1- and 2-Level Densities for Rational Families of Elliptic Curves: Evidence for the Underlying Group Symmetries

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ABSTRACT

Following Katz-Sarnak [KS1], [KS2], Iwaniec-Luo-Sarnak [ILS], and Rubinstein [Ru], we use the 1- and 2-level densities to study the distribution of low lying zeros for one-parameter rational families of elliptic curves over $\mathbb{Q}(t)$. Modulo standard conjectures, for small support the densities agree with Katz and Sarnak's predictions. Further, the densities confirm that the curves' L-functions behave in a manner consistent with having r zeros at the critical point, as predicted by the Birch and Swinnerton-Dyer conjecture. By studying the 2-level densities of some constant sign families, we find the first examples of families of elliptic curves where we can distinguish SO(even) from SO(odd) symmetry.

1. Introduction

1.1 n-Level Correlations and Densities

Assuming GRH, the zeros of any L-function lie on the critical line, and therefore it is possible to investigate statistics of the normalized zeros. The general philosophy, born out in many examples (see [CFKRS]), is that the behavior of random matrices / ensembles of random matrices behave similar to that of L-functions / families of L-functions. By a family $\mathcal{F}$ we mean a collection of geometric objects and their associated L-functions, where the geometric objects have similar properties.

We expect there is a symmetry group $\mathcal{G}(\mathcal{F})$ (one of the classical compact groups $U(N)$, $SU(N)$, $USp(2N)$, SO(even) and SO(odd)) which can be associated to a family of L-functions, and that the behavior of eigenvalues of matrices in $\mathcal{G}(\mathcal{F})$ should (after appropriate normalizations) equal the behavior of zeros of L-functions.

Iwaniec, Luo and Sarnak [ILS] consider (among other examples) all cuspidal newforms of a given level and weight. Rubinstein [Ru] considers twists by fundamental discriminants D of a fixed modular form.

We study the family of all elliptic curves and various one-parameter families of elliptic curves. Thus, in our case the notion of family is the standard one from geometry: we have a collection of curves over a base, and the geometry is much clearer in our examples than in [ILS] and [Ru].

Let $\{\alpha_j\}$ be an increasing sequence of numbers tending to infinity, such as eigenvalues or zeros normalized to have mean spacing 1. For a compact box $B \subset \mathbb{R}^{n-1}$, define the *n-level correlation* by

$$\lim_{N \to \infty} \frac{\#\left\{(\alpha_{j_1} - \alpha_{j_2}, \ldots, \alpha_{j_{n-1}} - \alpha_{j_n}) \in B, j_i \in \{1, \ldots, N\}, j_i \neq j_k\right\}}{N} \tag{1.1}$$

Note that the n-level correlations are unaffected by removing finitely many zeros. Instead of using a box, one can study a smoothed version with a test function on $\mathbb{R}^n$ (see [RS]).

For test functions whose Fourier Transform has small support, Montgomery [Mon] proved the 2- and Hejhal [Hej] proved the 3-level correlations for the zeros of $\zeta(s)$ are the same as that of the GUE, and Rudnick-Sarnak [RS] proved the n-level correlations for all automorphic cuspidal L-functions are the same as the GUE. The universality is due to the fact that the correlations are controlled by the second moment of the a_p's, and while there are many possible limiting distributions, all have the same second moment.

Katz and Sarnak [KS1] prove the classical compact groups have the same n-level correlations. In particular, we cannot use the n-level correlations to distinguish GUE behavior, $U(N)$, from the other classical compact groups. We are led to investigate another statistic which will depend on the underlying group.

For L-functions of elliptic curves, the order of vanishing of $L(s, E)$ at $s = \frac{1}{2}$ is conjecturally equal to the geometric rank of the Mordell-Weil group (Birch and Swinnerton-Dyer conjecture). If we force the Mordell-Weil group to be large, we expect many zeros exactly at $s = \frac{1}{2}$, and this might influence the behavior of the neighboring zeros. Hence we are led to study the distribution of the first few, or low lying, zeros, and the fascinating possibility that there could be a difference in statistics for zeros near $\frac{1}{2}$ than for zeros higher up.

Let $f(x)$ be an even Schwartz function whose Fourier Transform is supported in a neighborhood of the origin. We assume f is of the form $\prod_{i=1}^{n} f_i(x_i)$. The n-level density for the family $\mathcal{F}$ with test function f is

$$D_{n,\mathcal{F}}(f) = \frac{1}{|\mathcal{F}|} \sum_{E \in \mathcal{F}} \sum_{\substack{j_1, \ldots, j_n \\ j_i \neq \pm j_k}} f_1\left(\frac{\log N_E}{2\pi} \gamma_E^{(j_1)}\right) \cdots f_n\left(\frac{\log N_E}{2\pi} \gamma_E^{(j_n)}\right), \tag{1.2}$$

where $\gamma_E^{(j_i)}$ runs through the non-trivial zeros of the curve E, and N_E is its conductor. We rescale the zeros by $\log N_E$ as this is the order of the number of zeros with imaginary part less than a large absolute constant (see [ILS]). As f_i is Schwartz, most of the contribution is due to the zeros near the critical point. We use the Explicit Formula (Equation 2.3) to relate sums of test functions over zeros to sums over primes of $a_E(p)$ and $a_E^2(p)$.

Katz and Sarnak [KS1] determine the $N \to \infty$ limits for the n-level densities of eigenvalues near 1 for the classical compact groups (see Section 3); their calculations can be modified to determine the densities of classical compact groups with a forced number of eigenvalues at 1. Forcing eigenvalues at 1 corresponds to L-functions with zeros forced at the critical point.

1.2 Results

To any geometric family in the function field case, the results of Katz and Sarnak ([KS1], [KS2]) state the n-level density of zeros near $\frac{1}{2}$ depends only on a symmetry group attached to the family. In particular, for generic families of elliptic curves the relevant symmetry is orthogonal. One can further analyze the distributions depending on the signs of the functional equations. As the families of elliptic curves are self-dual, we expect the densities to be controlled by the distribution of signs (all even: $SO(\text{even})$; all odd: $SO(\text{odd})$; equidistributed: O).

For an elliptic curve E_t, let $D(t)$ be the product of the irreducible polynomial factors of the discriminant $\Delta(t)$, and let $C(t)$ be the conductor. Let B be the largest square dividing $D(t)$ for all t. Pass to a subsequence $ct + t_0$, and call $t \in [N, 2N]$ *good* if $D(ct + t_0)$ is square-free, except for primes $p|B$ where the power of such $p|D(t)$ is independent of t.

The main result is Theorem 5.8:

Rational Surfaces Density Theorem: *Consider a one-parameter family of elliptic curves of rank r over $\mathbb{Q}(t)$ which constitutes a rational surface. Assume GRH, $j(E_t)$ non-constant, and if $\Delta(t)$ has an irreducible polynomial factor of degree at least 4, assume the ABC Conjecture.*

After passing to a subsequence, for t good, $C(t)$ is a polynomial. Let f_i be an even Schwartz function of small but non-zero support σ_i ($\sigma_1 < \min(\frac{1}{2}, \frac{2}{3m})$ for the 1-level density, $\sigma_1 + \sigma_2 < \frac{1}{3m}$ for the 2-level density).

The 1-level density agrees with the orthogonal densities plus a term which equals the contributions from r zeros at the critical point. The 2-level density agrees with $SO(even)$, O, and $SO(odd)$ depending on whether the signs are all even, equidistributed in the limit, or all odd, plus a term which equals the contribution from r zeros at the critical point. Thus, for small support, the densities of the zeros agree with Katz and Sarnak's predictions. Further, the densities confirm that the curves' L-functions behave in a manner consistent with having r zeros at the critical point, as predicted by the Birch and Swinnerton-Dyer conjecture.

The ABC Conjecture is used to handle large prime divisors of polynomials of degree 4 or more (see [Gr]). In place of ABC, one could assume the Square-Free Sieve Conjecture.

For the 1-level densities, the three orthogonal densities agree for test functions with support less than 1, split (ie, are distinguishable) for support greater than 1, but are all distinguishable from U and Sp for any support. Hence, unlike the n-level correlations, the 1-level density is already sufficient to observe non-GUE and non-symplectic behavior.

The polynomial growth of the conductor in families of elliptic curves makes it difficult to evaluate the sums over primes for test functions with moderate support. Converting to our language, for small support the 1-level densities for many families have been shown equal to the Katz-Sarnak predictions: all elliptic curves (Brumer and Heath-Brown [Br], [BHB5], support less than $\frac{2}{3}$); twists of a given curve (support less than 1); one-parameter families (Silverman [Si3], small support).

None of these are sufficient to distinguish the three orthogonal candidates. Further, previous investigations have rescaled each curve's zeros by the average of the logarithms of the conductors. This greatly simplifies the calculations; however, the normalization is no longer natural for each curve, as each curve can sit in infinitely many families, each with a different average spacing. By using local normalizations for each curve's zeros, the n-level density for a family becomes the average of the n-level densities for each curve.

The utility of the 2-level density is that, even for test functions with arbitrarily small support, the three candidate orthogonal symmetries *are* distinguishable, and in a very satisfying way. The three candidates differ by a factor which encodes the distribution of sign in the family, and all differ from the GUE's 2-level density.

We will study several families of constant sign, and we will see that the densities are as expected. Thus, for these constant sign families, the 2-level density reflects the predicted symmetry, which is invisible through the 1-level density because of support considerations.

Similar to the universality Rudnick and Sarnak [RS] found in studying n-level correlations, our universality follows from the sums of $a_t^2(p)$ in our families (the second moments). For non-constant $j(E_t)$, this follows from a Sato-Tate law proved by Michel [Mi] (Theorem 2.3).

1.3 Structure of the Paper
First, we calculate sums of the Fourier coefficients of elliptic curves. We quote the predicted densities, and then calculate useful expansions for the 1- and 2-level densities for families of elliptic curves

over $\mathbb{Q}(t)$. We derive the density results, conditional on the evaluation of many elliptic curve sums. We calculate these sums for one-parameter rational families of elliptic curves. We conclude with several examples (four constant sign families, a rank 1 and a rank 6 rational family).

We need excellent control over the conductors to evaluate the above sums; the estimation is so delicate that if the log of conductors are of size $m \log N$, fluctuations of size $O(1)$ yield error terms greater than the expected main terms.

The key observation is that the error terms can be controlled if the conductors are monotone. By straightforward sieving and applications of Tate's algorithm (to calculate the conductors), given a one-parameter rational family of elliptic curves, we may pass to a positive percent sub-family where the conductors are monotone. Proofs of these results are given in the appendices.

In this paper, we concentrate on rational elliptic surfaces, because here Tate's conjecture is known. Rosen and Silverman [RSi] show Tate's conjecture implies certain sums over primes are related to the rank of the family over $\mathbb{Q}(t)$. This will allow us to interpret some of our density terms as the contributions from r critical point zeros.

The modifications needed to handle the family of all elliptic curves, parametrized by

$$y^2 = x^3 + ax + b, \quad a \in [-N^2, N^2], \ b \in [-N^3, N^3], \tag{1.3}$$

are straightforward, and can be found in [Mil].

Finally, if instead we normalize by the average of the logarithms of the conductors, we obtain the same results, but with significantly less work. This is done for one-parameter families and the family of all elliptic curves in [Mil].

2. Elliptic Curve Preliminaries

2.1 Definitions

Consider a one-parameter family $\mathcal{E}$ of elliptic curves E_t over $\mathbb{Q}(t)$:

$$\mathcal{E} : y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t), \quad a_i(t) \in \mathbb{Z}[t]. \tag{2.1}$$

For each curve E_t, let $\Delta(t)$ be its discriminant and $C(t)$ its conductor. Let $D(t)$ denote the product of the irreducible polynomial factors dividing $\Delta(t)$. We will take $t \in [N, 2N]$ such that $D(t)$ is square-free.

Let $a_t(p) = a_{E_t}(p) = p + 1 - N_{t,p}$, where $N_{t,p}$ is the number of solutions of E_t mod p (including ∞). If $y^2 = x^3 + A(t)x + B(t)$, then

$$a_t(p) = -\sum_{t(p)} \left(\frac{x^3 + A(t)x + B(t)}{p} \right). \tag{2.2}$$

2.2 Assumptions

We assume the following at various points:

Generalized Riemann Hypothesis (for Elliptic Curves) *Let $L(s, E)$ be the (normalized) L-function of an elliptic curve E. The non-trivial zeros ρ of $L(s, E)$ have $Re(\rho) = \frac{1}{2}$.*

Occasionally we assume the RH for the Riemann Zeta-function and Dirichlet L-functions.

Birch and Swinnerton-Dyer Conjecture [BSD1], [BSD2] *Let E be an elliptic curve of geometric rank r over $\mathbb{Q}$ (the Mordell-Weil group is $\mathbb{Z}^r \oplus T$). Then the analytic rank (the order of vanishing of the L-function at the critical point) is also r.*

We assume the above only for interpretation purposes.

Tate's Conjecture for Elliptic Surfaces [Ta] *Let $\mathcal{E}/\mathbb{Q}$ be an elliptic surface and $L_2(\mathcal{E}, s)$ be the L-series attached to $H^2_{\acute{e}t}(\mathcal{E}/\overline{\mathbb{Q}}, \mathbb{Q}_l)$. $L_2(\mathcal{E}, s)$ has a meromorphic continuation to $\mathbb{C}$ and $-\mathrm{ord}_{s=1} L_2(\mathcal{E}, s)$ = rank $NS(\mathcal{E}/\mathbb{Q})$, where $NS(\mathcal{E}/\mathbb{Q})$ is the $\mathbb{Q}$-rational part of the Néron-Severi group of $\mathcal{E}$. Further, $L_2(\mathcal{E}, s)$ does not vanish on the line $Re(s) = 1$.*

Most of the one-parameter families that we investigate are rational surfaces, in which case Tate's conjecture is known (see [RSi]).

ABC Conjecture *Fix $\epsilon > 0$. For co-prime positive integers a, b and c with $c = a + b$ and $N(a, b, c) = \prod_{p|abc} p$, $c \ll_\epsilon N(a, b, c)^{1+\epsilon}$.*

The full strength of ABC is never needed; rather, we need a consequence of ABC, the Square-Free Sieve (see [Gr]):

Square-Free Sieve Conjecture *Fix an irreducible polynomial $f(t)$ of degree at least 4. As $N \to \infty$, the number of $t \in [N, 2N]$ with $f(t)$ divisible by p^2 for some $p > \log N$ is $o(N)$.*

For irreducible polynomials of degree at most 3, the above is known, complete with a better error than $o(N)$ ([Ho], chapter 4).

