'_N| + 2 \ = \ 2|L'_N| + 2 & \text{otherwise.} \end{cases} \tag{2}$$

Using (1), we can characterize all $N$ with $\tau(N) \leq 9$; equivalently, $|S'_N|, |L'_N| \leq 3$.

1. If $\tau(N) = 2$ or 3, (1) gives that $N = p$ or $p^2$ for some prime $p$,
2. If $\tau(N) = 4$ or 5, $N = pq, p^3, p^4$ for some primes $p < q$,
3. If $\tau(N) = 6$ or 7, $N = p^5, pq^2, p^2q, p^6$ for some primes $p < q$,
4. If $\tau(N) = 8$ or 9, $N = pqr, pq^3, p^3q, p^7, p^2q^2, p^8$ for some primes $p < q < r$.

### 2.1 Regarding $S'_N$

**Proposition 1.** *If* $|S'_N| \geq 2$*, then* $N$ *cannot have two (not necessarily distinct) prime factors* $p_1$ *and* $p_2$ *at least* $\sqrt{N}$

*Proof.* Suppose that $N$ has two prime factors $p_1$ and $p_2$ at least $\sqrt{N}$. Then $p_1p_2 \geq N$ and $p_1p_2$ divides $N$; hence, $N = p_1p_2$, which contradicts that $|S'_N| \geq 2$.

**Proposition 2.** *If all elements of* $S'_N$ *are divisible by some prime* $p$ *and* $|S'_N| \geq 4$*, then either* $N = p^k$ *or* $N = p^k q$ *for some* $k \geq 1$ *and some prime* $q > p^k$*.*

*Proof.* If all divisors (except 1) of $N$ are divisible by $p$, then $n = p^k$ for some $k \geq 1$. Assume that $N$ has a prime factor $q \neq p$. Then $q \geq \sqrt{N}$. Proposition 1 implies that $N$ cannot have another prime factor at least $\sqrt{N}$. Hence, $N = p^k q$ for some $k \geq 1$ and $q > p^k$.

Let $p < q < r < s$ be distinct prime numbers. Write $S'_N = \{d_2, d_3, d_4, d_5, \ldots\}$. (We start with $d_2$ since the smallest divisor of $N$ is usually denoted by $d_1 = 1$, which is excluded from $S'_N$.)

**Lemma 1.** *Suppose that the first 4 numbers in $S'_N$ are $p < q < p^2 < r$. If $N$ is small recurrent with $U(p, q, a, b)$, the following hold:*

*i)* $\gcd(a, b) = 1$.
*ii) if $d_{2i} \in S'_N$, then $p|d_{2i}$; however, if $d_{2i-1} \in S'_N$, then $p \nmid d_{2i-1}$.*
*iii) for $d_i, d_{2i-1} \in S'_N$, we have* $\gcd(b, d_i) = \gcd(a, d_{2i-1}) = 1$.
*iv) for $d_i, d_{i+1} \in S'_N$, we have* $\gcd(d_i, d_{i+1}) = 1$.
*v) for $d_{2i-1}, d_{2i+1} \in S'_N$, we have* $\gcd(d_{2i-1}, d_{2i+1}) = 1$.

*Proof.* i) Since $r = ap^2 + bq$, we have $\gcd(a, b)|r$. Observe that $\gcd(a, b) = r$ contradicts $p^2 = aq + bp$. Hence, $\gcd(a, b) = 1$.

ii) We prove by induction. The claim is true for $i \leq 2$. Assume that the claim holds for $i = j \geq 2$. We show that it holds for $i = j + 1$. Since $p^2 = aq + bp$, we know that $p|a$. By item i), $p \nmid b$. Write

$$\begin{aligned} d_{2(j+1)} &= ad_{2(j+1)-1} + bd_{2(j+1)-2} = ad_{2j+1} + bd_{2j} \\ &= (a^2 + b)d_{2j} + abd_{2j-1}. \end{aligned}$$

By the inductive hypothesis, $p|d_{2j}$. Since $p|a$, we obtain $p|d_{2(j+1)}$. Furthermore, write

$$d_{2(j+1)-1} = ad_{2(j+1)-2} + bd_{2(j+1)-3} = ad_{2j} + bd_{2j-1}.$$

Since $p \nmid bd_{2j-1}$ by the inductive hypothesis and $p|ad_{2j}$, we obtain $p \nmid d_{2(j+1)-1}$. This completes our proof.

iii) Suppose that $k = \gcd(b, d_i) > 1$ for some $i$. If $i = 2$, then $p|b$, which contradicts that $p|a$ and $\gcd(a, b) = 1$. If $i = 3$, then $q|b$, which contradicts $p^2 = aq + bp$. If $i \geq 4$, write $d_i = ad_{i-1} + bd_{i-2}$. Since $k|bd_{i-2}$ and $\gcd(a, b) = 1$, we get $k|d_{i-1}$ and so, $k|\gcd(b, d_{i-1})$. By induction, we obtain $k|\gcd(b, d_3)$, which has been shown to be impossible.

Next, suppose that $k = \gcd(a, d_{2i-1}) > 1$ for some $i$. If $i = 2$, then $q|a$, contradicting $r = ap^2 + bq$. If $i \geq 3$,

$$k \mid d_{2i-1},\ d_{2i-1} = ad_{2i-2} + bd_{2i-3}, \text{ and } \gcd(a, b) = 1 \implies k \mid d_{2i-3}.$$

By induction, $k|d_3$; that is, $\gcd(a, d_3) > 1$, which has been shown to be impossible.

iv) The claim holds for $i \leq 4$. For $i \geq 5$, using the recurrence $d_{i+1} = ad_i + bd_{i-1}$, we have

$$\gcd(d_i, d_{i+1}) = \gcd(d_i, bd_{i-1}) \stackrel{\text{iii)}}{=} \gcd(d_i, d_{i-1}).$$

By induction, we obtain $\gcd(d_i, d_{i+1}) = 1$.

v) The claim holds for $i \leq 2$. For $i \geq 3$, we have

$$\gcd(d_{2i-1}, d_{2i+1}) = \gcd(d_{2i-1}, ad_{2i}) \stackrel{\text{iii)}}{=} \gcd(d_{2i-1}, d_{2i}) \stackrel{\text{iv)}}{=} 1.$$

This completes our proof.

**Lemma 2.** *Suppose that the first* 4 *numbers in* $S'_N$ *are* $p < q < r < p^2$. *If* $N$ *is small recurrent with* $U(p,\ q,\ a,\ b)$*, the following hold:*

*i)* $\gcd(a, b) = 1$ *and* $p \nmid a$.
*ii) For all* $d_i \in S'_N$, $\gcd(b, d_i) = 1$.
*iii) For all* $d_i, d_{i+1} \in S'_N$, $\gcd(d_i, d_{i+1}) = 1$.
*iv) Let* $d_i \in S'_N$. *Then* $p|d_i$ *if and only if* $i \equiv 2 \bmod 3$.

*Proof.* i) We have $\gcd(a, b)$ divides $r$ because $r = aq+bp$. Furthermore, $\gcd(a, b)$ divides $p$ because $p^2 = ar + bq$. Hence, $\gcd(a, b) = 1$.

Since $r = aq + bp$ and $r$ is a prime, $p \nmid a$.

ii) Suppose that $k = \gcd(b, d_i) > 1$ for some $i$. If $i = 2$, then $k = p$ and $p|b$. It follows from $p^2 = ar + bq$ that $p|a$, which contradicts that $\gcd(a, b) = 1$. If $i = 3$, then $k = q$ and $q|b$. It follows from $r = aq + bp$ that $q|r$, a contradiction. Hence, $i \geq 4$. Write

$$1 \ < \ \gcd(b, d_i) \ = \ \gcd(b, ad_{i-1} + bd_{i-2}) \ = \ \gcd(b, d_{i-1}).$$

By induction, we obtain $\gcd(b, d_3) > 1$, which has been shown to be impossible.

iii) The claim holds for $i \leq 4$. Let $d_i, d_{i+1} \in S'_N$ and $i \geq 5$. Then

$$\gcd(d_i, d_{i+1}) \ = \ \gcd(d_i, ad_i + bd_{i-1}) \ = \ \gcd(d_i, bd_{i-1}) \ \overset{\text{ii)}}{=} \ \gcd(d_i, d_{i-1}).$$

By induction, $\gcd(d_i, d_{i+1}) = 1$.

iv) The claim holds for $i \leq 4$. Suppose that the claim holds for all $i \leq j$ for some $j \geq 5$. We show that it also holds for $i = j + 1$ under the assumption that $d_{j+1} \in S'_N$. We have

$$p^2 \ = \ ar + bq \ = \ a(aq + bp) + bq \ = \ (a^2 + b)q + abp.$$

Hence, $p|(a^2 + b)$. Write

$$d_{j+1} \ = \ ad_j + bd_{j-1} \ = \ a(ad_{j-1} + bd_{j-2}) + bd_{j-1} \ = \ (a^2 + b)d_{j-1} + abd_{j-2}.$$

Therefore, $p|d_{j+1}$ if and only if $p|d_{j-2}$. By the inductive hypothesis, we know that $p|d_{j+1}$ if and only if $j + 1 \equiv 2 \bmod 3$.

The proofs of the next two lemmas are similar to those of Lemmas 1 and 2. Thus, we move their proofs to the Appendix.

**Lemma 3.** *Suppose that the first* 4 *numbers in* $S'_N$ *are* $p < q < r < pq$. *If* $N$ *is small recurrent with* $U(p, q, a, b)$*, the following hold:*

*i)* $\gcd(a, b) = 1$ *and* $p \nmid a$.
*ii) For all* $d_i \in S'_N$, $\gcd(b, d_i) = 1$.
*iii) For all* $d_i, d_{i+1} \in S'_N$, $\gcd(d_i, d_{i+1}) = 1$.
*iv) Let* $d_i \in S'_N$. *Then* $p|d_i$ *if and only if* $i \equiv 2 \bmod 3$.
*v) Let* $d_i \in S'_N$. *Then* $q|d_i$ *if and only if* $i$ *is odd.*

**Lemma 4.** *Suppose that the first* 4 *numbers in* $S'_N$ *are* $p < q < r < s$. *If* $N$ *is small recurrent with* $U(p, q, a, b)$, *the following hold:*

*i)* $\gcd(a, b) = 1$.
*ii)* *For all* $d_i \in S'_N$ *with* $i \geq 3$, $\gcd(b, d_i) = 1$.
*iii)* *For* $d_i, d_{i+1} \in S'_N$, $\gcd(d_i, d_{i+1}) = 1$.
*iv* *For* $d_i \in S'_N$, $\gcd(a, d_i) = 1$.
*v)* *For* $d_i, d_{i+2} \in S'_N$, $\gcd(d_i, d_{i+2}) = 1$.

## 3 Small recurrent numbers

We first find all small recurrent numbers with $|S'_N| \geq 4$. Then we check which $N$ is small recurrent out of all $N$ with $|S'_N| \leq 3$ at the end of this section. As we rely heavily on case analysis, we underline possible forms of $N$ throughout our analysis for the ease of later summary.

### 3.1 The case $|S'_N| \geq 4$

Let $d_2 = p$ for some prime $p$. Then $d_3$ is either $p^2$ or $q$ for some prime $q > p$. If $d_3 = p^2$, according to Proposition 2, we know that

$$\underline{N = p^k \text{ or } N = p^k q \text{ for some } k \geq 1 \text{ and } q > p^k}.$$

Assume, for the rest of this subsection, that $d_3 = q$ for some prime $q > p$. Then $d_4 = pq, p^2, r$ for some prime $r > q$.

**When $d_4 = pq$** Since $S'_N$ satisfies $U(p, q, a, b)$, we get $pq = aq + bp$. So, $p|a$ and $q|b$. Write $a = pm$ for some $m \in \mathbb{Z}$ and get $b = (1 - m)q$. Since $p^2 \nmid N$ and $q$ divides $d_5 = apq + bq$, we can write

$$d_5 = p^s q^t r_1^{\ell_1} \cdots r_k^{\ell_k},$$

where $s \leq 1$, $t \geq 1$, and $r_i$'s are primes strictly greater than $pq$. If some $\ell_i \geq 1$, then $pq < r_i < d_5$ and $r_i \in S'_N$, a contradiction. Hence, $\ell_i = 0$ for all $i \leq k$ and $d_5 = p^s q^t$. Since $d_5 > pq$ and $s \leq 1$, we know that $t \geq 2$.

a) If $s = 1$, $d_5 = pq^2 > q^2 > d_4$ and $q^2 | d_5$, so $q^2 \in S'_N$, a contradiction.
b) If $s = 0$, $d_5 = q^t$ for some $t \geq 2$. Since $d_5$ is the next number after $d_4$ in increasing order, $d_5 = q^2$. Using the linear recurrence, we obtain $q^2 = apq + bq$, so $q = ap + b = p^2 m + (1 - m)q$. It follows that $p^2 m = mq$. We arrive at $m = 0$, $a = 0$, and $b = q$. Hence, all elements of $S'_N$ are divisible by either $p$ or $q$. If $N$ has a prime factor $r \geq \sqrt{N}$, by Proposition 1, $r$ is unique. We conclude that $\underline{N = pq^k \text{ or } pq^k r \text{ for some } k \geq 2,\ p < q, \text{ and } pq^k < r}$.

**When $d_4 = p^2$** The first few divisors of $N$ are $1 < p < q < p^2$. We have $p^2 = aq + bp$, so $p|a$. Write $a = pm$ and get $b = p - mq$. We argue for possible forms of $d_5$. Let $r$ be the largest prime factor of $d_5$. If $r > q$, then $r > p^2$ and $d_5 = r$. Otherwise, if $r \leq q$, then $d_5 = p^\ell q^k$ for some $\ell, k \geq 0$. Suppose that $k \geq 2$. We get

$$d_5 \ \geq \ q^2 \ > \ pq \ > \ d_4 \text{ and } pq \mid N,$$

a contradiction. Hence, $k \leq 1$. If $k = 0$, then

$$d_5 \ = \ p^3 \ > \ pq \ > \ d_4 \text{ and } pq \mid N,$$

another contradiction. Therefore, $k = 1$ and $d_5 = pq$. We conclude that either $d_5 = r$ for some $r > q$ or $d_5 = pq$.

a) If $d_5 = pq$, then $bq + ap^2 = pq$. So, $(p - mq)q + mp^3 = pq$, which gives $mq^2 = mp^3$. Hence, $m = 0$, $a = 0$, and $b = p$. We know that elements of $S'_N$ are divisible by either $p$ or $q$. If $N$ has a prime factor $r'$ at least $\sqrt{N}$, by Proposition 1, $r'$ is unique. Hence, either $N = p^\ell q^k$ or $p^\ell q^k r'$ for some prime $r' > p^\ell q^k$, $\ell \geq 2$, and $k \geq 1$.
Case a.i) $N = p^\ell q^k r'$. We claim that $k = 1$. Indeed, if $k \geq 2$, then

$$q^4 \ < \ q^2 r' \ < \ N \implies q^2 \ < \ \sqrt{N} \implies q^2 \in S'_N.$$

Since $b = p$, we know that $p|d$ for all $d \geq d_4$ and $d \in S'_N$, which contradicts $q^2 \in S'_N$. Hence,

$$\underline{N \ = \ p^\ell q r' \text{ for some } \ell \geq 2, \text{ some prime } r' > p^\ell q, \text{ and } \sqrt{q} < p < q.}$$