We use the Square-Free Sieve to handle the variations in the conductors. If our evaluation of the log of the conductors is off by as little as a small constant, the prime sums become untractable. This is why many works normalize by the average log-conductor.

Restricted Sign Conjecture (for the Family $\mathcal{F}$) *Consider a one-parameter family $\mathcal{F}$ of elliptic curves. As $N \to \infty$, the signs of the curves E_t are equidistributed for $t \in [N, 2N]$.*

The Restricted Sign conjecture often fails. First, there are families with constant $j(E_t)$ where all curves have the same sign.

Helfgott [He] has recently related the Restricted Sign conjecture to the Square-Free Sieve conjecture and standard conjectures on sums of Moebius:

Polynomial Moebius *Let $f(t)$ be a non-constant polynomial such that no fixed square divides $f(t)$ for all t. Then $\sum_{t=N}^{2N} \mu(f(t)) = o(N)$.*

The Polynomial Moebius conjecture is known for linear $f(t)$.

Helfgott shows the Square-Free Sieve and Polynomial Moebius imply the Restricted Sign conjecture for many families. More precisely, let $M(t)$ be the product of the irreducible polynomials dividing $\Delta(t)$ and not $c_4(t)$.

Theorem: Equidistribution of Sign in a Family [He]: *Let $\mathcal{F}$ be a one-parameter family with $a_i(t) \in \mathbb{Z}[t]$. If $j(E_t)$ and $M(t)$ are non-constant, then the signs of E_t, $t \in [N, 2N]$, are equidistributed as $N \to \infty$. Further, if we restrict to good t, $t \in [N, 2N]$ such that $D(t)$ is good (usually square-free), the signs are still equidistributed in the limit.*

The above is only used to calculate $N(\mathcal{F}, -1)$, the percent of odd curves. Without this, we can still calculate the 1-level densities for small support, and all but one term in the 2-level densities, $N(\mathcal{F}, -1)f_1(0)f_2(0)$.

2.3 Explicit Formula

The starting point for working with zeroes of the L-functions of elliptic curves is the Explicit Formula (see [Mes]), which relates sums over zeros to sums over primes.

For an elliptic curve E with conductor N_E,

$$\sum_{\gamma_E^{(j)}} G\left(\gamma_E^{(j)} \frac{\log N_E}{2\pi}\right) = \widehat{G}(0) + G(0) - 2\sum_p \frac{\log p}{\log N_E} \frac{1}{p} \widehat{G}\left(\frac{\log p}{\log N_E}\right) a_E(p)$$

$$-2\sum_p \frac{\log p}{\log N_E} \frac{1}{p^2} \widehat{G}\left(\frac{2\log p}{\log N_E}\right) a_E^2(p)$$

$$+ O\left(\frac{\log\log N_E}{\log N_E}\right). \tag{2.3}$$

2.4 Sums of $a_t(p)$

Using the Explicit Formula, we will find that we need to handle sums like

$$\sum_{t=N}^{2N} a_t^{r_1}(p_1) \cdots a_t^{r_n}(p_n). \tag{2.4}$$

We record these results for later use. Define

$$A_{r,\mathcal{F}}(p) = \sum_{t(p)} a_t^r(p). \tag{2.5}$$

LEMMA 2.1. *Let $p_1, \ldots, p_n$ be distinct primes and $r_i \geqslant 1$. Then*

$$\sum_{t(p_1\cdots p_n)} \prod_{i=1}^n a_t^{r_i}(p_i) = \prod_{i=1}^n A_{r_i,\mathcal{F}}(p_i). \tag{2.6}$$

The proof is a straightforward induction, using the fact that $a_{t+mp}(p) = a_t(p)$.

Lemma 2.1 is our best analogue to the Petersson formula, which is used in [ILS] to obtain large support for the density functions.

$\frac{A_{1,\mathcal{F}}(p)}{p}$ is bounded independent of p ([De]). Rosen and Silverman [RSi] proved the following conjecture of Nagao [Na]:

THEOREM 2.2 ROSEN-SILVERMAN. *For a one-parameter family $\mathcal{E}$ of elliptic curves over $\mathbb{Q}(t)$, if Tate's conjecture is true, then*

$$\lim_{X\to\infty} \frac{1}{X} \sum_{p\leqslant X} -\frac{A_{1,\mathcal{F}}(p)}{p} \log p = \operatorname{rank} \mathcal{E}(\mathbb{Q}(t)) \tag{2.7}$$

Tate's conjecture is known for rational surfaces (see [RSi]). An elliptic surface $y^2 = x^3 + A(t)x + B(t)$ is rational iff one of the following is true: (1) $0 < \max\{3\deg A, 2\deg B\} < 12$; (2) $3\deg A = 2\deg B = 12$ and $\text{ord}_{t=0} t^{12}\Delta(t^{-1}) = 0$.

THEOREM 2.3 MICHEL [MI]. *Consider a one-parameter family over $\mathbb{Q}(t)$ with non-constant $j(E_t)$. Then*

$$A_{2,\mathcal{F}}(p) = p^2 + O(p^{\frac{3}{2}}). \tag{2.8}$$

2.5 Sieving and Conductors

To evaluate the sums of $\prod_i a_t^{r_i}(p_i)$, it is necessary to restrict t to arithmetic progressions; in order to bound some of the error terms, we will see that the conductors $C(t)$ must be monotone.

Let

$$\mathcal{T}_{sqfree} = \left\{ t \in [N, 2N] : \ D(t) \text{ is sqfree} \right\}$$
$$\mathcal{T}_N = \left\{ t \in [N, 2N] : \ d^2 \nmid D(t) \text{ for } 2 \leqslant d \leqslant \log^l N \right\}. \tag{2.9}$$

Clearly $\mathcal{T}_{sqfree} \subset \mathcal{T}_N$. We show $\mathcal{T}_N$ is a union of arithmetic progressions, and $|\mathcal{T}_N - \mathcal{T}_{sqfree}| = o(N)$.

Thus, except for $o(N)$ values of t, we can write t good (where the conductors are monotone) as a union of arithmetic progressions. For proofs, see Theorems A.5 and B.2.

3. 1- and 2-Level Density Kernels for the Classical Compact Groups

By [KS1], the n-level densities for the classical compact groups are

$$W_{n,O^+}(x) = \mathbf{det}(K_1(x_i, x_j))_{i,j \leqslant n}$$
$$W_{n,O^-}(x) = \mathbf{det}(K_{-1}(x_i, x_j))_{i,j \leqslant n} + \sum_{k=1}^{n} \delta(x_k)\mathbf{det}(K_{-1}(x_i, x_j))_{i,j \neq k}$$
$$= (W_{n,O^-})_1(x) + (W_{n,O^-})_2(x)$$
$$W_{n,O}(x) = \frac{1}{2}W_{n,O^+}(x) + \frac{1}{2}W_{n,O^-}(x)$$
$$W_{n,U}(x) = \mathbf{det}(K_0(x_i, x_j))_{i,j \leqslant n}$$
$$W_{n,Sp}(x) = \mathbf{det}(K_{-1}(x_i, x_j))_{i,j \leqslant n} \tag{3.1}$$

where $K(y) = \frac{\sin \pi y}{\pi y}$, $K_\epsilon(x, y) = K(x - y) + \epsilon K(x + y)$ for $\epsilon = 0, \pm 1$, O^+ denotes the group $SO(\text{even})$ and O^- the group $SO(\text{odd})$.

3.1 1-Level Densities

Let $I(u)$ be the characteristic function of $[-1, 1]$.

THEOREM 3.1 1-LEVEL DENSITIES.

$$\widehat{W_{1,O^+}}(u) = \delta(u) + \frac{1}{2}I(u)$$
$$\widehat{W_{1,O}}(u) = \delta(u) + \frac{1}{2}$$
$$\widehat{W_{1,O^-}}(u) = \delta(u) - \frac{1}{2}I(u) + 1$$

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$$\widehat{W_{1,Sp}}(u) = \delta(u) - \frac{1}{2}I(u)$$

$$\widehat{W_{1,U}}(u) = \delta(u). \tag{3.2}$$

For functions whose Fourier Transforms are supported in $[-1,1]$, the three orthogonal densities are indistinguishable, though they are distinguishable from U and Sp. To detect differences between the orthogonal groups using the 1-level density, one needs to work with functions whose Fourier Transforms are supported beyond $[-1,1]$.

3.2 2-Level Densities

THEOREM 3.2 $\mathcal{G} = SO(\text{EVEN})$, O, OR $SO(\text{ODD})$. Let $c(\mathcal{G}) = 0$, $\frac{1}{2}$, 1 for $\mathcal{G} = SO(\text{even})$, O, $SO(\text{odd})$. For even functions supported in $|u_1| + |u_2| < 1$

$$\int \int \widehat{f_1}(u_1)\widehat{f_2}(u_2)\widehat{W_{2,\mathcal{G}}}(u)du_1 du_2$$

$$= \left[\widehat{f_1}(0) + \frac{1}{2}f_1(0)\right]\left[\widehat{f_2}(0) + \frac{1}{2}f_2(0)\right] + 2\int |u|\widehat{f_1}(u)\widehat{f_2}(u)du$$

$$- 2\widehat{f_1 f_2}(0) - f_1(0)f_2(0) + c(\mathcal{G})f_1(0)f_2(0). \tag{3.3}$$

For arbitrarily small support, the three 2-level densities differ. One increases by a factor of $\frac{1}{2}f_1(0)f_2(0)$ moving from $\widehat{W_{2,O^+}}$ to $\widehat{W_{2,O}}$ to $\widehat{W_{2,O^-}}$.

THEOREM 3.3 $\mathcal{G} = Sp$.

$$\int \int \widehat{f_1}(u_1)\widehat{f_2}(u_2)\widehat{W_{2,Sp}}(u)du_1 du_2$$

$$= \left[\widehat{f_1}(0) + \frac{1}{2}f_1(0)\right]\left[\widehat{f_2}(0) + \frac{1}{2}f_2(0)\right] + 2\int |u|\widehat{f_1}(u)\widehat{f_2}(u)du$$

$$-2\widehat{f_1 f_2}(0) - f_1(0)f_2(0) - f_1(0)\widehat{f_2}(0) - \widehat{f_1}(0)f_2(0) + 2f_1(0)f_2(0). \tag{3.4}$$

THEOREM 3.4 $\mathcal{G} = U$.

$$\int \int \widehat{f_1}(u_1)\widehat{f_2}(u_2)\widehat{W_{2,U}}du_1 du_2 = \widehat{f_1}(0)\widehat{f_2}(0) + \int |u|\widehat{f_1}(u)\widehat{f_2}(u)du - \widehat{f_1 f_2}(0). \tag{3.5}$$

For test functions with arbitrarily small support, the 2-level densities for the classical compact groups are mutually distinguishable.

4. Expansions for the 1- and 2-Level Densities for Elliptic Curve Families

For $i = 1$ and 2, let f_i be an even Schwartz function whose Fourier Transform is supported in $(-\sigma_i, \sigma_i)$ and $f(x_1, x_2) = f_1(x_1)f_2(x_2)$, $\widehat{f}(u_1, u_2) = \widehat{f_1}(u_1)\widehat{f_2}(u_2)$.

4.1 1-Level Density: $D_{1,\mathcal{F}}(f)$

$$D_{1,\mathcal{F}}(f) = \frac{1}{|\mathcal{F}|}\sum_{E \in \mathcal{F}}\sum_{\gamma_E^{(j)}} f_1\left(\gamma_E^{(j)}\frac{\log N_E}{2\pi}\right)$$

$$= \widehat{f_1}(0) + f_1(0) - 2\sum_p \frac{1}{p}\frac{1}{|\mathcal{F}|}\sum_{E \in \mathcal{F}}\frac{\log p}{\log N_E}\widehat{f_1}\left(\frac{\log p}{\log N_E}\right)a_E(p)$$

$$-2\sum_p \frac{1}{p^2}\frac{1}{|\mathcal{F}|}\sum_{E \in \mathcal{F}}\frac{\log p}{\log N_E}\widehat{f_1}\left(\frac{2\log p}{\log N_E}\right)a_E^2(p)$$

$$+O\Big(\frac{\log\log N_E}{\log N_E}\Big). \tag{4.1}$$

As the 1-level density sums are sub-calculations which arise in the 2-level investigations, we postpone their determination for now.

4.2 2-Level Density: $D_{2,\mathcal{F}}(f)$ and $D_{2,\mathcal{F}}^*(f)$

Recall the 2-level density $D_{2,\mathcal{F}}(f)$ is the sum over all indices j_1, j_2 with $j_1 \neq \pm j_2$.

DEFINITION 4.1. $D_{2,\mathcal{F}}^*(f)$ differs from the 2-level density $D_{2,\mathcal{F}}(f)$ in that j_1 may equal $\pm j_2$.

We first calculate $D_{2,\mathcal{F}}^*(f)$, and then subtract off the contribution from $j_1 = \pm j_2$. Assuming GRH, we may write the zeros as $1 + i\gamma^{(j)}$, with $\gamma^{(j)} = -\gamma^{(-j)}$.

$$
\begin{aligned}
D_{2,\mathcal{F}}^*(f) &= \frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_{j_1}\sum_{j_2} f_1(L\gamma_E^{(j_1)})f_2(L\gamma_E^{(j_2)}) \\
&= \frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\prod_{i=1}^{2}\left[\widehat{f}_i(0) + f_i(0) - 2\sum_{p_i}\frac{\log p_i}{\log N_E}\frac{1}{p_i}\widehat{f}_i\Big(\frac{\log p_i}{\log N_E}\Big)a_E(p_i)\right. \\
&\qquad\left. -2\sum_{p_i}\frac{\log p_i}{\log N_E}\frac{1}{p_i^2}\widehat{f}_i\Big(2\frac{\log p_i}{\log N_E}\Big)a_E^2(p_i) + O\Big(\frac{\log\log N_E}{\log N_E}\Big)\right] \\
&= \frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\prod_{i=1}^{2}\left[\widehat{f}_i(0) + f_i(0) + S_{i,1} + S_{i,2}\right]. \tag{4.2}
\end{aligned}
$$

We use Theorem D.1 to drop the error terms, as they do not contribute in the limit as $|\mathcal{F}| \to \infty$. The astute reader will notice Theorem D.1 requires us to know the 1-level density, and we have postponed that calculation; however, in the process of calculating the 2-level density we will determine the needed sums for the 1-level density (without using Theorem D.1 to evaluate them). Thus, there is no harm in removing the error terms.