Case a.ii) $N = p^\ell q^k$. As above, $q^2 \notin S'_N$. If $k \geq 2$, then

$$q^2 \ \geq \ \sqrt{N} \implies q^4 \ \geq \ N \ = \ p^\ell q^k \ = \ p^{\ell-2} p^2 q^k \ > \ q^{k+1}.$$

Hence, $k < 3$, which implies that $k = 2$. In this case, $N = p^\ell q^2$ and

$$q^2 \ > \ p^\ell \ > \ q^{\ell/2} \implies \ell \ \leq 3.$$

We conclude that one of the following holds:
- $\underline{N = p^\ell q \text{ for some } \ell \geq 2 \text{ and } \sqrt{q} < p < q}$,
- $\underline{N = p^2 q^2 \text{ for some } p < q < p^2}$,
- $\underline{N = p^3 q^2 \text{ for some } p^{3/2} < q < p^2}$.

b) $d_5 = r$.

**Proposition 3.** *Suppose that the first* 4 *numbers in* $S'_N$ *are* $p < q < p^2 < r$. *Then* $|S_N| \leq 7$. *As a result,* $|S'_N| \leq 6$.

*Proof.* Assume that $|S_N| \geq 2i$ for some $i \geq 4$. We obtain a contradiction by showing that $|S_N| \geq 2i + 2$. By Lemma 1 item ii), $p \nmid d_{2i-1}$, $p|d_{2i-2}$, and $p \nmid d_{2i-3}$. By Lemma 1 item v), $\gcd(d_{2i-1}, d_{2i-3}) = 1$, so $p^2 d_{2i-1} d_{2i-3}$ divides $N$. Hence, $pd_{2i-3} \in S'_N$.

If $pd_{2i-3} = d_{2i-2}$, then

$$pd_{2i-3} = ad_{2i-3} + bd_{2i-4} \implies d_{2i-3} \mid bd_{2i-4},$$

which contradicts Lemma 1 items iii) and iv).
If $pd_{2i-3} = d_{2i}$, then

$$\begin{aligned} pd_{2i-3} &= ad_{2i-1} + bd_{2i-2} = a(ad_{2i-2} + bd_{2i-3}) + bd_{2i-2} \\ &= (a^2 + b)d_{2i-2} + abd_{2i-3}. \end{aligned}$$

Therefore, $d_{2i-3}$ divides $a^2 + b$. It is easy to check that for $d_j \in S'_N$, the sequence $d_j \bmod a^2 + b$ is congruent to

$$1, p, q, p^2, abp, abq, abp^2, (ab)^2p, (ab)^2q, (ab)^2p^2, \ldots.$$

Hence, we can write

$$d_{2i-3} = (a^2 + b)\ell + a^k b^k s,$$

for some $\ell \in \mathbb{Z}$, some $k \geq 1$, and some $s \in \{p, q, p^2\}$. Since $d_{2i-3}|(a^2 + b)$, $d_{2i-3}|a^k b^k s$. By Lemma 1 item iii), $d_{2i-3}|s$; that is, $d_{2i-3} \leq p^2$. However, $d_{2i-3} \geq d_5 > d_4 = p^2$, a contradiction.
We conclude that $pd_{2i-3} \geq d_{2i+2}$. Since $pd_{2i-3} \in S'_N$, we know that $d_{2i+2} \in S'_N$ and $|S_N| \geq 2i + 2$.

**Proposition 4.** *Suppose that the first 4 numbers in $S'_N$ are $p < q < p^2 < r$. If $N$ is small recurrent, then $|S'_N| \neq 4, 6$.*

*Proof.* If $|S'_N| = 4$, then (2) gives $\tau(N) = 10$ or 11. Note that $N$ has three distinct prime factors $p, q, r$ and the power of $p$ is at least 2. Since $2^3 \cdot 3 > 11$, $N$ cannot have another prime factor besides $p, q, r$. Write $N = p^a q^b r^c$, for some $a \geq 2, b \geq 1, c \geq 1$. However, neither $(a + 1)(b + 1)(c + 1) = 10$ nor $(a + 1)(b + 1)(c + 1) = 11$ has a solution. Therefore, $|S'_N| \neq 4$. A similar argument gives $|S'_N| \neq 6$.

By Propositions 3 and 4, we know that $|S'_N| = 5$; that is, $\tau(N) = 12$ or 13. Using the same reasoning as in the proof of Proposition 4, we know that $\tau(N) = 12$ and $N = p^2qr$, where $p < q < p^2 < r$.

**When $d_4 = r$ for some $r > q$** The possible values for $d_5$ are $p^2, pq, s$ for some prime $s > r$.

a) If $d_5 = p^2$, we can generalize the method by Chentouf.

**Proposition 5.** *If $N$ is small recurrent and the first four numbers of $S'_N$ are $p < q < r < p^2$, then $|S_N| \leq 7$. As a result, $|S'_N| \leq 6$.*

*Proof.* Suppose that $|S_N| \geq 8$. We show that $|S_N| \geq 3i+2$ for all $i \in \mathbb{N}$, which is a contradiction. The claim holds for $i = 2$. Assume that $|S_N| \geq 3j+2$ for some $j \geq 2$. By Lemma 2, $p \nmid d_{3j}d_{3j+1}$ and $\gcd(d_{3j}, d_{3j+1}) = 1$. Hence, $p^2 d_{3j}d_{3j+1}$ divides $N$, which implies that $pd_{3j} \in S'_N$.
If $pd_{3j} = d_{3j+2} = ad_{3j+1} + bd_{3j}$, then $d_{3j}$ divides $ad_{3j+1}$. By Lemma 2, $d_{3j}|a$. Observe that for $d_i \in S'_N$, the sequence $d_i \bmod a$ is

$$1, p, q, bp, bq, b^2p, b^2q, \ldots.$$

Write $d_{3j} = a\ell + b^k s$, for some $\ell \in \mathbb{Z}$, some $k \geq 0$, and some $s \in \{p, q\}$. We see that $d_{3j}|b^k s$ for some $k \geq 0$ and $s \in \{p, q\}$. By Lemma 2, $d_{3j} \leq q$. However,

$$d_{3j} \ \geq \ d_6 \ > \ d_3 \ = \ q,$$

a contradiction.
If $pd_{3j} > d_{3j+2}$, then $pd_{3j} \geq d_{3(j+1)+2}$ by Lemma 2. Therefore, $|S_N| \geq 3(j+1)+2$.

**Proposition 6.** *There is no small recurrent $N$ whose the first four numbers of $S'_N$ are $p < q < r < p^2$.*

*Proof.* By Proposition 5, $|S'_N| \in \{4, 5, 6\}$. If $|S'_N| = 4$, then $\tau(N) = 10$ or $11$, none of which can be written as a product of at least three integers, each of which is at least 2. This contradicts (2) and the fact that $N$ has three distinct prime factors. We arrive at the same conclusion when $|S'_N| = 6$. For $|S'_N| = 5$, we obtain $N = p^2qr$ for some primes $p < q < r < p^2$. However, this poses another contradiction. Observe that $(pq)^2 < p^2qr$, so the divisors in $S'_N$ are $p < q < r < p^2 < pq$. Since $pq = ap^2 + br$, we get $p|b$, which contradicts Lemma 2 item ii).

b) Suppose that $d_5 = pq$.

**Proposition 7.** *There is no small recurrent number $N$ such that the first four numbers of $S'_N$ are $p < q < r < pq$.*

*Proof.* Assume that $|S'_N| \geq 8$. Since $p^2 \notin S'_N$, $p$ divides $N$ exactly. By Lemma 3, $d_6$ is divisible neither by $p$ nor $q$. Hence, $d_6 = s$ for some prime $s > pq$. The divisor $d_7$ is divisble by $q$; hence, $d_7 = qr$ or $q^2$. The divisor $d_8$ is divisible by $p$ but not by $q$. So, $d_8 = pr$, which gives that $d_7$ must be $q^2$ because $d_7 < d_8$. Now $q|d_9$ and $p \nmid d_9 \implies d_9 = qr$. However, that $\gcd(d_8, d_9) = r$ contradicts Lemma 3. Therefore, $|S'_N| \in \{4, 5, 6, 7\}$. Using the same argument as in the proof of Proposition 6, we know that $|S'_N| \neq 4, 6$ and so, $|S'_N| \in \{5, 7\}$. By the above argument, if $|S'_N| \geq 5$, then $d_6$ is a prime greater than $pq$. Hence, $N$ has at least 4 distinct prime factors, so $\tau(N)$ can be written as a product of at least 4 integers greater than 1. Clearly, (1) rules out the case $|S'_N| = 5$. If $|S'_N| = 7$, the above argument shows that $q^2|N$; hence, $\tau(N)$ can be written as a product of at least 4 integers greater than 1, one of which is greater than 2. This cannot happen as $\tau(N) \in \{16, 17\}$.

c) Suppose that $d_5 = s$.

**Proposition 8.** *There is no small recurrent number $N$ such that the first four numbers of $S'_N$ are $p < q < r < s$.*

*Proof.* Observe that $pq$ and $pr$ are in $S'_N$. Let $d_j = pv$ be the largest element of $S'_N$ that is divisible by $p$. Clearly, $v > p$ and $j \geq 7$. By Lemma 4, $d_j, d_{j-1}$, and $d_{j-2}$ are pairwise coprime. Hence, $pvd_{j-1}d_{j-2}$ divides $N$, so $pd_{j-2} \in S'_N$. If $pd_{j-2} = d_{j-1}$, then $p|d_{j-1}$ and so, $p|\gcd(d_{j-1}, d_j)$, which contradicts Lemma 4 item iii).
If $pd_{j-2} = d_j$, then $d_{j-2} = v$ and $\gcd(d_{j-2}, d_j) = v > 1$, which contradicts Lemma 4 item v).
Therefore, we have $pd_{j-2} > d_j$, which, however, contradicts that $d_j$ is the largest element of $S'_N$ that is divisible by $p$. We conclude that there is no small recurrent number $N$ such that the first four numbers of $S'_N$ are $p < q < r < s$.

From the above analysis, we arrive at the following proposition.

**Proposition 9.** *If $N$ is small recurrent and $|S'_N| \geq 4$, then $N$ belongs to one of the following forms.*

*(S1) $N = p^k$ or $N = p^kq$ for some $k \geq 1$ and $q > p^k$.*
*(S2) $N = pq^k$ or $pq^kr$ for some $k \geq 2$, $p < q$, and $pq^k < r$.*
*(S3) $N = p^kqr$ for some $k \geq 2$, some prime $r > p^kq$, and $\sqrt{q} < p < q$.*
*(S4) $N = p^kq$ for some $k \geq 2$ and $\sqrt{q} < p < q$.*
*(S5) $N = p^2q^2$ for some $p < q < p^2$.*
*(S6) $N = p^3q^2$ for some $p^{3/2} < q < p^2$.*
*(S7) $N = p^2qr$, where the first four numbers in $S'_N$ are $p < q < p^2 < r$.*

These forms together establish the necessary condition for a small recurrent $N$ to have $|S'_N| \geq 4$. We now refine each form (if necessary) to obtain a necessary and sufficient condition.

(S1)
- If $N = p^k$, then $N$ is small recurrent with $|S'_N| \geq 4$ if $k \geq 9$. In this case, $S'_N = \{p, p^2, p^3, \ldots, p^{\lfloor (k-1)/2 \rfloor}\}$ satisfies $U(p, p^2, p, 0)$.
- If $N = p^kq$ for some $k \geq 1$ and $q > p^k$, then $N$ is small recurrent with $|S'_N| \geq 4$ if $k \geq 4$. In this case, $S'_N = \{p, p^2, p^3, \ldots, p^k\}$ satisfies $U(p, p^2, p, 0)$.

(S2)
- If $N = pq^k$ for some $k \geq 2$ and $p < q$, then $N$ is small recurrent with $|S'_N| \geq 4$ if $\sqrt{N} = \sqrt{pq^k} > q^2$. Hence, $N = pq^k$ for some $k \geq 4$ and $p < q$. In this case, $S'_N = \{p, q, pq, q^2, \ldots\}$ satisfies $U(p, q, 0, q)$.
- If $N = pq^kr$ for some $k \geq 2$, $p < q$, and $r > pq^k$, then $N$ is small recurrent with $|S'_N| \geq 4$. In this case, $S'_N = \{p, q, pq, q^2, \ldots, pq^{k-1}, q^k, pq^k\}$ satisfies $U(p, q, 0, q)$.

(S3) If $N$ belongs to (S3), then $S'_N = \{p, q, p^2, pq, \dots, p^k, p^{k-1}q, p^kq\}$. Since $p^2 = aq + bp$, we know that $p|a$. Write $a = pm$ for some $m \in \mathbb{Z}$ and get $b = p - mq$. Hence,

$$pq \;=\; ap^2 + bq \;=\; p^3m + (p - mq)q \implies mq^2 = p^3m.$$

Therefore, $(m, a, b) = (0, 0, p)$. However, the largest element in $S'_N$, $p^kq$, is not equal to $p \cdot p^k$. We conclude that form (S3) does not give a small recurrent number.

(S4) If $N = p^kq$ for some $k \geq 2$ and $\sqrt{q} < p < q$, then the nontrivial divisors of $N$ in increasing order is $p < q < p^2 < pq < \cdots$. In order that $|S'_N| \geq 4$, we need $(pq)^2 < p^kq$, so $q < p^{k-2}$. Hence, $k \geq 4$. In this case, $S'_N = \{p, q, p^2, pq, \dots\}$ satisfies $U(p, q, 0, p)$.

(S5) If $N = p^2q^2$ for some $p < q < p^2$, then $\tau(N) = 9$. However, if $|S'_N| \geq 4$, then $\tau(N) \geq 10$ by (2). We conclude that form (S5) does not give a small recurrent number.

(S6) If $N = p^3q^2$ for some $p^{3/2} < q < p^2$, then $S'_N = \{p, q, p^2, pq, p^3\}$ satisfies $U(p, q, 0, p)$.

(S7) Let $N$ have form (S7). Since the first four numbers of $S'_N$ are $p < q < p^2 < r$ and $\tau(N) = 12$, we know that the fifth number in $S'_N$ must be $pq$. That $p < q < p^2 < r < pq$ satisfies some $U(p, q, a, b)$ gives $a = \frac{p(pq-r)}{q^2-p^3}$, $b = \frac{rq-p^4}{q^2-p^3}$, and $r = pq - \sqrt{(q^2 - p^3)(p^2 - q)}$. We conclude that a number of form (S7) is small recurrent if and only if $p < q < p^2 < r < pq$, $(q^2 - p^3)|(pq - r)$, $(q^2 - p^3)|(rq - p^4)$, and $r = pq - \sqrt{(q^2 - p^3)(p^2 - q)}$. An example is $(p, q, r) = (2, 3, 5)$. We do not know if $(2, 3, 5)$ is the only set of primes that satisfy all these conditions or not.

From the above analysis, we obtain the proposition, which is a refinement of Proposition 9.