There are five types of sums we need to investigate: $\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}} S_{i,1}$, $\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}} S_{i,2}$, $\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}} S_{1,1}S_{2,1}$, $\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}} S_{1,2}S_{2,2}$, and $\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}} S_{1,1}S_{2,2}$ ($i \neq j$). In $S_{i,j}$, i refers to which prime (p_1 or p_2), and j the power of $a_E(p_\alpha)$ (1 or 2). The first and the second are what we need to calculate the one-level densities.

4.2.1 $j_1 = \pm j_2$ Let $\rho = 1 + i\gamma_E^{(j)}$ be a zero. For a curve with even functional equation, we may label the zeros by

$$\cdots \leqslant \gamma_E^{(-2)} \leqslant \gamma_E^{(-1)} \leqslant 0 \leqslant \gamma_E^{(1)} \leqslant \gamma_E^{(2)} \leqslant \cdots, \gamma_E^{(-k)} = -\gamma_E^{(k)}, \tag{4.3}$$

while for a curve with odd functional equation we label the zeros by

$$\cdots \leqslant \gamma_E^{(-1)} \leqslant 0 \leqslant \gamma_E^{(0)} = 0 \leqslant \gamma_E^{(1)} \leqslant \cdots, \gamma_E^{(-k)} = -\gamma_E^{(k)}. \tag{4.4}$$

We exclude the contribution from $j_1 = \pm j_2$. If an elliptic curve has even functional equation, j_i ranges over all non-zero integers, and $\gamma_E^{(-j)} = -\gamma_E^{(j)}, j \neq -j$. Since the test functions are even, the sum over all pairs (j_1, j_2) with $j_1 = \pm j_2$ is twice the sum over all pairs (j, j), which is $D_{1,E}(f_1 f_2)$, ie, the 1-level density for a curve E with test function $f_1(x)f_2(x)$.

If an elliptic curve has odd functional equation, j_i ranges over all integers. The curve vanishes to odd order at the critical point $s = 1$. Except for one zero (labelled $\gamma_E^{(0)}$), for every non-zero j, $\gamma_E^{(-j)} = -\gamma_E^{(j)}$, and $j \neq -j$. Twice the sum over pairs (j, j) minus the contribution from the pair $(0, 0)$ equals the sum over all pairs (j_1, j_2) with $j_1 = \pm j_2$. Thus, the curves with odd sign contribute $D_{1,E}(f_1 f_2) - f_1(0) f_2(0)$.

Let $\epsilon_E = \pm 1$ be the sign of the functional equation for E, and define

DEFINITION 4.2. $N(\mathcal{F}, -1) = \frac{1}{|\mathcal{F}|} \sum_{E \in \mathcal{F}} \frac{1 - \epsilon_E}{2}$, ie, the percent of curves with odd sign.

Summing over $E \in \mathcal{F}$ yields $D_{1,\mathcal{F}}(f_1 f_2) - N(\mathcal{F}, -1) f_1(0) f_2(0)$ for $j_1 = \pm j_2$.

4.2.2 2-Level Density Expansion

LEMMA 4.3 2-LEVEL DENSITY EXPANSION.

$$D_{2,\mathcal{F}}(f) = \frac{1}{|\mathcal{F}|} \sum_{E \in \mathcal{F}} \prod_{i=1}^{2} \left[\widehat{f}_i(0) + f_i(0) + S_{i,1} + S_{i,2} \right]$$
$$- 2D_{1,\mathcal{F}}(f_1 f_2) + (f_1 f_2)(0) N(\mathcal{F}, -1) + O\left(\frac{\log \log N}{\log N} \right). \tag{4.5}$$

To evaluate the above, we only need to know the percent of curves with odd sign, not which curves are even or odd. For the 3 and higher level densities, we have to execute sums over the subset of curves with odd sign.

4.3 Useful Expansion for the 1- and 2-Level Densities for One Parameter Families

Let $\mathcal{E}$ denote a one-parameter family of elliptic curves E_t over $\mathbb{Q}(t)$, $t \in [N, 2N]$, and $\mathcal{F}$ denote a sub-family of $\mathcal{E}$. In the applications, $\mathcal{F}$ will be obtained by sieving to $D(t)$ good, where $D(t)$ is the product of the irreducible polynomial factors of $\Delta(t)$.

4.3.1 Needed Prime Sums

LEMMA 4.4 PRIME SUMS. *Let $C(N)$ be a power of N. By Lemmas C.2, C.3 and C.4,*

i) $\sum_p \frac{\log p}{\log C(N)} \frac{1}{p} \widehat{f}_1\left(\frac{\log p}{\log C(N)} \right) = \frac{1}{2} f_1(0) + O\left(\frac{1}{\log N} \right)$

ii) $\sum_p \frac{\log p}{\log C(N)} \frac{1}{p} \widehat{f}_1\left(2\frac{\log p}{\log C(N)} \right) = \frac{1}{4} f_1(0) + O\left(\frac{1}{\log N} \right)$

iii) $\sum_p \frac{\log^2 p}{\log^2 C(N)} \frac{1}{p} \widehat{f}_1 \widehat{f}_2\left(\frac{\log p}{\log C(N)} \right) = \frac{1}{2} \int_{-\infty}^{\infty} |u| \widehat{f}_1(u) \widehat{f}_2(u) du + O\left(\frac{1}{\log N} \right)$

If instead we are summing over primes congruent to $a \bmod m$, we use Lemma C.1 and C.5, and the right-hand sides are modified by $\frac{1}{\varphi(m)}$.

4.3.2 Expansions of Sums   We use the expansion from Lemma 4.3. Recall

$$S_{i,j} = -2 \sum_{p_i} \frac{\log p_i}{\log C(t)} \frac{1}{p_i^j} \widehat{f}_i\left(2^{j-1} \frac{\log p_i}{\log C(t)} \right) a_t^j(p_i). \tag{4.6}$$

In $S_{i,j}$, i refers to the prime (p_1, p_2) and j refers to the power of $a_t(p)$ $(a_t(p), a_t^2(p))$.

To determine the 1- and 2-level densities, there are eight sums over $t \in \mathcal{F}$ to evaluate: $\frac{1}{|\mathcal{F}|} \sum_{t \in \mathcal{F}} S_{1,1}$ and $\frac{1}{|\mathcal{F}|} \sum_{t \in \mathcal{F}} S_{2,1}$; $\frac{1}{|\mathcal{F}|} \sum_{t \in \mathcal{F}} S_{1,2}$ and $\frac{1}{|\mathcal{F}|} \sum_{t \in \mathcal{F}} S_{2,2}$; $\frac{1}{|\mathcal{F}|} \sum_{t \in \mathcal{F}} S_{1,1} S_{2,2}$ and $\frac{1}{|\mathcal{F}|} \sum_{t \in \mathcal{F}} S_{2,1} S_{1,2}$; $\frac{1}{|\mathcal{F}|} \sum_{t \in \mathcal{F}} S_{1,1} S_{2,1}$; $\frac{1}{|\mathcal{F}|} \sum_{t \in \mathcal{F}} S_{1,2} S_{2,2}$.

We have written the sums in pairs where the two sums are handled similarly. Substituting the definitions leads to five types of sums:

i) $-2\sum_p \frac{1}{p}\frac{1}{|\mathcal{F}|}\sum_{t\in\mathcal{F}}\frac{\log p}{\log C(t)}\widehat{f_1}\left(\frac{\log p}{\log C(t)}\right)a_t(p)$

ii) $-2\sum_p \frac{1}{p^2}\frac{1}{|\mathcal{F}|}\sum_{t\in\mathcal{F}}\frac{\log p}{\log C(t)}\widehat{f_1}\left(2\frac{\log p}{\log C(t)}\right)a_t^2(p)$

iii) $4\sum_{p_1}\sum_{p_2}\frac{1}{p_1 p_2^2}\frac{1}{|\mathcal{F}|}\sum_{t\in\mathcal{F}}\frac{\log p_1}{\log C(t)}\frac{\log p_2}{\log C(t)}\widehat{f_1}\left(\frac{\log p}{\log C(t)}\right)\widehat{f_2}\left(2\frac{\log p}{\log C(t)}\right)a_t(p_1)a_t^2(p_2)$

iv) $4\sum_{p_1}\sum_{p_2}\frac{1}{p_1 p_2}\frac{1}{|\mathcal{F}|}\sum_{t\in\mathcal{F}}\frac{\log p_1}{\log C(t)}\frac{\log p_2}{\log C(t)}\widehat{f_1}\left(\frac{\log p}{\log C(t)}\right)\widehat{f_2}\left(\frac{\log p}{\log C(t)}\right)a_t(p_1)a_t(p_2)$

v) $4\sum_{p_1}\sum_{p_2}\frac{1}{p_1^2 p_2^2}\frac{1}{|\mathcal{F}|}\sum_{t\in\mathcal{F}}\frac{\log p_1}{\log C(t)}\frac{\log p_2}{\log C(t)}\widehat{f_1}\left(2\frac{\log p}{\log C(t)}\right)\widehat{f_2}\left(2\frac{\log p}{\log C(t)}\right)a_t^2(p_1)a_t^2(p_2)$

In the above sums, we use Lemma C.7 to restrict to primes greater than $\log^l N$, $l < 2$. Label the five sums $\frac{1}{|\mathcal{F}|}\sum_{t\in\mathcal{F}}S(t;p)$ by $T_k(p)$ and $T_k(p_1,p_2)$. Trivially by Hasse some of the above do not contribute.

In the third sum, if $p_1 = p_2 = p$, we get $\ll \frac{1}{\log N}\sum_p \frac{p^{\frac{3}{2}}\log p}{p^3} = O(\frac{1}{\log N})$. In the fifth sum, if $p_1 = p_2 = p$ we get $\ll \frac{1}{\log N}\sum_p \frac{p^2 \log p}{p^4} = O(\frac{1}{\log N})$.

Thus, we only study the third and fifth sums when $p_1 \neq p_2$. The fourth sum has the potential to contribute when $p_1 = p_2$. Hence we break it into two cases: $p_1 \neq p_2$ and $p_1 = p_2$.

4.3.3 *Conditions on the Family to Evaluate the Sums*

$$\text{Conditions on the Family } \mathcal{F} \tag{4.7}$$

Let $T_k(p)$ and $T_k(p_1,p_2)$ $(= \frac{1}{|\mathcal{F}|}\sum_{t\in\mathcal{F}}S(t;p))$ equal

i) $\frac{\log p}{\log C(N)}\widehat{f_1}\left(\frac{\log p}{\log C(N)}\right)\left[-r + O\left(p^{-\alpha} + \frac{p^\beta}{|\mathcal{F}|} + \frac{1}{\log^\gamma N}\right)\right]$

ii) $\frac{\log p}{\log C(N)}\widehat{f_1}\left(2\frac{\log p}{\log C(N)}\right)\left[p + O\left(p^{1-\alpha} + \frac{p^\beta}{|\mathcal{F}|} + \frac{p}{\log^\gamma N}\right)\right]$

iii) $\frac{\log p_1}{\log C(N)}\frac{\log p_2}{\log C(N)}\widehat{f_1}\left(\frac{\log p_1}{\log C(N)}\right)\widehat{f_2}\left(2\frac{\log p_2}{\log C(N)}\right)\left[-rp_2 + O\left(p_1^{-\alpha_1}p_2^{1-\alpha_2} + \frac{p_1^{\beta_1}p_2^{\beta_2}}{|\mathcal{F}|} + \frac{p_2}{\log^\gamma N}\right)\right]$

iv) (a) $\frac{\log p_1}{\log C(N)}\frac{\log p_2}{\log C(N)}\widehat{f_1}\left(\frac{\log p_1}{\log C(N)}\right)\widehat{f_2}\left(\frac{\log p_2}{\log C(N)}\right)\left[r^2 + O\left(p_1^{1-\alpha_1}p_2^{1-\alpha_2} + \frac{p_1^{\beta_1}p_2^{\beta_2}}{|\mathcal{F}|} + \frac{1}{\log^\gamma N}\right)\right]$ if $p_1 \neq p_2$

   (b) $\frac{\log^2 p}{\log^2 C(N)}\widehat{f_1}\widehat{f_2}\left(\frac{\log p}{\log C(N)}\right)\left[p + O\left(p^{1-\alpha} + \frac{p^\beta}{|\mathcal{F}|} + \frac{p}{\log^\gamma N}\right)\right]$ if $p_1 = p_2 = p$

v) $\frac{\log p_1}{\log C(N)}\frac{\log p_2}{\log C(N)}\widehat{f_1}\left(2\frac{\log p_1}{\log C(N)}\right)\widehat{f_1}\left(2\frac{\log p_2}{\log C(N)}\right)\left[p_1 p_2 + O\left(p_1^{1-\alpha_1}p_2^{1-\alpha_2} + \frac{p_1^{\beta_1}p_2^{\beta_2}}{|\mathcal{F}|} + \frac{p_1 p_2}{\log^\gamma N}\right)\right]$

where $\alpha, \beta, \gamma > 0$, $\alpha_i, \beta_i \geq 0$ and whenever two α_i or β_i occur, at least one is positive.