**Proposition 10.** *Let $p, q, r$ denote prime numbers and $k$ be some positive integer. A positive integer $N$ is small recurrent with $|S'_N| \geq 4$ if and only if $N$ belongs to one of the following forms.*

1. *$N = p^k$ for some $k \geq 9$. In this case, $S'_N = \{p, p^2, p^3, \dots, p^{\lfloor (k-1)/2 \rfloor}\}$ satisfies $U(p, p^2, p, 0)$.*
2. *$N = p^kq$ for some $k \geq 4$ and $q > p^k$. In this case, $S'_N = \{p, p^2, p^3, \dots, p^k\}$ satisfies $U(p, p^2, p, 0)$.*
3. *$N = pq^k$ for some $k \geq 4$ and $p < q$. In this case, $S'_N = \{p, q, pq, q^2, \dots\}$ satisfies $U(p, q, 0, q)$.*
4. *$N = pq^kr$ for some $k \geq 2$, $p < q$, and $r > pq^k$. In this case, $S'_N = \{p, q, pq, q^2, \dots, pq^{k-1}, q^k, pq^k\}$ satisfies $U(p, q, 0, q)$.*
5. *$N = p^kq$ for some $k \geq 4$ and $\sqrt{q} < p < q$. In this case, $S'_N = \{p, q, p^2, pq, \dots\}$ satisfies $U(p, q, 0, p)$.*
6. *$N = p^3q^2$ for some $p^{3/2} < q < p^2$. In this case, $S'_N = \{p, q, p^2, pq, p^3\}$ satisfies $U(p, q, 0, p)$.*
7. *$N = p^2qr$, where $p < q < p^2 < r < pq$, $(q^2 - p^3)|(pq - r)$, $(q^2 - p^3)|(rq - p^4)$, and $r = pq - \sqrt{(q^2 - p^3)(p^2 - q)}$. In this case, $S'_N = \{p, q, p^2, r, pq\}$ satisfies $U\left(p, q, \frac{p(pq-r)}{q^2-p^3}, \frac{rq-p^4}{q^2-p^3}\right)$.*

### 3.2 The case $|S'_N| \leq 3$

If $|S'_N| \leq 3$, then $\tau(N) \leq 9$. We use the classifications of those $N$ from the introduction to obtain the following proposition.

**Proposition 11.** *Let $p, q, r$ denote prime numbers and $k$ be some positive integer. A positive integer $N > 1$ is small recurrent with $|S'_N| \leq 3$ if and only if $N$ belongs to one of the following forms.*

1. *$N = p^k$ for some $k \leq 8$. In this case, $S'_N = \{p, p^2, \ldots, p^{\lfloor (k-1)/2 \rfloor}\}$ satisfies $U(p, p^2, p, 0)$.*
2. *$N = pq$ for some $p < q$. In this case, $S'_N = \{p\}$.*
3. *$N = pq^2$ for some $p < q$. In this case, $S'_N = \{p, q\}$.*
4. *$N = p^2 q$ for some $p < q$. If $q < p^2$, then $S'_N = \{p, q\}$. If $q > p^2$, then $S'_N = \{p, p^2\}$.*
5. *$N = pq^3$ for some $p < q$. In this case, $S'_N = \{p, q, pq\}$.*
6. *$N = p^3 q$ for some $p < q$. If $p < q < p^2$, then $S'_N = \{p, q, p^2\}$. If $p^3 < q$, then $S'_N = \{p, p^2, p^3\}$. (The case $p^2 < q < p^3$ is eliminated because the three elements in $S'_N$ would be $p < p^2 < q$. However, there is no integral solution $(a, b)$ to $q = ap^2 + bp$.)*
7. *$N = p^2 q^2$ for some $p < q$. If $p < q < p^2$, then $S'_N = \{p, q, p^2\}$. (The case $p^2 < q$ is eliminated due to the same reason as in item (6).)*
8. *$N = pqr$ for some $p < q < r$. If $r < pq$ and there is an integral solution $(a, b)$ to $r = aq + bp$, then $S'_N = \{p, q, r\}$. If $r > pq$, then $S'_N = \{p, q, pq\}$.*

Combining Propositions 10 and 11, we obtain Theorem 1.

## 4 Large recurrent numbers

Now we characterize all positive integers $N$ whose $L'_N$ satisfies a linear recurrence of order at most two. By a simple observation, instead of working directly with divisors in $L'_N$, we work with divisors in $S'_N$. Again, the set of divisors of a positive integer $N$ is $1 = d_1 < d_2 < \cdots < d_{\tau(N)}$ and the set $S'_N = \{d_2, d_3, \ldots\}$.

### 4.1 The case $|L'_N| \geq 4$

Note that $|L'_N| \geq 4$ is equivalent to $|S'_N| \geq 4$.

**Lemma 5.** *For any $d \in L'_N$, we have $N/d \in S'_N$. If $N$ is large recurrent with $|L'_N| \geq 4$, then*

$$ad_{i+2} + bd_{i+1} = \frac{d_{i+1} d_{i+2}}{d_i}, \forall d_i, d_{i+1}, d_{i+2} \in S'_N. \quad (3)$$

*In particular, we have*

$$ad_4 + bd_3 = \frac{d_3 d_4}{d_2}. \quad (4)$$

*Proof.* If $d \in L'_N$, then $\sqrt{N} < d < N$. Then $1 < N/d < \sqrt{N}$, so $N/d \in S'_N$. Let

$$d'_i \;:=\; d_{\tau(n)+1-i} \;=\; \frac{N}{d_i} \in L'_N, \forall d_i \in S'_N.$$

If $N$ is large recurrent, then we have

$$d'_i \;=\; ad'_{i+1} + bd'_{i+2}, \forall d'_i, d'_{i+1}, d'_{i+2} \in L'_N.$$

Therefore,

$$\frac{N}{d_i} \;=\; a\frac{N}{d_{i+1}} + b\frac{N}{d_{i+2}}, \forall d_i, d_{i+1}, d_{i+2} \in S'_N,$$

which gives

$$ad_{i+2} + bd_{i+1} \;=\; \frac{d_{i+1}d_{i+2}}{d_i}, \forall d_i, d_{i+1}, d_{i+2} \in S'_N.$$

This completes our proof.

Since $d_2$ is a prime number $p$ and $d_3$ is either $p^2$ or a prime number $q > p$, we consider two cases.

**When $d_3 = p^2$** Then $d_4$ is either $p^3$ or a prime number $q > p^2$.

a) If $d_4 = p^3$, then (4) implies that $p^2 = ap + b$.

*Claim.* If $p \neq a$, then $S'_N = \{p, p^2, \ldots, p^k\}$ for some $k \geq 4$.

*Proof.* We need to show that if $d_i \in S'_N$, then $d_i = p^{i-1}$. Base case: the claim holds for $i \leq 4$. Suppose that there exists a $j \geq 4$ such that $d_i = p^{i-1}$ for all $i \leq j$. Using (3), we have

$$ad_{j+1} + bp^{j-1} \;=\; ad_{j+1} + bd_j \;=\; \frac{d_{j+1}d_j}{d_{j-1}} \;=\; \frac{d_{j+1}p^{j-1}}{p^{j-2}} \;=\; pd_{j+1},$$

which, combined with $p^2 = ap + b$, gives

$$(p-a)(d_{j+1} - p^j) \;=\; 0.$$

Since $p \neq a$, we obtain $d_{j+1} = p^j$, as desired.

By Proposition 2, we know that when $p \neq a$, either $N = p^k$ or $N = p^k q$ for some $k \geq 1$ and some prime $q > p^k$.