By Lemma 4.4 we can evaluate the eight $S_{i,j}$ sums for a family satisfying Conditions 4.7:

LEMMA 4.5 $S_{i,j}$ SUMS. *If the family satisfies Conditions 4.7, then (up to lower order terms which do not contribute for small support),*

i) $\frac{1}{|\mathcal{F}|}\sum_{t\in\mathcal{F}}S_{i,1} = rf_i(0)$

ii) $\frac{1}{|\mathcal{F}|}\sum_{t\in\mathcal{F}}S_{i,2} = -\frac{1}{2}f_i(0)$

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iii) $\frac{1}{|\mathcal{F}|} \sum_{t \in \mathcal{F}} S_{1,1} S_{2,2} + S_{2,1} S_{1,2} = -\frac{1}{2} r f_1(0) f_2(0) + -\frac{1}{2} r f_1(0) f_2(0)$

iv) $\frac{1}{|\mathcal{F}|} \sum_{t \in \mathcal{F}} S_{1,1} S_{2,1} = r^2 f_1(0) f_2(0) + 2 \int_{-\infty}^{\infty} |u| \widehat{f_1}(u) \widehat{f_2}(u) du$

v) $\frac{1}{|\mathcal{F}|} \sum_{t \in \mathcal{F}} S_{1,2} S_{2,2} = \frac{1}{4} f_1(0) f_2(0)$

4.3.4 1- and 2-Level Densities, Assuming Certain Conditions on the Family Substituting Lemma 4.5 into the 1- and 2-level density expansions we obtain

LEMMA 4.6 1- AND 2-LEVEL DENSITIES. *Assume $|\mathcal{F}|$ is a positive multiple of N and $\mathcal{F}$ satisfies conditions 4.7. Up to lower order correction terms (which vanish as $|\mathcal{F}| \to \infty$), for even Schwartz functions with small support,*

$$D_{1,\mathcal{F}}(f) = \widehat{f_1}(0) + \frac{1}{2} f_1(0) + r f_1(0) \tag{4.8}$$

and

$$D_{2,\mathcal{F}}(f) = \prod_{i=1}^{2} \left[\widehat{f_i}(0) + \frac{1}{2} f_i(0) \right] + 2 \int_{-\infty}^{\infty} |u| \widehat{f_1}(u) \widehat{f_2}(u) du$$
$$-2 \widehat{f_1 f_2}(0) - f_1(0) f_2(0) + (f_1 f_2)(0) N(\mathcal{F}, -1)$$
$$+ (r^2 - r) f_1(0) f_2(0) + r \widehat{f_1}(0) f_2(0) + r f_1(0) \widehat{f_2}(0). \tag{4.9}$$

Let $D_{1,\mathcal{F}}^{(r)}(f_1)$ and $D_{2,\mathcal{F}}^{(r)}(f_1)$ be the 1- and 2-level densities from which the contributions of r family zeros at the critical point have been subtracted. Then

$$D_{1,\mathcal{F}}^{(r)}(f_1) = \widehat{f_1}(0) + \frac{1}{2} f_1(0) \tag{4.10}$$

and

$$D_{2,\mathcal{F}}^{(r)}(f_1) = \prod_{i=1}^{2} \left[\widehat{f_i}(0) + \frac{1}{2} f_i(0) \right] + 2 \int_{-\infty}^{\infty} |u| \widehat{f_1}(u) \widehat{f_2}(u) du$$
$$-2 \widehat{f_1 f_2}(0) - f_1(0) f_2(0) + (f_1 f_2)(0) N(\mathcal{F}, -1). \tag{4.11}$$

Thus, removing the contribution from r family zeros, for test functions of small support the 2-level density of the remaining zeros agrees with $SO(even)$ if all curves are even, O if half are even and half odd, and $SO(odd)$ if all are odd.

Proof: The 1-level density is immediate from substitution. Substituting for the eight $S_{i,j}$ sums for $D_{2,\mathcal{F}}(f)$ yields (up to lower order terms which don't contribute for small support)

$$D_{2,\mathcal{F}}(f) = = \prod_{i=1}^{2} \left[\widehat{f_i}(0) + f_i(0) \right]$$
$$+ \left[\widehat{f_1}(0) + f_1(0) \right] r f_2(0) + \left[\widehat{f_2}(0) + f_2(0) \right] r f_1(0)$$
$$+ r^2 f_1(0) f_2(0) + 2 \int_{-\infty}^{\infty} |u| \widehat{f_1}(u) \widehat{f_2}(u) du$$
$$+ \left[\widehat{f_1}(0) + f_1(0) \right] \left(-\frac{1}{2} f_2(0) \right) + \left[\widehat{f_2}(0) + f_2(0) \right] \left(-\frac{1}{2} f_1(0) \right)$$
$$- \frac{1}{2} r f_1(0) f_2(0) - \frac{1}{2} r f_1(0) f_2(0) + \frac{1}{4} f_1(0) f_2(0)$$

$$- 2D_{1,\mathcal{F}}(f_1 f_2) + (f_1 f_2)(0)N(\mathcal{F}, -1) + O\Big(\frac{\log \log N}{\log N}\Big)$$

$$= \prod_{i=1}^{2} \left[\widehat{f_i}(0) + \frac{1}{2}f_i(0)\right] + 2\int_{-\infty}^{\infty} |u|\widehat{f_1}(u)\widehat{f_2}(u)du$$

$$+2rf_1(0)f_2(0) + r\widehat{f_1}(0)f_2(0) + rf_1(0)\widehat{f_2}(0) - rf_1(0)f_2(0) + r^2 f_1(0)f_2(0)$$
$$-2D_{1,\mathcal{F}}(f_1 f_2) + (f_1 f_2)(0)N(\mathcal{F}, -1). \tag{4.12}$$

Substituting

$$D_{1,\mathcal{F}}(f_1 f_2) = \widehat{f_1 f_2}(0) + \frac{1}{2}f_1(0)f_2(0) + rf_1(0)f_2(0) \tag{4.13}$$

yields

$$D_{2,\mathcal{F}}(f) = \prod_{i=1}^{2} \left[\widehat{f_i}(0) + \frac{1}{2}f_i(0)\right] + 2\int_{-\infty}^{\infty} |u|\widehat{f_1}(u)\widehat{f_2}(u)du$$

$$+rf_1(0)f_2(0) + r\widehat{f_1}(0)f_2(0) + rf_1(0)\widehat{f_2}(0) + r^2 f_1(0)f_2(0)$$
$$-2\widehat{f_1 f_2}(0) - f_1(0)f_2(0) - 2rf_1(0)f_2(0) + (f_1 f_2)(0)N(\mathcal{F}, -1)$$

$$= \prod_{i=1}^{2} \left[\widehat{f_i}(0) + \frac{1}{2}f_i(0)\right] + 2\int_{-\infty}^{\infty} |u|\widehat{f_1}(u)\widehat{f_2}(u)du$$

$$-2\widehat{f_1 f_2}(0) - f_1(0)f_2(0) + (f_1 f_2)(0)N(\mathcal{F}, -1)$$
$$+(r^2 - r)f_1(0)f_2(0) + r\widehat{f_1}(0)f_2(0) + rf_1(0)\widehat{f_2}(0). \tag{4.14}$$

If the family has rank r over $\mathbb{Q}(t)$, there is a natural interpretation of these terms. By the Birch and Swinnerton-Dyer conjecture (used only for interpretation purposes) and Silverman's Specialization Theorem, for all t sufficiently large, each curve's L-function has at least r zeros at the critical point. We isolate the contributions from r family zeros.

Assume there are r family zeros at the critical point. Let $L_t = \frac{\log C(t)}{2\pi}$. Recall the 1-level density is $D_{1,\mathcal{F}}(f) = \widehat{f}(0) + \frac{1}{2}f(0) + rf(0)$. Let j_i range over all zeros of a curve, and j_i' range over all but the r family zeros.

$$D_{2,\mathcal{F}}(f) = \frac{1}{|\mathcal{F}|}\sum_{t\in\mathcal{F}}\sum_{j_1}\sum_{j_2} f_1(L_t\gamma_{E_t}^{(j_1)})f_2(L_t\gamma_{E_t}^{(j_2)})$$

$$- 2D_{1,\mathcal{F}}(f_1 f_2) + (f_1 f_2)(0)N(\mathcal{F}, -1)$$

$$= \frac{1}{|\mathcal{F}|}\sum_{t\in\mathcal{F}} \Big(rf_1(0) + \sum_{j_1'} f_1(L_t\gamma_{E_t}^{(j_1')})\Big)\Big(rf_2(0) + \sum_{j_2'} f_2(L_t\gamma_{E_t}^{(j_2')})\Big)$$

$$- 2D_{1,\mathcal{F}}(f_1 f_2) + (f_1 f_2)(0)N(\mathcal{F}, -1)$$

$$= \frac{1}{|\mathcal{F}|}\sum_{t\in\mathcal{F}}\sum_{j_1'}\sum_{j_2'} f_1(L_t\gamma_{E_t}^{(j_1')})f_2(L_t\gamma_{E_t}^{(j_2')})$$

$$+rf_1(0)D_{1,\mathcal{F}}(f_2) + D_{1,\mathcal{F}}(f_1)rf_2(0) - r^2 f_1(0)f_2(0)$$
$$-2D_{1,\mathcal{F}}(f_1 f_2) + (f_1 f_2)(0)N(\mathcal{F}, -1)$$

$$= \frac{1}{|\mathcal{F}|}\sum_{t\in\mathcal{F}}\sum_{j_1'}\sum_{j_2'} f_1(L_t\gamma_{E_t}^{(j_1')})f_2(L_t\gamma_{E_t}^{(j_2')}) + (f_1 f_2)(0)N(\mathcal{F}, -1)$$

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$$+rf_1(0)\Big(\widehat{f_2}(0) + (r + \frac{1}{2})f_2(0)\Big) + \Big(\widehat{f_1}(0) + (r + \frac{1}{2})f_1(0)\Big)rf_2(0)$$

$$-r^2 f_1(0)f_2(0) - 2\Big(\widehat{f_1f_2}(0) + \frac{1}{2}f_1(0)f_2(0) + rf_1(0)f_2(0)\Big)$$

$$= \left[\frac{1}{|\mathcal{F}|} \sum_{t\in\mathcal{F}} \sum_{j_1'} \sum_{j_2'} f_1(L_t\gamma_{E_t}^{(j_1')}) f_2(L_t\gamma_{E_t}^{(j_2')}) \right.$$

$$\left. -2\Big(\widehat{f_1f_2}(0) + \frac{1}{2}f_1(0)f_2(0)\Big) + (f_1 f_2)(0)N(\mathcal{F}, -1) \right]$$

$$+rf_1(0)\widehat{f_2}(0) + r\widehat{f_1}(0)f_2(0) + (r^2 - r)f_1(0)f_2(0)$$

$$= D_{2,\mathcal{F}}^{(r)}(f_1) + rf_1(0)\widehat{f_2}(0) + r\widehat{f_1}(0)f_2(0) + (r^2 - r)f_1(0)f_2(0). \tag{4.15}$$

We isolate

LEMMA 4.7. *The contribution from r critical point zeros is*

$$rf_1(0)\widehat{f_2}(0) + r\widehat{f_1}(0)f_2(0) + (r^2 - r)f_1(0)f_2(0). \tag{4.16}$$

5. Calculation of the 1- and 2-Level Densities for Elliptic Curve Families

Let $\mathcal{E}$ be a one-parameter family of elliptic curves E_t with discriminants $\Delta(t)$ and conductors $C(t)$. For many families, we can evaluate the conductors exactly if we sieve to a subfamily $\mathcal{F}$ defined as the $t \in [N, 2N]$ with $D(t)$ good, where $D(t) = a_k t^k + \cdots + a_0$ ($a_k \geqslant 1$) is the product of the irreducible polynomial factors of $\Delta(t)$. Usually good will mean square-free, although occasionally it will mean square-free except for a fixed set of primes, and for these special primes, the power of $p|D(t)$ is independent of t.

Let our family $\mathcal{F}$ be the set of good $t \in [N, 2N]$ where the conductors are given by a monotone polynomial in t. We use this polynomial for the conductors at non-good t; this is permissible as these curves are not in our family, and do not originally appear in our sums.

For each d, let

$$T(d) = \{t \in [N, 2N] : d^2|D(t)\}. \tag{5.1}$$

Let $S(t)$ be some quantity associated to the elliptic curve E_t. We study

$$\sum_{\substack{t=N \\ D(t)\ good}}^{2N} S(t) = \sum_{d=1}^{(2a_k N)^{\frac{k}{2}}} \mu(d) \sum_{t\in T(d)} S(t). \tag{5.2}$$

In particular, setting $S(t) = 1$ yields the cardinality of the family. In all the families we investigate, $|\mathcal{F}| = c_\mathcal{F} N + o(N)$, $c_\mathcal{F} > 0$.

Let $t_1(d), \ldots, t_{\nu(d)}(d)$ be the incongruent roots of $D(t) \equiv 0 \bmod d^2$. The presence of $\mu(d)$ allows us to restrict to d square-free. For small d, we may take the $t_i(d) \in [N, N + d^2)$. For such d,

$$\sum_{t\in T(d)} S(t) = \sum_{i=1}^{\nu(d)} \sum_{t'=0}^{[N/d^2]} S\Big(t_i(d) + t'd^2\Big) + O\Big(\nu(d)||S||_\infty\Big). \tag{5.3}$$

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The error piece is from boundary effects for the last value of t'. $T(d)$ restricts us to $t \in [N, 2N]$; as each $t_i(d) \geqslant N$, and at most one is exactly N, it is possible in summing to $t' = [N/d^2]$ we've added an extra term.

5.1 Assumptions for Sieving

We evaluate the sums under the following assumptions:

i) For square-free $D(t)$, the conductors $C(t)$ are given by a monotone polynomial in t.

ii) A positive percent of $t \in [N, 2N]$ have $D(t)$ square-free; ie, $|\mathcal{F}| = c_{\mathcal{F}} N + o(N)$.

We constantly use Lemma A.2 ($\nu(d) \ll d^\epsilon$ for square-free d) and

$$\sum_{\substack{t=N \\ D(t) \; good}}^{2N} 1 = \sum_{d=1}^{\log^l N} \mu(d) \sum_{\substack{t=N \\ D(t) \equiv 0(d^2)}}^{2N} 1 \; + \; o(N) = c_{\mathcal{F}} N + o(N), \; c_{\mathcal{F}} > 0. \tag{5.4}$$

We show the family satisfies Conditions 4.7. We evaluate the sums over $t \in \mathcal{F}$ below and then execute the summation over the prime(s). $\widehat{f}_i$ is supported in $(-\sigma_i, \sigma_i)$. There are no contributions (for σ_i sufficiently small) in the prime sum(s) for sufficiently small error terms.

5.2 Definition of Terms for Sieving

Recall $A_{r,\mathcal{F}}(p) = \sum_{t(p)} a_t^r(p)$. For distinct primes, by Lemma 2.1

$$\sum_{t(p_1 \cdots p_n)} \prod_{j=1}^{n} a_t^{r_i}(p_j) = \prod_{j=1}^{n} A_{r_i, \mathcal{F}}(p_i). \tag{5.5}$$

By Lemma C.7, we may assume all of our primes (in the expansion from the Explicit Formula in the n-level densities) are at least $\log^l N$, $l \in [1, 2]$. We can incorporate these errors into our existing error terms; the result will still be a lower order term which will not contribute for small support.