Now suppose that $p = a$. Then $b = 0$. We can write elements in $L'_N$ as $\{g_1, g_2, pg_2, p^2g_2, \ldots, p^kg_2\}$ for some $k \geq 2$. Correspondingly, the set $S'_N$ is $\{p, p^2, \ldots, p^k, p^{k+1}, p^{k+1}g_2/g_1\}$. If $p^{k+1}g_2/g_1$ is a power of $p$, then we have the same conclusion about $N$ as when $p \neq a$. If $p^{k+1}g_2/g_1$ is not a power of $p$, then

$$\frac{p^{k+1}g_2}{g_1} \;=\; q, \text{ for some prime } q > p \implies g_1 \;=\; p^{k+1}\frac{g_2}{q}.$$

Note that $g_2/q \in \mathbb{N}$. Furthermore, we claim that $g_2/q = p$. Indeed, since $1 < g_2/q < g_1$, we know that $g_2/q \in S'_N$. If $g_2/q = q$, then

$$pq \;<\; p^{k+1}\frac{g_2}{q} \;=\; g_1,$$

which implies that $pq \in S'_N$, a contradiction. If $g_2/q = p^j$ for some $j > 1$, then

$$p^{k+2} \;<\; p^{k+1+j} \;=\; p^{k+1}\frac{g_2}{q} \;=\; g_1,$$

which implies that $p^{k+2} \in S'_N$, another contradiction. Therefore, $g_2/q = p$, and we obtain $g_1 = p^{k+2}$ and $g_2 = pq$. Hence, $N = p^{k+2}q$ for some $k \geq 2$ and $p^{k+1} < q < p^{k+2}$.

b) If $d_4 = q$, we claim that $a \neq p$. Suppose otherwise. Applying (4) to $d_2, d_3$, and $d_4$ gives $aq + bp^2 = pq$. Hence, $a = p$ implies that $b = 0$. However, applying (3) to $d_3, d_4$, and $d_5$ gives $(p^3 - q)d_5 = 0$, a contradiction. Therefore, $a \neq p$. By (4), we have

$$d_4 \;=\; \frac{bp^2}{p-a} \;=\; q \;\Longrightarrow\; q|b \;\Longrightarrow\; b = kq \text{ for some } k \in \mathbb{Z}\backslash\{0\}.$$

Hence, $a = p - kp^2$. By (3) applied to $d_3, d_4$, and $d_5$,

$$d_5 \;=\; \frac{kp^2q^2}{q - p^3 + kp^4}, \tag{5}$$

which implies that $p^2|d_5$ since $\gcd(p^2, q - p^3 + kp^4) = 1$. Hence, $d_5 = p^3$, and (5) gives $k = p(p^3 - q)/(p^5 - q^2)$.

Case b.i) If $S'_N$ has exactly four elements, which are $p, p^2, q, p^3$, then $\tau(N) = 10$, which implies that $N = p^4q$. Hence, $L'_N = \{pq, p^4, p^2q, p^3q\}$ with $a = pq(p^2 - q)/(p^5 - q^2)$ and $b = pq(p^3 - q)/(p^5 - q^2)$. We conclude that $N = p^4q$ $(p^2 < q < p^3)$, $(p^5 - q^2)|(p^2 - q)$, and $(p^5 - q^2)|(p^3 - q)$.

Case b.ii) If $|S'_N| > 4$, then (3) gives

$$d_6 \;=\; \frac{bp^3q}{p^3 - aq} \;\Longrightarrow\; q|d_6 \;\Longrightarrow\; d_6 \;=\; pq.$$

However, since $(a, b) = (p(1 - kp), kq)$, we have

$$pq \;=\; d_6 \;=\; \frac{bp^3q}{p^3 - aq} \;=\; \frac{kp^2q^2}{p^2 - q(1 - kp)},$$

which gives $p^2 = q$, a contradiction.

We summarize our result when $d_3 = p^2$.

**Proposition 12.** *A number $N$ is large recurrent with $|L'_N| \geq 4$ and $(d_2, d_3) = (p, p^2)$ for some prime $p$ if and only if $N$ belongs to one of the following forms.*

1. *$N = p^k$ for some $k \geq 9$. Then $L'_N = \{p^{\lceil (k-1)/2 \rceil + 1}, p^{\lceil (k-1)/2 \rceil + 2}, \ldots, p^{k-1}\}$ satisfies $U(p^{\lceil (k-1)/2 \rceil + 1}, p^{\lceil (k-1)/2 \rceil + 2}, p, 0)$.*
2. *$N = p^k q$ for some $k \geq 4$ and $q > p^k$. In this case, $L'_N = \{q, pq, p^2q, \ldots, p^{k-1}q\}$ satisfies $U(q, pq, p, 0)$.*
3. *$N = p^k q$ for some $k \geq 4$ and $p^{k-1} < q < p^k$. Then*

$$L'_N = \{p^k, pq, p^2q, \ldots, p^{k-1}q\}$$

*satisfies $U(p^k, pq, p, 0)$.*
4. *$N = p^4 q$ with $p^2 < q < p^3$, $(p^5 - q^2)|(p^2 - q)$, and $(p^5 - q^2)|(p^3 - q)$. In this case, $L'_N = \{pq, p^4, p^2q, p^3q\}$.*

**When $d_3 = q$** By (3),

$$p(ad_4 + bq) = d_4 q \implies p|d_4.$$

Write $d_4 = kp$ for some integer $k$. Since $d_2 = p$ and $d_3 = q$, $d_4$ must be either $p^2$ or $pq$.

a) If $d_4 = p^2$, (3) gives $ap^2 + bq = pq$. Hence, $q|a$ and $p|b$. Write $a = mq$ and $b = np$ for some integers $m, n$ to get $mp + n = 1$. By (3), we see that

$$d_5 = \frac{bp^2q}{p^2 - aq} \implies q|d_5 \implies d_5 = pq.$$

Therefore,

$$\frac{bp^2q}{p^2 - aq} = \frac{np^3q}{p^2 - mq^2} = \frac{(1-mp)p^3q}{p^2 - mq^2} = pq \implies m(p^3 - q^2) = 0,$$

which gives $m = 0$ and so, $(a, b) = (0, p)$. By (3), $d_{i+2} = pd_i$ for all $d_i, d_{i+2} \in S'_N$ and

$$S'_N = \{p, q, p^2, pq, \ldots\}.$$

If $|S'_N| = 4$, then $\tau(N) = 10$ and $\underline{N = p^4q \text{ for } p < q < p^2}$. Suppose that $|S'_N| \geq 5$, then $p^3 \in S'_N$.

Case a.i) If $q^2|N$, let $k \geq 2$ and $\ell \geq 3$ be the largest power such that $q^k|N$ and $p^k|N$, respectively. Since $q^2 \notin S'_N$, we know that

$$q^4 \geq N \geq p^3q^k > q^{k+3/2} \implies k < 5/2.$$

It follows that $k = 2$. That $q^2 < p^2q$ implies that

$$(p^2q)^2 > N \geq p^\ell q^2 \implies 3 \geq \ell \geq 3.$$

Hence, $\ell = 3$. If $N$ does not have any other prime divisors besides $p$ and $q$, then $\underline{N = p^3q^2 \text{ for } p < q < p^2}$. If $N$ has a prime divisor $r \neq p, q$, then $r > \sqrt{N}$. So, $r$ must be the unique prime divisor different from $p$ and $q$. We have $N = p^3q^2r$ for $p < q < p^2$ and $r > p^3q^2$. Then $q^2 \in S'_N$, a contradiction.

Case a.ii) If $q^2 \nmid N$ and $N$ has no prime divisors other than $p$ and $q$, then $N = p^k q$ some for $k \geq 2$ and $p < q < p^2$.