$S(t)$ will equal $\widetilde{a}_P(t) G_P(t)$, where for distinct primes p_1 and p_2

$$\widetilde{a}_P(t) = a_t^{r_1}(p_1) a_t^{r_2}(p_2)$$

$$G_P(t) = \prod_{\substack{j=1 \\ r_j \neq 0}}^{2} \frac{\log p_j}{\log C(t)} f_j \left(2^{r_j - 1} \frac{\log p_j}{\log C(t)} \right)$$

$$(r_1, r_2) \in \left\{ (1,0), (0,1), (2,0), (1,1), (0,2), (1,2), (2,1), (2,2) \right\}. \tag{5.6}$$

Thus $\widetilde{a}_P(t) G_P(t)$ is merely a convenient way of encoding the eight sums we need to examine for the 1 and 2-level densities.

Actually, this is slightly off. We have to study

$$\prod_{\substack{j=1 \\ r_j \neq 0}}^{2} \frac{1}{p_j^{r_j}} \frac{\log p_j}{\log C(t)} g_j \left(2^{r_j - 1} \frac{\log p_j}{\log C(t)} \right) a_t^{r_j}(p_j). \tag{5.7}$$

If both r_j's are non-zero and the two primes are equal, we obtain

$$\frac{1}{p^{r_1+r_2}}\Big(\frac{\log p}{\log C(t)}\Big)^2 \times \cdots \times a_t^{r_1+r_2}(p). \tag{5.8}$$

For example, if $r_1 = r_2 = 1$ we would get $(\frac{\log p}{\log C(t)})^2 \times \cdots \times a_t^2(p)$. Thus, the definition of G_P needs to be slightly modified. We want to deal with distinct primes p_1 and p_2. There will be no contribution for equal primes if $r_1 + r_2 \geqslant 3$; simply bound each $a_t(p)$ by Hasse. There is a contribution if $r_1 = r_2 = 1$. By modifying the definition of G_P we may regard it as a case where $r = (2,0)$; however, we have $(\frac{\log p}{\log C(t)})^2$ instead of $(\frac{\log p}{\log C(t)})$, and instead of $f_1(\cdots)$ we will have $f_1 f_2(\cdots)$. Note we evaluate the test functions at $\frac{\log p}{\log C(t)}$ and not $2\frac{\log p}{\log C(t)}$. We have

$$G_P(t) = \prod_{\substack{j=1 \\ r_j \neq 0}}^{2} \Big(\frac{\log p_j}{\log C(t)}\Big)^{\kappa(r)} g_j\Big(2^{r_j-\kappa(r)}\frac{\log p_j}{\log C(t)}\Big), \tag{5.9}$$

where $\kappa(r)$ is 2 if $r = (2,0)$ and this arises from $p_1 = p_2 = p$ and $\kappa(r) = 1$ otherwise; $g_j = f_j$ unless $r = (2,0)$ arising from $p_1 = p_2 = p$, in which case $g_1 = f_1 f_2$.

We may now assume the primes are distinct. Define

$$P = \prod_{\substack{j=1 \\ r_j \neq 0}}^{2} p_j, \quad r = (r_1, r_2), \ r_j \in \{0,1,2\}$$

$$S_c(r, P) = \sum_{t(P)} \widetilde{a}_P(t) = \sum_{t(P)} a_t^{r_1}(p_1) a_t^{r_2}(p_2) = A_{r_1,\mathcal{F}}(p_1) A_{r_2,\mathcal{F}}(p_2), \tag{5.10}$$

where for convenience we set $A_0(p) = 1$. We often have incomplete sums of $\widetilde{a}_P(t) \bmod P$. Let $S_I(r, P)$ denote a generic incomplete sum. By Hasse,

$$S_I(r, P) \leqslant P \cdot 2^{r_1}\sqrt{p_1^{r_1}} \cdot 2^{r_2}\sqrt{p_2^{r_2}} = 2^{r_1+r_2}p_1^{1+\frac{r_1}{2}} \cdot p_2^{1+\frac{r_2}{2}} = 2^r P^{1+\frac{r}{2}}, \tag{5.11}$$

where the last expression is a convenient abuse of notation:

$$2^r = 2^{r_1+r_2}, \quad P^r = p_1^{r_1} \cdot p_2^{r_2}. \tag{5.12}$$

For fixed i and d, we evaluate the arguments at $t = t_i(d) + t'd^2$. Let

$$\widetilde{a}_{d,i,P}(t') = \widetilde{a}_P\Big(t_i(d) + t'd^2\Big), \quad G_{d,i,P}(t') = G_P\Big(t_i(d) + t'd^2\Big). \tag{5.13}$$

5.3 Ranges and Contributions of Sums over Primes

Each prime sum is to (approximately) $C(N)^{\frac{\sigma_j}{2^{r_j}-\kappa(r)}} \approx N^{\frac{m\sigma_j}{2^{r_j}-\kappa(r)}}$, as $C(t)$ is a degree m polynomial. We assume $\sigma_j < \frac{1}{2}$ as we do not worry about $p^2 > N$. This is harmless, as handling the error terms forces the support to be significantly less than $\frac{1}{2}$.

LEMMA 5.1 CONTRIBUTIONS FROM SUMS OVER PRIMES. For $r_j = 1$, summing $\frac{p^{\frac{1}{2}}}{|\mathcal{F}|}$ does not contribute for $\sigma_j < \frac{2}{3m}$. For $r_j = 2$, summing $\frac{1}{|\mathcal{F}|}$ does not contribute for $\sigma_j < \frac{2}{m}$ for $\kappa(r) = 1$ and $\frac{1}{m}$

for $k(r) = 2$. As we often have two sums, dividing the above supports by 2 ensures all errors are manageable: write $\frac{1}{|\mathcal{F}|}$ as $\frac{1}{\sqrt{|\mathcal{F}|}}\frac{1}{\sqrt{|\mathcal{F}|}}$.

5.3.1 *Expected Result*   To simplify the proof, we assume

$$A_{1,\mathcal{F}}(p) = -rp + O(1)$$
$$A_{2,\mathcal{F}}(p) = p^2 + O(p^{\frac{3}{2}}). \tag{5.14}$$

For a general rational surface, $A_{1,\mathcal{F}}(p) \neq -rp + O(1)$. A careful book-keeping of the arguments below show that we only need to be able to handle sums such as

$$\sum_p \frac{\log p}{\log X} f\left(\frac{\log p}{\log X}\right)\frac{A_{1,\mathcal{F}}(p)}{p^2}. \tag{5.15}$$

For surfaces where Tate's conjecture is known, we may replace $A_{1,\mathcal{F}}(p)$ in the above sum with the rank of the family over $\mathbb{Q}(t)$ (see Lemma C.6 and [RSi]). For notational simplicity, in the proof below we assume $A_{1,\mathcal{F}}(p) = -rp + O(1)$, and content ourselves with noting a similar proof works in general.

$A_{r_j}(p_j) = c_j \cdot p_j^{r_j}$ plus lower order terms not contributing for any support. (This is not quite true. For families where the curves have complex multiplication, often $a_t(p)$ vanishes for half the primes, and has double the expected contribution for the other primes. This case is handled similarly, using Lemmas C.1 and C.5).

Hence $S_c(r, P) = c_1 c_2 p_1^{r_1} p_2^{r_2} = c_1 c_2 P^r$ plus lower terms. For each pair (d, i) we expect (if we can manage the conductors) to have approximately $\frac{N/d^2}{P}$ complete sums of $S_c(r, P) = c_1 c_2 P^r$. We hit this with $\frac{1}{N}\frac{\log p_j}{\log C(t)}\frac{1}{p_j^{r_j}}$ for each non-zero r_j. We have approximately $\frac{\log p_j}{\log C(t)}\frac{1}{P^r}$.

A sum like $\sum_{p_j}\frac{\log p_j}{\log C(t)}\frac{1}{p_j}g\left(\frac{\log p_j}{\log C(t)}\right)$ contributes; if we had an additional $\frac{1}{\log N}$ there would be no net contribution.

Thus, we expect terms of the size P^r to contribute, and $\frac{P^r}{\log N}$ to not contribute.

We rewrite Conditions 4.7 in a more tractable form, using $A_{1,\mathcal{F}}(p)$, $A_{2,\mathcal{F}}(p)$ and $S_c(r, P)$. Assume the family satisfies Equation 5.14 (or the related equation if $a_t(p)$ vanishes for half the primes). Then

i) $P = p$, $\widetilde{a}_P(t) = a_t(p)$: $\frac{S_c(r,P)}{P} = \frac{-rp+O(1)}{p} = -r + O(\frac{1}{p})$

ii) $P = p$, $\widetilde{a}_P(t) = a_t^2(p)$: $\frac{S_c(r,P)}{P} = \frac{p^2+O(p^{\frac{3}{2}})}{p} = p + O(\sqrt{p})$

iii) $P = p_1 p_2$, $\widetilde{a}_P(t) = a_t(p_1)a_t^2(p_2)$: $\frac{S_c(r,P)}{P} = \frac{-rp_1 p_2^2 + O(p_1 p_2^{\frac{3}{2}})}{p_1 p_2} = -rp_2 + O(\sqrt{p_2})$

iv) $P = p_1 p_2$, $\widetilde{a}_P(t) = a_t(p_1)a_t(p_2)$:
   (a) $\frac{S_c(r,P)}{P} = \frac{r^2 p_1 p_2 + O(p_1+p_2)}{p_1 p_2} = r^2 + O(\sqrt{p_1} + \sqrt{p_2})$ if $p_1 \neq p_2$
   (b) $\frac{S_c(r,P)}{P} = \frac{p^2+O(p^{\frac{3}{2}})}{p} = p + O(\sqrt{p})$ if $p_1 = p_2 = p$

v) $P = p_1 p_2$, $\widetilde{a}_P(t) = a_t^2(p_1)a_t^2(p_2)$: $\frac{S_c(r,P)}{P} = \frac{p_1^2 p_2^2 + O(p_1^{\frac{3}{2}} p_2^{\frac{3}{2}})}{p_1 p_2} = p_1 p_2 + O(\sqrt{p_1 p_2})$

We have proved

LEMMA 5.2 CONDITIONS TO EVALUATE THE FIVE TYPES OF SUMS. *Assume the family satisfies Equation 5.14. If, up to lower order terms, the five sums (Equation 4.7) are $G_P(N)\frac{S_c(r,P)}{P}$, then the family satisfies Conditions 4.7.*

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5.4 Taylor Expansion of $G_{d,i,P}(t')$

Fix i and d. We calculate the first order Taylor Expansion of $G_{d,i,P}(t') = G_P(t_i(d) + t'd^2)$. $G_{d,i,P}$ involves t' only through expressions like $\frac{\log p_j}{\log C(t)}$, where $t = t_i(d) + t'd^2$. Let $C(t) = h_m t^m + \cdots + h_0$.

The derivative of $G_{d,i,P}$ in t' will involve nice functions times factors like

$$
\begin{aligned}
\frac{d}{dt'}\frac{\log p_j}{\log C(t)} &= -\frac{\log p_j}{\log^2 C(t)}\frac{d}{dt'}\log C(t_i(d) + t'd^2) \\
&= -\frac{\log p_j}{\log^2 C(t)}\frac{mh_m t^{m-1}d^2 + \cdots}{h_m t^{m-1}\cdot(t_i(d)+t'd^2)+\cdots} \\
&\leqslant \left(\frac{10m}{|h_m|}\max_{0\leqslant k\leqslant m-1}|m-k|\cdot|h_{m-k}|\right)\frac{\log p_j}{\log^2 C(t)}\frac{d^2}{t_i(d)+t'd^2},
\end{aligned}
\tag{5.16}
$$

provided N is sufficiently large.

As $p_j \leqslant C(t)^\sigma$, where σ is related to the support of G, $\frac{\log p_j}{\log C(t)} \leqslant \sigma$. As $C(t)$ is of size a power of t, we have

LEMMA 5.3 TAYLOR EXPANSION OF $G_{d,i,P}$.

$$
G_{d,i,P}(t') = G_{d,i,P}(0) + O\left(\frac{1}{\log N}\right).
\tag{5.17}
$$

The constant above does not depend on p_j, d or i.

By the Mean Value Theorem $\exists \xi \in [0,t']$, corresponding to $t_\xi = t_i(d) + \xi d^2 \in [N, 2N+d^2] \subset [N, 2.1N]$, such that

$$
G_{d,i,P}(t') = G_{d,i,P}(0) + \frac{d}{dt'}G_{d,i,P}\Big|_{t'=\xi}(t'-0).
\tag{5.18}
$$

First, we have derivatives of $\frac{\log p_j}{\log C(t)}$, which can be universally bounded from the support of G. Second, we evaluate G and its derivative at $2^{r_j - \kappa(r)}\frac{\log p_j}{\log C(t_\xi)}$. We see it is sufficient to universally bound functions like $\frac{d}{dt'}g\left(\frac{\log p}{\log C(t)}\right)$.

$\log C(t_\xi) \approx \log C(N)$. Evaluating the derivative at ξ, by Equation 5.16 we have something bounded by $\frac{1}{\log C(t_\xi)}\frac{d^2}{t_i(d)+\xi d^2}$. We then multiply by $t'-0$. Thus we are bounded by $\frac{1}{\log C(N)}\frac{t'd^2}{t_i(d)+\xi d^2}$. As $t_i(d) \geqslant N$ and $t'd^2 \leqslant N$, the bound is at most $\frac{1}{\log C(N)}$.

LEMMA 5.4 FURTHER TAYLOR EXPANSION OF $G_{d,i,P}$.

$$
G_{d,i,P}(t') = G_P(N) + O\left(\frac{1}{\log N}\right).
\tag{5.19}
$$

The constant above does not depend on p_j, d or i.

The proof is similar to the previous lemma. $G_{d,i,P}(0) = G_P\left(t_i(d)\right)$, $t_i(d) \in [N, N+d^2]$. Thus, to replace $G_{d,i,P}(0)$ with $G_P(N)$ involves Taylor Expanding $G_P(t)$ around $t=N$. □

This allows us to replace all the conductors of curves with $D(t)$ good with the value from $t=N$ with small error. This is very convenient, as $G_P(N)$ has no t', i or d dependence. Consequently, we will be able to move it past all summations except over primes, which will allow us to take advantage of cancellations in t-sums of the $a_t(p)$'s.

5.5 Removing the $\nu(d)\|S\|_\infty$ Term for $d < \log^l N$

$$\sum_{t \in T(d)} S(t) = \sum_{i=1}^{\nu(d)} \sum_{t'=0}^{[N/d^2]} S\Big(t_i(d) + t'd^2\Big) + O\Big(\nu(d)\|S\|_\infty\Big). \tag{5.20}$$

We show the $O\Big(\nu(d)\|S\|_\infty\Big)$ piece does not contribute for $d < \log^l N$. Using Hasse to trivially bound $\|S\|_\infty$ gives $2^r P^r$. We hit this with $\frac{1}{P^r}$ and sum over the primes, which will be at most $O(N^\sigma)$. We now sum over $d < \log^l N$, getting

$$\ll N^\sigma \sum_{d=1}^{\log^l N} \nu(d) \ll N^\sigma \sum_{d=1}^{\log^l N} d^\epsilon \ll N^\sigma \log^{l(1+\epsilon)} N. \tag{5.21}$$

We then divide by the cardinality of the family, which is assumed to be a multiple of N. There is no contribution for $\sigma_1 + \sigma_2 < 1$.

5.6 Sieving

Let B be the largest square which divides $D(t)$ for all t. Recall by t good we mean $D(t)$ is square-free except for primes dividing B, and for $p|B$, the power of $p|D(t)$ is independent of t. By Theorem A.5, possibly after passing to a subsequence, we can approximate t good by

$$\sum_{\substack{t \in [N,2N] \\ t \ good}} S(t) = \sum_{\substack{d=1 \\ (d,B)=1}}^{\log^l N} \mu(d) \sum_{\substack{t \in [N,2N] \\ D(t) \equiv 0(d^2)}} S(t) + O\Big(\sum_{t \in \mathcal{T}} S(t)\Big), \tag{5.22}$$

where the set of good t is $c_{\mathcal{F}} N + o(N)$, $c_{\mathcal{F}} > 0$, $\mathcal{T}$ is the set of $t \in [N, 2N]$ such that $D(t)$ is divisible by the square of a prime $p > \log^l N$ and $|\mathcal{T}| = o(N)$.

5.7 Contributions from $d < \log^l N$

We would like to use Lemma 5.4 to replace $G_{d,i,P}(t')$ with $G_P(N)$ plus a manageable error. This works for pairs such as $r = (2,0)$ or $r = (2,2)$ but fails for pairs such as $r = (1,0)$. There, we need to evaluate $\frac{1}{|\mathcal{F}|} \sum_{E \in \mathcal{F}} \frac{1}{p} S(r,p)$. Replacing $\widetilde{a}_p(t)$ with $|a_t(p)| \leqslant 2\sqrt{p}$ gives

$$\ll \frac{1}{|\mathcal{F}|} \frac{N}{p} \sqrt{p}, \tag{5.23}$$

which is disastrous when we sum over p. The reason we must trivially bound $\widetilde{a}_P(t)$ is the Taylor Expansion. We evaluate the derivative at $\xi(t') = \xi(p_j, i, d; t')$. The dependence of the other parameters prevents us from obtaining complete sums (mod P) and using that cancellation for control. We need to keep the cancellation from summing $\widetilde{a}_P(t)$.

We use Partial Summation twice. Note we may always replace a $G_{d,i,P}(t')$ with a $G_P(N)$ at a cost of $\frac{1}{\log N}$.

Let $\widetilde{A}_P(u) = \sum_{t'=0}^{u} \widetilde{a}_P(t')$. As $(p_i, d) = 1$ (**this is why we are assuming $d \leqslant \log^l N$ and $p_i \geqslant \log^l N$**), every time t' increases by P we have a complete sum of the $\widetilde{a}_P$'s. Thus,

$$\widetilde{A}_P(u) = \Big[\frac{u}{P}\Big] S_c(r,P) + O\Big(P^{1+\frac{r}{2}}\Big) = \frac{u}{P} S_c(r,P) + O\Big(P^R\Big)$$

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$$R = 1 + \frac{r}{2}, \quad P^R = \prod_{\substack{j=1 \\ r_j \neq 0}}^{2} p_j^{1+\frac{r_j}{2}}. \tag{5.24}$$

In the above, the first error term is from our bound for the incomplete sum of at most P terms, each term bounded by $\sqrt{p_1^{r_1} p_2^{r_2}} = P^{\frac{r}{2}}$. Dropping the greatest integer brackets costs at most $S_c(r, P) = O(P^r)$. $P^r = p_1^{r_1} p_2^{r_2}$, and $P^{1+\frac{r}{2}} = p_1^{1+\frac{r_1}{2}} p_2^{1+\frac{r_2}{2}}$. As $r_j \in \{0, 1, 2\}$, $r_j \leqslant 1 + \frac{r_j}{2}$. Thus, we may incorporate the error from removing the greatest integer brackets into the $O(P^R)$ term.

$$S(d, i, r, P) = \sum_{t'=0}^{[N/d^2]} \widetilde{a}_{d,i,P}(t') G_{d,i,P}(t')$$

$$= \left(\frac{[N/d^2]}{P} S_c(r, P) + O\left(P^R\right) \right) G_{d,i,P}([N/d^2])$$

$$- \sum_{u=0}^{[N/d^2]-1} \left(\frac{u}{P} S_c(r, P) + O\left(P^R\right) \right) \left(G_{d,i,P}(u) - G_{d,i,P}(u+1) \right)$$

$$S(r, P) = \sum_{d=1}^{\log^l N} \mu(d) \sum_{i=1}^{\nu(d)} S(d, i, r, P) = \sum_{w=1}^{4} \sum_{d=1}^{\log^l N} \mu(d) \sum_{i=1}^{\nu(d)} S_w(d, i, r, P). \tag{5.25}$$

5.7.1 *First Sum:* $\frac{[N/d^2]}{P} S_c(r, P) G_{d,i,P}([N/d^2])$ Summing over i and d yields

$$S_1(r, P) = \sum_{d=1}^{\log^l N} \mu(d) \sum_{i=1}^{\nu(d)} \frac{[N/d^2]}{P} S_c(r, P) G_{d,i,P}([N/d^2])$$

$$= \frac{S_c(r, P)}{P} \sum_{d=1}^{\log^l N} \mu(d) \sum_{i=1}^{\nu(d)} \left[\frac{N}{d^2} \right] \left(G_P(N) + O\left(\frac{1}{\log N} \right) \right)$$

$$= \frac{S_c(r, P) G_P(N)}{P} \sum_{d=1}^{\log^l N} \mu(d) \sum_{i=1}^{\nu(d)} \sum_{t'=0}^{[N/d^2]} \left(1 + O\left(\frac{1}{\log N} \right) \right)$$

$$= \frac{S_c(r, P) G_P(N)}{P} \sum_{d=1}^{\log^l N} \mu(d) \left(O(\nu(d)) + \sum_{\substack{t=N \\ D(t) \equiv 0(d^2)}}^{2N} 1 \right) \left(1 + O\left(\frac{1}{\log N} \right) \right)$$

$$= \frac{S_c(r, P) G_P(N)}{P} |\mathcal{F}| + \frac{S_c(r, P)}{P} \cdot o(N). \tag{5.26}$$

In the last line, the error term follows from Equation 5.4 (which gives the d, t-sums are $|\mathcal{F}| + o(N)$) and Lemma A.2 (which gives $\nu(d) \ll d^\epsilon$). Dividing by $|\mathcal{F}| = c_{\mathcal{F}} N + o(N)$, the error term will not contribute when we sum over primes, leaving us with $\frac{S_c(r,P) G_P(N)}{P}$.

5.7.2 *Second Sum:* $O(P^R) G_{d,i,P}([N/d^2])$ Summing over i and d yields

$$S_2(r, P) \ll \sum_{d=1}^{\log^l N} |\mu(d)| \sum_{i=1}^{\nu(d)} P^R |G_{d,i,P}([N/d^2])|$$

$$\ll P^R \sum_{d=1}^{\log^l N} |\mu(d)| \sum_{i=1}^{\nu(d)} ||G||_\infty$$

$$\ll P^R \sum_{d=1}^{\log^l N} |\mu(d)| \sum_{i=1}^{\nu(d)} 1. \tag{5.27}$$

As $\nu(d) \ll d^\epsilon$, we obtain

$$S_2(r, P) \ll P^R \log^{l(1+\epsilon)} N \leqslant P^R \log^{2l} N = P^{1+\frac{r}{2}} \log^{2l} N. \tag{5.28}$$

We divide by $|\mathcal{F}| = c_\mathcal{F} N + o(N)$, hit it with $\frac{1}{P^r}$ and then sum over the primes. By Lemma 5.1, for small support ($\sigma = \sigma_1 + \sigma_2 < \frac{2}{3m}$) there is no contribution.

5.7.3 *Third Sum:* $\sum_{u=0}^{[N/d^2]-1} \frac{u}{P} S_c(r, P) \Big(G_{d,i,P}(u) - G_{d,i,P}(u+1)\Big)$ We apply Partial Summation, where $a_u = G_{d,i,P}(u) - G_{d,i,P}(u+1)$ and $b_u = \frac{u}{P} S_c(r, P)$. Thus

$$S_3(d, i, r, P) = \left(G_{d,i,P}(0) - G_{d,i,P}\Big([N/d^2]\Big)\right) \frac{[N/d^2] - 1}{P} S_c(r, P)$$

$$- \sum_{u=0}^{[N/d^2]-2} \Big(G_{d,i,P}(0) - G_{d,i,P}(u+1)\Big) \frac{1}{P} S_c(r, P). \tag{5.29}$$

Using the Taylor Expansion, we gain a $\frac{1}{\log N}$ in the first term, making it of size $\frac{S_c(r,P)}{P} \frac{[N/d^2]}{\log N} \ll \frac{S_c(r,P)}{P} \frac{|\mathcal{F}|}{d^2 \log N}$.

For the second term, we have $< [N/d^2]$ summands, each $\ll \frac{1}{\log N} \frac{S_c(r,P)}{P}$. We again obtain a term of size $\frac{S_c(r,P)}{P} \frac{|\mathcal{F}|}{d^2 \log N}$.

We sum over i and d.

$$S_3(r, P) \ll \sum_{d=1}^{\log^l N} |\mu(d)| \sum_{i=1}^{\nu(d)} \frac{S_c(r, P)}{P} \frac{|\mathcal{F}|}{d^2 \log N}$$

$$\ll \frac{S_c(r, P)}{P} \frac{|\mathcal{F}|}{\log N} \sum_{d=1}^{\log^l N} \sum_{i=1}^{\nu(d)} \frac{1}{d^2}$$

$$\ll \frac{S_c(r, P)}{P} \frac{|\mathcal{F}|}{\log N} \sum_{d=1}^{\log^l N} \frac{\nu(d)}{d^2}. \tag{5.30}$$

As $\nu(d) \ll d^\epsilon$, $S_3(r, P) \ll \frac{S_c(r,P)}{P} \frac{|\mathcal{F}|}{\log N}$.

5.7.4 *Fourth Sum:* $\sum_{u=0}^{[N/d^2]-1} O(P^R) \Big(G_{d,i,P}(u) - G_{d,i,P}(u+1)\Big)$ Using the Taylor Expansion for $G_{d,i,P}(u) - G_{d,i,P}(u+1)$ is insufficient. That gives $\frac{NP^R}{d^2 \log N}$. Summing over i and d is manageable, giving $O(P^R \frac{|\mathcal{F}|}{\log N})$. Dividing by the cardinality of the family yields $O(\frac{P^R}{\log N})$.

The problem is in summing over the primes, as we no longer have $\frac{1}{|\mathcal{F}|}$. We multiply by $\frac{1}{P^r}$. We recall the definitions of r and R and unwind the above.

Consider the case $r = (1, 0)$. Then $P = p_1 = p$, $R = 1 + \frac{r_1}{2} = \frac{3}{2}$, and $\frac{1}{P^r} = \frac{1}{p}$. We have

$$\sum_{p=\log^l N}^{N^{m\sigma}} \frac{1}{p} \frac{p^{\frac{3}{2}}}{\log N} \gg N^{m\sigma}. \tag{5.31}$$

As $N \to \infty$, this term diverges. We need significantly better cancellation in

$$S_4(r, P) = \sum_{d=1}^{\log^l N} \mu(d) \sum_{i=1}^{\nu(d)} \sum_{u=0}^{[N/d^2]-1} O(P^R)\Big(G_{d,i,P}(u) - G_{d,i,P}(u+1)\Big). \tag{5.32}$$

Taking absolute values and using the maximum of the $O(P^R)$ terms gives

$$S_4(r, P) \ll P^R \sum_{d=1}^{\log^l N} \sum_{i=1}^{\nu(d)} \sum_{u=0}^{[N/d^2]-1} \Big|G_{d,i,P}(u) - G_{d,i,P}(u+1)\Big|. \tag{5.33}$$

The constant is independent of P. Taking the maximum of the P^R term involves the maximum of either the incomplete sum or one complete sum. Using Hasse, the constant is at most $2^{r_1+r_2}$. Thus, the constant in Equation 5.33 does not depend on P.

If exactly one of the r_j's is non-zero, then

$$G_{d,i,P}(u) - G_{d,i,P}(u+1) = g\Big(\frac{\log p}{\log C(t_i(d) + ud^2)}\Big) - g\Big(\frac{\log p}{\log C(t_i(d) + (u+1)d^2)}\Big) \tag{5.34}$$

for some Schwartz function g of compact support.