Case a.iii) If $q^2 \nmid N$ and there exists a prime divisor $r$ other than $p$ or $q$, then $r > \sqrt{N}$ and $r$ is the unique prime different from $p$ and $q$. Therefore, $N = p^k qr$ for some $k \geq 2$ and $p < q < p^2 < p^k q < r$. Note that the two largest elements in $S'_N$ are $p^{k-1}q$ and $p^k q$. Let $d$ be the third largest divisor in $S'_N$. The relation $d_{i+2} = pd_i$ for all $d_i, d_{i+2} \in S'_N$ gives that $dp = p^k q$ and so, $d = p^{k-1}q$, which contradicts that $p^{k-1}q$ is the second largest in $S'_N$.

b) If $d_4 = pq$, then $p^2 \nmid N$ since $p^2 < pq$. By (4),

$$ap = q - b. \tag{6}$$

We see that $d_5$ is equal to $q^2$ or $r$, for some prime $r > pq$.

Case b.i) If $d_5 = q^2$, then (3) gives

$$bp = (p - a)q. \tag{7}$$

From (6) and (7), we obtain $a(p^2 - q) = 0$, so $(a, b) = (0, q)$. By (3), $d_{i+2} = qd_i$ for all $d_i, d_{i+2} \in S'_N$. Using [1, Proposition 5], we conclude that $N = pq^k$ or $N = pq^k r$ for some $k \geq 2$ and $p < q < pq^k < r$.

Case b.ii) If $d_5 = r$, then we claim that $|S'_N| > 4$. If not, $|S'_N| = 4$ implies that $\tau(N) = 10$, which contradicts that $N$ has three distinct prime divisors. By (3), we see that

$$pq(ad_6 + br) = d_6 r,$$

so $pq|d_6$. So, $d_6 \in \{pq^2, pqr\}$. If $d_6 = pq^2$, then $q^2 < d_6$, but $q^2$ does not appear before $d_6$ in $S'_N$, a contradiction. If $d_6 = pqr$, then $pr < d_6$, but $pr$ does not appear before $d_6$ in $S'_N$, again a contradiction.

**Proposition 13.** *A number $N$ is large recurrent with $|L'_N| \geq 4$ and $(d_2, d_3) = (p, q)$ for some primes $p < q$ if and only if $N$ belongs to one of the following forms.*

1. *$N = p^3 q^2$ for $p < q < p^2$. In this case, $L'_N = \{q^2, p^2 q, pq^2, p^3 q, p^2 q^2\}$ satisfies $U(q^2, p^2 q, 0, p)$.*
2. *$N = p^k q$ some for $k \geq 4$ and $p < q < p^2$. In this case,*

$$L'_N = \begin{cases} \{p^{k/2+1}, p^{k/2}q, p^{k/2+2}, \ldots, p^{k-1}q\} & \text{if } 2|k, \\ \{p^{(k-1)/2}q, p^{(k+3)/2}, p^{(k+1)/2}q, \ldots, p^{k-1}q\} & \text{if } 2 \nmid k. \end{cases}$$

   *Observe that $L'_N$ satisfies $U(p^{k/2+1}, p^{k/2}q, 0, p)$ and $U(p^{(k-1)/2}q, p^{(k+3)/2}, 0, p)$ for even and odd $k$, respectively.*
3. *$N = pq^k$ for some $k \geq 4$ and $p < q$. In this case,*

$$L'_N = \begin{cases} \{pq^{k/2}, q^{k/2+1}, \ldots, q^k\} & \text{if } 2|k, \\ \{q^{(k+1)/2}, pq^{(k+1)/2}, \ldots, q^k\} & \text{if } 2 \nmid k. \end{cases}$$

   *Observe that $L'_N$ satisfies $U(pq^{k/2}, q^{k/2+1}, 0, q)$ and $U(q^{(k+1)/2}, pq^{(k+1)/2}, 0, q)$ for even and odd $k$, respectively.*

4. $N = pq^k r$ *for some* $k \geq 2$ *and* $p < q < pq^k < r$. *In this case,* $L'_N = \{r, pr, qr, pqr, q^2r, \ldots, q^k r\}$ *satisfies* $U(r, pr, 0, q)$.

Combining Propositions 12 and 13, we obtain the following.

**Proposition 14.** *A number* $N$ *is large recurrent with* $|L'_N| \geq 4$ *if and only if* $N$ *belongs to one of the following forms.*

1. $N = p^k$ *for some* $k \geq 9$. *Then* $L'_N = \{p^{\lceil (k-1)/2 \rceil + 1}, p^{\lceil (k-1)/2 \rceil + 2}, \ldots, p^{k-1}\}$ *satisfies* $U(p^{\lceil (k-1)/2 \rceil + 1}, p^{\lceil (k-1)/2 \rceil + 2}, p, 0)$.
2. $N = p^k q$ *for some* $k \geq 4$ *and* $q > p^k$. *In this case,* $L'_N = \{q, pq, p^2q, \ldots, p^{k-1}q\}$ *satisfies* $U(q, pq, p, 0)$.
3. $N = p^k q$ *for some* $k \geq 4$ *and* $p^{k-1} < q < p^k$. *Then*
$$L'_N = \{p^k, pq, p^2q, \ldots, p^{k-1}q\}$$
*satisfies* $U(p^k, pq, p, 0)$.
4. $N = p^k q$ *some for* $k \geq 4$ *and* $p < q < p^2$. *In this case,*
$$L'_N = \begin{cases} \{p^{k/2+1}, p^{k/2}q, p^{k/2+2}, \ldots, p^{k-1}q\} & \text{if } 2|k, \\ \{p^{(k-1)/2}q, p^{(k+3)/2}, p^{(k+1)/2}q, \ldots, p^{k-1}q\} & \text{if } 2 \nmid k. \end{cases}$$
*Observe that* $L'_N$ *satisfies* $U(p^{k/2+1}, p^{k/2}q, 0, p)$ *and* $U(p^{(k-1)/2}q, p^{(k+3)/2}, 0, p)$ *for even and odd* $k$, *respectively.*
5. $N = p^4 q$ *with* $p^2 < q < p^3$, $(p^5 - q^2)|(p^2 - q)$, *and* $(p^5 - q^2)|(p^3 - q)$. *In this case,* $L'_N = \{pq, p^4, p^2q, p^3q\}$.
6. $N = p^3 q^2$ *for* $p < q < p^2$. *In this case,* $L'_N = \{q^2, p^2q, pq^2, p^3q, p^2q^2\}$ *satisfies* $U(q^2, p^2q, 0, p)$.
7. $N = pq^k$ *for some* $k \geq 4$ *and* $p < q$. *In this case,*
$$L'_N = \begin{cases} \{pq^{k/2}, q^{k/2+1}, \ldots, q^k\} & \text{if } 2|k, \\ \{q^{(k+1)/2}, pq^{(k+1)/2}, \ldots, q^k\} & \text{if } 2 \nmid k. \end{cases}$$
*Observe that* $L'_N$ *satisfies* $U(pq^{k/2}, q^{k/2+1}, 0, q)$ *and* $U(q^{(k+1)/2}, pq^{(k+1)/2}, 0, q)$ *for even and odd* $k$, *respectively.*
8. $N = pq^k r$ *for some* $k \geq 2$ *and* $p < q < pq^k < r$. *In this case,* $L'_N = \{r, pr, qr, pqr, q^2r, \ldots, q^k r\}$ *satisfies* $U(r, pr, 0, q)$.