If both of the r_j's are non-zero, we may write $G_{d,i,P}(u)$ as the product of two functions, say g_1 and g_2. Thus

$$G_{d,i,P}(u) = \prod_{j=1}^{2} g_j\Big(\frac{\log p_j}{\log C(t_i(d) + ud^2)}\Big) \tag{5.35}$$

Recall

$$\begin{aligned}|a_1 a_2 - b_1 b_2| &= |a_1 a_2 - b_1 a_2 + b_1 a_2 - b_1 b_2| \\ &\leqslant |a_1 a_2 - b_1 a_2| + |b_1 a_2 - b_1 b_2| = |a_2| \cdot |a_1 - b_1| + |b_1| \cdot |a_2 - b_2|.\end{aligned} \tag{5.36}$$

We apply the above to our function $G_{d,i,P}(u) = g_1(d, i, p_1; u)g_2(d, i, p_2; u)$. Each $g_j(d, i, p_j; u)$ can be bounded independently of d, i, p_j and u, as each g_j is a Schwartz function defined in terms of the n-level density test functions. Let $B = \max_j ||g_j||_\infty$. Then

$$\begin{aligned}S_4(d, i, r, P)(u) &= G_{d,i,P}(u) - G_{d,i,P}(u+1) \\ &= \prod_{\substack{j=1 \\ r_j \neq 0}}^{2} g_j\Big(\frac{\log p_i}{\log C(t_i(d) + ud^2)}\Big) - \prod_{\substack{j=1 \\ r_j \neq 0}}^{2} g_j\Big(\frac{\log p_j}{\log C(t_i(d) + (u+1)d^2)}\Big) \\ &\leqslant \sum_{\substack{j=1 \\ r_j \neq 0}}^{2} B \cdot \Big|g_j\Big(\frac{\log p_j}{\log C(t_i(d) + ud^2)}\Big) - g_j\Big(\frac{\log p_j}{\log C(t_i(d) + (u+1)d^2)}\Big)\Big|.\end{aligned} \tag{5.37}$$

We sum the above over u, i and d. Let $t_{i,d}(u) = t_i(d) + ud^2$.

$$S_4(r,P) \leqslant 2^r P^R \sum_{d=1}^{\log^l N} |\mu(d)| \sum_{i=1}^{\nu(d)} \sum_{u=0}^{[N/d^2]-1} S_4(d,i,r,P)(u)$$

$$\leqslant 2^r P^R \sum_{d=1}^{\log^l N} \sum_{i=1}^{\nu(d)} \sum_{\substack{j=1 \\ r_j \neq 0}}^{2} B \sum_{u=0}^{[N/d^2]-1} \left| g_j\Big(\frac{\log p_j}{\log C(t_{i,d}(u))}\Big) - g_j\Big(\frac{\log p_j}{\log C(t_{i,d}(u+1))}\Big) \right|. \quad (5.38)$$

We show the u-sums are bounded independent of p_j, i, d, and N. We may add

$$\left| g_j(0) - g_j\Big(\frac{\log p_j}{\log C(t_i(d))}\Big) \right| + \left| g_j\Big(\frac{\log p_j}{\log C(t_i(d) + [N/d^2]d^2)}\Big) - g_j(1000\sigma) \right|. \quad (5.39)$$

As each g_j is a Schwartz function, they are of bounded variation. Let $x_u(d,i,p_j) = \frac{p_j}{\log N_{t_i(d)+ud^2}}$. **As the conductors are monotone increasing**, $x_u(d,i,p_j) > x_{u+1}(d,i,p_j)$. Thus, we have a partition of $[0, 1000\sigma]$, and we may now apply theorems on bounded variation to bound the u-sum independent of p_j, i, d and N, obtaining $\ll 1000\sigma$.

The above is an exercise in the bounded variation of $g(x)$ on $[0, \sigma]$. If we were to regard this as a problem in the bounded variation of $g_{j;p_j,d,i}$ we would have u ranging over at least $\left[0, [N/d^2]\right]$. Even though we would gain a $\frac{1}{\log N}$ from the derivatives, the bounded variation bound depends on the size of the interval, which here is of length $[N/d^2]$. We could also argue that each g_j has continuous, bounded first derivative on $[0, 1000\sigma]$. By the Mean Value Theorem, the u-sum is $\ll \|g_j'\|_\infty \cdot |1000\sigma - 0|$.

Thus, the u and the j-sums are universally bounded. We are left with $\ll P^R$. Summing over i and d gives $\ll P^R \log^{l(1+\epsilon)} N$. We multiply by $\frac{1}{P^r}$ and sum over the primes. The prime sums give $N^{h(\sigma)}$; dividing by the cardinality of the family (a multiple of N), we find there is no contribution for small support.

Note: *if our conductors are not monotone, we cannot apply theorems on bounded variation. The problem is we could transverse $[0, 1000\sigma]$ (or a large subset of it) up to $\frac{N}{d^2}$ times. This is why S_4 is the most difficult of the error pieces, and why we needed to obtain polynomial expressions for the conductors for good t.*

5.7.5 *Summary of Contributions for $d < \log^l N$*

LEMMA 5.5 CONTRIBUTIONS FOR $d < \log^l N$. *Based on our Sieving Assumptions for the family (for good $D(t)$ the conductors are given by a monotone polynomial in t, a positive percent of $t \in [N, 2N]$ give $D(t)$ good), the main term contribution from $d < \log^l N$ is $\frac{S_c(r,P)}{P} G_P(N)|\mathcal{F}|$. The error terms are either of size $\frac{S_c(r,P)}{P} o(|\mathcal{F}|)$, which won't contribute when we sum over primes, or are such that their sum over primes will not contribute.*

5.8 Contributions from $t \in \mathcal{T}$

5.8.1 *Preliminaries*   We are left with estimating the contributions from the troublesome set

$$\mathcal{T} = \Big\{ t \in [N, 2N] : \exists d > \log^l N \text{ with } d^2 | D(t) \Big\} \quad (5.40)$$

We will show in Theorem A.5 that $|\mathcal{T}| = o(N)$. By Cauchy-Schwartz

$$\left| \sum_{t \in \mathcal{T}} S(t) \right| \leqslant \left(\sum_{t \in \mathcal{T}} S^2(t) \right)^{\frac{1}{2}} \left(\sum_{t \in \mathcal{T}} 1 \right)^{\frac{1}{2}} \leqslant \left(\sum_{t=N}^{2N} S^2(t) \right)^{\frac{1}{2}} o\left(\sqrt{N} \right). \tag{5.41}$$

We then sum over the primes, and need to show the sum over t is $O(N)$. As it stands, however, this is not sufficient to control the error. Quick sketch: assume $S(t) = a_t(p) g(\frac{\log p}{\log C(t)})$. Ignoring the t-dependence in the conductors, we have

$$\sum_{t=N}^{2N} S(t) \approx g^2 \left(\frac{\log p}{\log C(N)} \right) \frac{N}{p} \sum_{t(p)} a_t^2(p)$$

$$\approx g^2 \left(\frac{\log p}{\log C(N)} \right) \frac{N}{p} p^2 = O(Np). \tag{5.42}$$

Taking the square-root, we hit it with $\frac{1}{p}$ and sum over $p \leqslant N^\sigma$, which is not $O(\sqrt{N})$.

$S(t)$ is the product of at most two terms involving factors such as $a_t^{r_j}(p_j)$. We hit this with factors $p_j^{-r_j}$ and sum over p. Thus, instead of $S(t)$ consider $S_1(t)S_2(t)$, where $S_j(t)$ incorporates the sum over primes to the jth power and all relevant factors.

$$S = \sum_{t=N}^{2N} \left[\prod_{\substack{j=1 \\ r_j \neq 0}}^{2} \sum_{p_j \geqslant \log^l N} p_j^{-r_j} g_j \left(\frac{\log p_j}{\log C(t)} \right) a_t^{r_j}(p_j) \right]^2$$

$$= \sum_{t=N}^{2N} \prod_{w=1}^{2} \prod_{\substack{j=1 \\ r_j \neq 0}}^{2} \sum_{p_{jw} \geqslant \log^l N} p_{jw}^{-r_{jw}} g_{jw} \left(\frac{\log p_{jw}}{\log C(t)} \right) a_t^{r_{jw}}(p_{jw}). \tag{5.43}$$

We proceed similarly as in the $d \leqslant \log^l N$ case, except now there are no d and i, and we have potentially four factors instead of one or two. On expanding, we combine terms where we have the same prime occurring multiple times. There are several types of sums: four distinct primes (four factors), three distinct primes (three factors), ..., all primes the same (one factor). We do the worst case, when there are four factors; the other cases are handled similarly.

5.8.2 A Specific Case: Four Distinct Primes   Assume we have four distinct primes. Relabelling, we have $p^{-r_i} a_t^{r_i}(p_i)$ for $i = 1$ to 4. Let $P = \prod_{i=1}^{4} p_i$. Interchange the t-summation with the p_i-summations. As before, we apply partial summation to $\sum_{t=N}^{2N} \prod_{i=1}^{4} a_t^{r_i}(p_i) \cdot g_i(p_i, t) p^{-r_i} = \sum_{N}^{2N} a(P, t) \cdot b(P, t)$, the only change being the addition of the factors $\prod_i p^{-r_i}$. Now $A(u) = \sum_{t=N}^{u} a(P, t)$ $= \frac{u-N}{P} S_c(r, P) + O(\prod_{i=1}^{4} p_i^{1+\frac{r_i}{2}})$, $S_c(r, P) = \prod_{i=1}^{4} A_{r_i, \mathcal{F}}(p_i)$ by Lemma 2.1. Let $P^R = \prod_{i=1}^{4} p_i^{1+\frac{r_i}{2}}$; the error in the partial summation is $O(P^R)$.

As in Equation 5.25 we have

$$S = \prod_{i=1}^{4} \sum_{p_i} \sum_{t=N}^{2N} a_t^{r_i}(p_i) \cdot p^{-r_i} G(P, t)$$

$$= \prod_{i=1}^{4} \sum_{p_i} \left(\frac{N}{P} S_c(r, P) + O(P^R) \right) p_i^{-r_i} G(P, 2N)$$

$$-\prod_{i=1}^{4}\sum_{p_i}\sum_{u=N}^{2N-1}\left(\frac{u-N}{P}S_c(r,P)+O(P^R)\right)p_i^{-r_i}\left(G(P,u)-G(P,u+1)\right). \qquad (5.44)$$

For $r \geqslant 2$ by Hasse $A_{r,\mathcal{F}}(p) \leqslant 2^r p^{1+\frac{r}{2}}$. For $r=1$, $A_{1,\mathcal{F}}(p) \ll p$ by [De]. Hence $\forall r$, $A_{r,\mathcal{F}}(p) \ll p^r$.

$$\prod_{i=1}^{4}\frac{S_c(P)}{p_i}p_i^{-r_i} \;\ll\; \prod_{i=1}^{4}\frac{A_{r_i,\mathcal{F}}(p_i)}{p_i^{1+r_i}} \;\ll\; \prod_{i=1}^{4}\frac{p_i^{r_i}}{p_i^{1+r_i}} \;=\; \prod_{i=1}^{4}\frac{1}{p_i}. \qquad (5.45)$$

We can immediately handle the first sum. Inserting absolute values yields something like

$$\prod_{i=1}^{4}\sum_{p_i}\frac{\log p_i}{\log C(2N)}\left|g_i\left(\frac{\log p_i}{\log C(2N)}\right)\right|\frac{1}{p_i} \ll \prod_{i=1}^{4}O(1) \qquad (5.46)$$

where the last result (the sums over the primes) follows from Corollary C.2.

Pulling out the prime factors and using partial summation again, the third sum is handled similarly.

The second and fourth pieces are more difficult, and result in significantly decreased support. We analyze this loss later. For now, we need only note that the second sum is $\prod_i \sum_{p_i} p_i^{r_i/2}$. For test functions of small support, this sum is $o(N)$.

There is a slight obstruction in applying the same argument to the fourth sum, namely, that $G(P,u)$ could be the product of four factors. Similar to the identity $|a_1a_2 - b_1b_2| \leqslant |a_1| \cdot |a_1 - b_1| + |b_1| \cdot |a_2 - b_2|$, we have

$$\begin{aligned}|a_1a_2a_3a_4 - b_1b_2b_3b_4| \leqslant \;\; & |a_2a_3a_4| \cdot |a_1 - b_1| + |b_1a_3a_4| \cdot |a_2 - b_2| \\ & + |b_1b_2a_4| \cdot |a_3 - b_3| + |b_1b_2b_3| \cdot |a_4 - b_4| \\ \leqslant \;\; & \prod_{j=1}^{4}\left(|a_j| + |b_j| + 1\right)\sum_{i=1}^{4}|a_i - b_i| \qquad (5.47)\end{aligned}$$

The rest of the proof in this case is identical to the fourth sum in the $d \leqslant \log^l N$ case.

Note: as we have always inserted absolute values before summing over primes, it is permissible to extend from the primes are distinct to all possible 4-tuples.

5.8.3 *Handling the Other Cases*   The other cases (especially cases where some primes are equal) are handled similarly. The only real change is if we have less than four factors, and this only affects the Fourth Sum. For example, if we have three factors instead of 4, set $a_4 = b_4 = 1$ in Equation 5.47.

5.9 Determining the Admissible Supports of the Test Functions

The largest errors arise from $r_i = 1$ terms, using Hasse to trivially bound partial sums of $a_t(p)$ by $p^{3/2}$ (at most p terms, each term at most $2\sqrt{p}$). Let $C(t)$ be a polynomial of degree m for good t. We assume all supports are at most $\frac{1}{2}$ (as otherwise p^2 could exceed N, changing some of our arguments above). In the 1-level densities, we encounter errors like

$$\sum_{p=\log^l N}^{N^{\sigma m}}\frac{1}{p}\frac{\log p}{\log N^m}g\left(\frac{\log p}{\log N^m}\right)p^{\frac{3}{2}} \ll \sum_{p=\log^l N}^{N^{\sigma m}}p^{\frac{1}{2}} \ll N^{\frac{3\sigma m}{2}}. \qquad (5.48)$$

We divide by $|\mathcal{F}|$, a multiple of N. The errors are negligible for $\sigma < \min\left(\frac{2}{3m}, \frac{1}{2}\right)$.

In the 2-level density, the worst case (not including the Cauchy-Schwartz arguments to handle the over-counting of almost square-free numbers) was when we had two $r_i = 1$ terms. We have two functions of support σ_1 and σ_2, and we obtain

$$\prod_{i=1}^{2} \sum_{p_i=\log^l N}^{N^{\sigma_i m}} \frac{1}{p} \frac{\log p_i}{\log N^m} g\left(\frac{\log p_i}{\log N^m}\right) p_i^{\frac{3}{2}} \ll \prod_{i=1}^{2} \sum_{p_i=\log^l N}^{N^{\sigma m}} p_i^{\frac{1}{2}} \ll N^{\frac{3(\sigma_1+\sigma_2)m}{2}} \tag{5.49}$$

We divide by a multiple of N and see the errors are negligible for $\sigma_1 + \sigma_2 < \min\left(\frac{2}{3m}, \frac{1}{2}\right)$. Thus, for $\sigma_1 = \sigma_2$, the support of each test function is half that from the 1-level density.

In applying Cauchy-Schwartz, we decrease further the allowable support. The worst case is where we have four distinct primes with $r_i = 1$. We sum as before, and obtain $N^{3(\sigma_1+\sigma_2)m}$ (there is no factor of 2 as two of the primes are associated to test functions with support σ_1 and two to σ_2). We take the square-root, and this must be $O(\sqrt{N})$. Thus, we now find $\sigma_1 + \sigma_2 < \frac{1}{2}\frac{2}{3m}$. Setting $\sigma_1 = \sigma_2$ yields the support is one-quarter that of the 1-level density.