### 4.2 The case $|L'_N| \leq 3$

If $|L'_N| \leq 3$, then $\tau(N) \leq 9$. We use the classifications of those $N$ from the introduction to obtain the following proposition

**Proposition 15.** *Let* $p, q, r$ *denote prime numbers and* $k$ *be some positive integer. A positive integer* $N > 1$ *is small recurrent with* $|L'_N| \leq 3$ *if and only if* $N$ *belongs to one of the following forms.*

1. *$N = p^k$ for some $k \leq 8$. In this case, $L'_N = \{p^{\lceil (k-1)/2 \rceil + 1}, \ldots, p^{k-1}\}$ satisfies $U(p^{\lceil (k-1)/2 \rceil + 1}, p^{\lceil (k-1)/2 \rceil + 2}, p, 0)$.*
2. *$N = pq$ for some $p < q$. In this case, $L'_N = \{q\}$.*
3. *$N = pq^2$ for some $p < q$. In this case, $L'_N = \{pq, q^2\}$.*
4. *$N = p^2 q$ for some $p < q$. If $q < p^2$, then $L'_N = \{p^2, pq\}$. If $q > p^2$, then $L'_N = \{q, pq\}$.*
5. *$N = pq^3$ for some $p < q$. In this case, $L'_N = \{q^2, pq^2, q^3\}$.*
6. *$N = p^3 q$ for some $p < q$. If $p < q < p^2$, then $L'_N = \{pq, p^3, p^2 q\}$. If $p^3 < q$, then $L'_N = \{q, pq, p^2 q\}$. If $p^2 < q < p^3$, then $L'_N = \{p^3, pq, p^2 q\}$.*
7. *$N = p^2 q^2$ for some $p < q$. If $p < q < p^2$, then $L'_N = \{q^2, p^2 q, pq^2\}$. The case $p^2 < q$ is impossible as it gives $L'_N = \{p^2 q, pq^2, q^2\}$ and there is no integral solution $(a, b)$ to $apq^2 + bp^2 q = q^2$.*
8. *$N = pqr$ for some $p < q < r$. If $r > pq$, then $L'_N = \{r, pr, qr\}$. The case $r < pq$ is impossible as it gives $L'_N = \{pq, pr, qr\}$ and there is no integral solution $(a, b)$ to $apr + bpq = qr$.*

Combining Propositions 14 and 15, we obtain Theorem 2.

## 5 Appendix

*Proof (Proof of Lemma 3).* i) Since $r = aq + bp$ and $pq = ar + bq$, we know that $\gcd(a, b) | r$ amd $\gcd(a, b) | pq$, respectively. Hence, $\gcd(a, b) = 1$.

Since $r = aq + bp$ and $r$ is a prime, $p \nmid a$.

ii) Suppose that $k = \gcd(b, d_i) > 1$ for some $d_i \in S'_N$. If $d_i = d_2$, then $p | b$. Since $pq = ar + bq$, we get $p | a$, which contradicts $\gcd(a, b) = 1$. If $d_i = d_3$, then $q | b$. Since $r = aq + bp$, we get $q | r$, a contradiction. If $d_i > d_3$, then write

$$\gcd(b, d_i) \;=\; \gcd(b, ad_{i-1} + bd_{i-2}) \;\overset{\text{i)}}{=}\; \gcd(b, d_{i-1}),$$

which, by induction, gives $1 < \gcd(b, d_i) = \gcd(b, d_3)$, which has been shown to be impossible.

iii) The claim holds for $i \leq 4$. Let $d_i, d_{i+1} \in S'_N$ for some $i \geq 5$. We have

$$\gcd(d_i, d_{i+1}) \;=\; \gcd(d_i, ad_i + bd_{i-1}) \;\overset{\text{ii)}}{=}\; \gcd(d_i, d_{i-1}).$$

By induction, we obtain $\gcd(d_i, d_{i+1}) = 1$.

iv) The claim holds for $i \leq 5$. Assume that it holds for all $i \leq j$ for some $j \geq 5$. We show that it holds for $i = j + 1$. We have

$$pq \;=\; ar + bq \;=\; a(aq + bp) + bq \;=\; (a^2 + b)q + abp.$$

Hence, $p | (a^2 + b)$. Write

$$d_{j+1} \;=\; ad_j + bd_{j-1} \;=\; a(ad_{j-1} + bd_{j-2}) + bd_{j-1} \;=\; (a^2 + b)d_{j-1} + abd_{j-2}.$$

Since $p | (a^2 + b)$ and $\gcd(p, ab) = 1$, we know that $p | d_{j+1}$ if and only if $p | d_{j-2}$. By the inductive hypothesis, $p | d_{j-2}$ if and only if $j - 2 \equiv 2 \bmod 3$, or equivalently, $j + 1 \equiv 2 \bmod 3$. By induction, we have the desired conclusion.

v) The claim holds for $i \leq 5$. Assume that it holds for all $i \leq j$ for some $j \geq 5$. We show that it holds for $i = j + 1$. That $pq = ar + bq$ implies that $q|a$. Write

$$d_{j+1} \ = \ ad_j + bd_{j-1}.$$

By ii), $q|d_{j+1}$ if and only if $q|d_{j-1}$. By the inductive hypothesis, $q|d_{j-1}$ if and only if $j + 1 \equiv 1 \bmod 2$. This completes our proof.

*Proof (Proof of Lemma 4).* i) Same as the proof of Lemma 3 item i).

ii) Suppose, for a contradiction, that $\gcd(b, d_i) > 1$ for some $i \geq 3$. If $i = 3$, then $q|b$. We have $r = aq + bp$. Since $q|b$, we get $q|r$, a contradiction. If $i \geq 4$, write

$$\gcd(b, d_i) \ = \ \gcd(b, ad_{i-1} + bd_{i-2}) \ = \ \gcd(b, d_{i-1}).$$

By induction, $1 < \gcd(b, d_i) = \gcd(b, d_3)$, which has been shown to be impossible.

iii) The claim holds for $i \leq 4$. Pick $i \geq 5$. We have

$$\gcd(d_i, d_{i+1}) \ = \ \gcd(d_i, ad_i + bd_{i-1}) \ = \ \gcd(d_i, bd_{i-1}) \ \overset{\text{ii)}}{=} \ \gcd(d_i, d_{i-1}).$$

By induction, we obtain $\gcd(d_i, d_{i+1}) = \gcd(d_4, d_5) = 1$.

iv) Assume that $\gcd(a, d_i) > 1$ for some $i \geq 2$. If $i = 2$, then $p|a$, which contradicts the primality of $r$ and the linear recurrence $r = aq+bp$. If $i = 3$, then $q|b$, which contradicts the primality of $s$ and the linear recurrence $s = ar + bq$. Assume that $i \geq 4$. Write

$$\gcd(a, d_i) \ = \ \gcd(a, ad_{i-1} + bd_{i-2}) \ \overset{\text{i)}}{=} \ \gcd(a, d_{i-2}).$$

By induction, either $1 < \gcd(a, d_i) = \gcd(a, d_2)$ or $1 < \gcd(a, d_i) = \gcd(a, d_3)$, neither of which is possible.

v) The claims holds for $i \leq 3$. Pick $i \geq 4$ and suppose that $k = \gcd(d_i, d_{i+2}) > 1$. Since $d_{i+2} = ad_{i+1} + bd_i$, $k$ divides $ad_{i+1}$. By iii), $\gcd(k, d_{i+1}) = 1$, so $k|a$. However, $\gcd(a, d_i) > 1$ contradicts iv).

## References


1. A. A Chentouf, Linear recurrences of order at most two in small divisors, *J. Integer Seq.* **25** (2022).
2. H. V. Chu, When the large divisors of a natural number are in arithmetic progression, *J. Integer Seq.* **23** (2020).
3. H. V. Chu, When the nontrivial, small divisors of a natural number are in arithmetic progression, *Quaest. Math.* **45** (2022), 969–977.
4. D. E. Iannucci, When the small divisors of a natural number are in arithmetic progression, *Integers* **18** (2018).