5.10 1- and 2-Level Densities

Assume the original family has rank r over $\mathbb{Q}(t)$. The Birch and Swinnerton-Dyer conjecture and Silverman's Specialization Theorem imply, for all t sufficiently large, each curve's L-function has r family zeros at the critical point.

The Birch and Swinnerton-Dyer conjecture is only used for interpretation purposes. The results below are derived independently of this conjecture; however, assuming this allows us to interpret some of the n-level density terms as contributions from expected family zeros.

DEFINITION 5.6 NON-FAMILY DENSITY. *Let* $D_{n,\mathcal{F}}^{(r)}(f)$ *be the n-level density from the non-family zeros (ie, the trivial contributions from r family zeros have been removed).*

THEOREM 5.7 $D_{n,\mathcal{F}}(f)$ AND $D_{n,\mathcal{F}}^{(r)}(f)$, $n = 1$ OR 2. *For any one-parameter family of rank r over* $\mathbb{Q}(t)$ *satisfying*

i) *for good t(relative to $D(t)$), the conductors $C(t)$ are a monotone polynomial in t;*
ii) *up to $o(N)$, the good $t \in [N, 2N]$ are obtainable by sieving up to $d = \log^l N$; further, the number of such t is $|\mathcal{F}| = c_\mathcal{F} N + o(N)$, $c_\mathcal{F} > 0$;*
iii) $A_{1,\mathcal{F}}(p) = -rp + O(1)$, $A_{2,\mathcal{F}}(p) = p^2 + O(p^{\frac{3}{2}})$.

Then for f_i even Schwartz functions of small but non-zero support σ_i,

$$D_{1,\mathcal{F}}(f) = \widehat{f_1}(0) + \frac{1}{2}f_1(0) + rf_1(0)$$

$$D_{1,\mathcal{F}}^{(r)}(f_1) = \widehat{f_1}(0) + \frac{1}{2}f_1(0) \tag{5.50}$$

and

$$D_{2,\mathcal{F}}(f) = \prod_{i=1}^{2}\left[\widehat{f_i}(0) + \frac{1}{2}f_i(0)\right] + 2\int_{-\infty}^{\infty}|u|\widehat{f_1}(u)\widehat{f_2}(u)du$$
$$-2\widehat{f_1 f_2}(0) - f_1(0)f_2(0) + (f_1 f_2)(0)N(\mathcal{F}, -1)$$
$$+(r^2 - r)f_1(0)f_2(0) + r\widehat{f_1}(0)f_2(0) + rf_1(0)\widehat{f_2}(0)$$

$$D_{2,\mathcal{F}}^{(r)}(f_1) = \prod_{i=1}^{2}\left[\widehat{f_i}(0) + \frac{1}{2}f_i(0)\right] + 2\int_{-\infty}^{\infty}|u|\widehat{f_1}(u)\widehat{f_2}(u)du$$

$$-2\widehat{f_1 f_2}(0) - f_1(0)f_2(0) + (f_1 f_2)(0)N(\mathcal{F}, -1). \qquad (5.51)$$

Removing the contribution from r family zeros, for small support the 2-level density of the remaining zeros agrees with $SO(\text{even})$, O or $SO(\text{odd})$ if the signs are all even, equidistributed, or all odd. If Tate's conjecture is true for the surface, we may interpret r as the rank of $\mathcal{E}$ over $\mathbb{Q}(t)$.

Let $m = \deg C(t)$. For the 1-level density, $\sigma < \min(\frac{1}{2}, \frac{2}{3m})$. For the 2-level density, $\sigma_1 + \sigma_2 < \frac{1}{3m}$. For families where $\Delta(t)$ has no irreducible factors of degree 4 or more, the sieving is unconditional, otherwise the results are conditional on ABC or the Square-Free Sieve conjecture.

Proof: When we sieve we obtain $\frac{S_c(r,P)G_P(N)}{P}$ plus lower order terms. By Theorem 5.2, the family satisfies Conditions 4.7. Thus Lemma 4.6 is applicable. □

As remarked, we do not need to assume $A_{1,\mathcal{F}}(p) = -rp + O(1)$. A more cumbersome proof (using Lemma C.6) handles $A_{1,\mathcal{F}}(p)$ for surfaces where Tate's conjecture is known.

To apply Theorem 5.7, we need

  i) *the conductors are monotone polynomials for $D(t)$ good;*
 ii) *a positive percent of $D(t)$ are good, and all but $o(N)$ of the good t may be taken in the required arithmetic progressions;*
iii) *knowledge of $A_{1,\mathcal{F}}(p)$ and $A_{2,\mathcal{F}}(p)$.*

For rational surfaces, by passing to a subsequence the above conditions are satisfied. By changing $t \to ct + t_0$, Tate's algorithm yields $C(t)$ is a monotone polynomial for $D(t)$ good (Theorem B.2). By Theorem A.5, $|\mathcal{F}| = c_{\mathcal{F}}N + o(N)$, $c_{\mathcal{F}} > 0$ (ie, a positive percent of $D(t)$ are good). If Tate's conjecture is true, Rosen-Silverman (Theorem 2.2) gives $A_{1,\mathcal{F}}(p)$; if $j(E_t)$ is non-constant, Michel's Theorem (Theorem 2.3) gives $A_{2,\mathcal{F}}(p)$. We have proved

THEOREM 5.8 RATIONAL SURFACES DENSITY THEOREM. *Consider a one-parameter family of elliptic curves of rank r over $\mathbb{Q}(t)$ that is a rational surface. Assume GRH, $j(E_t)$ is non-constant, and the ABC or Square-Free Sieve conjecture if $\Delta(t)$ has an irreducible polynomial factor of degree at least 4. Let f_i be an even Schwartz function of small but non-zero support σ_i and $m = \deg C(t)$. For the 1-level density, $\sigma < \min(\frac{1}{2}, \frac{2}{3m})$. For the 2-level density, $\sigma_1 + \sigma_2 < \frac{1}{3m}$. Assume the Birch and Swinnerton-Dyer conjecture for interpretation purposes.*

Let $M(t)$ be the product of the irreducible polynomials dividing $\Delta(t)$ and not $c_4(t)$. If $M(t)$ is non-constant, then the signs of E_t, t good, are equidistributed as $N \to \infty$ (see [He]). In this case, $N(\mathcal{F}, -1) = \frac{1}{2}$.

After passing to a subsequence,

$$D_{1,\mathcal{F}}(f_1) = \widehat{f_1}(0) + \frac{1}{2}f_1(0) + rf_1(0)$$

$$D_{1,\mathcal{F}}^{(r)}(f_1) = \widehat{f_1}(0) + \frac{1}{2}f_1(0). \qquad (5.52)$$

and

$$D_{2,\mathcal{F}}(f) = \prod_{i=1}^{2}\left[\widehat{f_i}(0) + \frac{1}{2}f_i(0)\right] + 2\int_{-\infty}^{\infty}|u|\widehat{f_1}(u)\widehat{f_2}(u)du$$

$$-2\widehat{f_1 f_2}(0) - f_1(0)f_2(0) + (f_1 f_2)(0)N(\mathcal{F}, -1)$$

$$+(r^2 - r)f_1(0)f_2(0) + r\widehat{f_1}(0)f_2(0) + rf_1(0)\widehat{f_2}(0)$$

$$D_{2,\mathcal{F}}^{(r)}(f_1) = \prod_{i=1}^{2}\left[\widehat{f_i}(0) + \frac{1}{2}f_i(0)\right] + 2\int_{-\infty}^{\infty}|u|\widehat{f_1}(u)\widehat{f_2}(u)du$$

$$-2\widehat{f_1 f_2}(0) - f_1(0)f_2(0) + (f_1 f_2)(0)N(\mathcal{F}, -1). \tag{5.53}$$

The 2-level non-family density is $SO(even)$ ($SO(odd)$, O) if all curves are even (odd, the signs are equidistributed).

Thus, for small support, the 1- and 2-level non-family density agrees with the predictions of Katz and Sarnak; further, the densities confirm that the curves' L-functions behave in a manner consistent with having r zeros at the critical point, as predicted by the Birch and Swinnerton-Dyer conjecture.

6. Examples

6.1 Constant Sign Families

We consider several families where the sign of the functional equation is always positive or negative. We verify the Katz-Sarnak predictions, assuming only GRH.

6.1.1 $\mathcal{F} : y^2 = x^3 + 2^4(-3)^3(9t + 1)^2$, $9t + 1$ Square-Free Let $\mathcal{F} : y^2 = x^3 + 2^4(-3)^3(9t + 1)^2$, $t \in [N, 2N]$, $9t + 1$ square-free. Note $y^2 = x^3 + 2^4(-3)^3 D^2$ is equivalent to $y^3 = x^3 + Dz^3$. Birch and Stephens [BS] calculate the sign of the functional equation for $y^3 = x^3 + Dz^3$, D cube-free. It is

$$\epsilon_{E_D} = -w_3 \prod_{p \neq 3} w_p, \tag{6.1}$$

where $w_3 = -1$ if $D \equiv \pm 1, \pm 3(9)$ and 1 otherwise, $w_p = -1$ if $p|D, p \equiv 2(3)$ and 1 otherwise, and D is cube-free.

Consider $D = D(t) = 9t + 1$. Mod 9 it is 1, so $-w_3$ is 1. Assume a prime congruent to 2 mod 3 divides $9t+1$. If there were only one such prime, the remaining primes would be congruent to 1 mod 3, and the product over all primes dividing $9t + 1$ would be congruent to 2 mod 3, a contradiction. Hence the number of primes congruent to 2 mod 3 dividing $9t + 1$ is even. For $9t + 1$ square-free, this proves the functional equation is even.

Applying Tate's algorithm (see [Mil]), we find the conductors $C(t)$ are $3^3(9t + 1)^2$ for $9t + 1$ square-free. $\delta_D = 1$, $k = 1$, $a_k = 9$ so $\mathcal{P} = \{2, 3\}$. As $\nu(2) = 1$ and $\nu(3) = 0$, by Theorem A.5 $c_{\mathcal{F}} > 0$.

For $p \equiv 2(3)$, $x \to x^3$ is an automorphism and $a_t(p) = 0$. Therefore in the sequel we assume all primes are congruent to 1 mod 3, for any sum involving a prime congruent to 2 mod 3 is zero.

For $p > 3$ and $p \equiv 1 \bmod 3$, direct calculation gives

$$\begin{aligned} A_{1,\mathcal{F}}(p) &= 0 \\ A_{2,\mathcal{F}}(p) &= 2p^2 - 2p = 2p^2 + O(p). \end{aligned} \tag{6.2}$$

From Michel's Theorem, Theorem 2.3, we expect $A_{2,\mathcal{F}}(p) = p^2 + O(p^{\frac{3}{2}})$; however, his theorem is only applicable for non-constant $j(E_t)$. As $j(E_t)$ is constant, we must directly compute $A_{2,\mathcal{F}}(p)$. Further, as $a_t(p)$ trivially vanishes for half of the primes, we expect and observe twice the predicted contribution at the other primes. Finally, we will see later that the correction term to $A_{2,\mathcal{F}}(p)$ contributes a potential lower order term to the density functions.

By Dirichlet's Theorem for Primes in Arithmetic Progressions (using Lemma C.1 instead of Corollaries C.2 and C.3), we see the factors of 2 compensate for the restriction to primes congruent to 1 mod 3, and this will be harmless in the applications.

Thus, the family satisfies the conditions of Theorem 5.8 with $r = 0$. We verify (for small support) the Katz-Sarnak predictions. As all the signs are even, conditional only on GRH, we observe SO(even) symmetry, which is distinguishable from SO(odd) and O symmetry.

6.1.2 $\mathcal{F} : y^2 = x^3 \pm 4(4t+2)x$, $4t+2$ Square-Free Let $\mathcal{F} : y^2 = x^3 + 4(4t+2)x$, $4t+2$ square-free. We need to study sums of $\left(\frac{x^3 \pm 4(4t+2)x}{p}\right)$. For $p > 2$, changing variables by $t \to t - 2^{-1}$, $t \to \pm 16^{-1}t$, we are led to study sums of $\left(\frac{x^3+tx}{p}\right)$. If $p \equiv 3 \bmod 4$ then $\left(\frac{-1}{p}\right) = -1$. Changing variables $x \to -x$ shows $a_t(p) = -\sum_{x(p)} \left(\frac{f_t(x)}{p}\right)$ vanishes; therefore, in the sequel we only consider $p \equiv 1 \bmod 4$.

Birch and Stephens [BS] calculate the sign of the functional equation for this family. For general D, D not divisible by 4 or any fourth power, the sign of the functional equation for the curve $y^2 = x^3 + 4Dx$ is

$$w_\infty w_2 \prod_{p^2||D} w_p, \tag{6.3}$$

where $w_\infty = \text{sgn}(-D)$, $w_2 = -1$ if $D \equiv 1, 3, 11, 13 \bmod 16$ and 1 otherwise, $w_p = -1$ for $p \equiv 3(4)$, and $w_p = 1$ for other $p \geqslant 3$.

By restricting to positive, even, square-free D, we force the sign of the functional equation to be odd. Hence $\epsilon_D = -1$ if $D = 4t+2$, D square-free. If we had taken $D = -(4t+2)$, $4t+2$ square-free, we would have found $\epsilon_D = +1$.

From Tate's algorithm, for $D(t) = \pm(4t+2)$ square-free, $C(t) = 2^6(4t+2)^2$. $\delta_D = 1$, $k = 1$, $a_k = 4$ so $\mathcal{P} = \{2\}$. As $\nu(2) = 0$, by Theorem A.5 $c_\mathcal{F} > 0$.

For $p > 2$ and $p \equiv 1 \bmod 4$, direct calculation gives

$$
\begin{aligned}
A_{1,\mathcal{F}}(p) &= 0 \\
A_{2,\mathcal{F}}(p) &= 2p^2 - 2p = 2p^2 + O(p).
\end{aligned} \tag{6.4}
$$

For the family $\mathcal{F}_\pm : y^2 = x^3 \pm 4(4t+2)x$, $4t+2$ square-free, all curves in $\mathcal{F}_-$ have even sign, all curves in $\mathcal{F}_+$ have odd sign. The families satisfy the conditions of Theorem 5.8 with $r = 0$. We verify (for small support) the Katz-Sarnak predictions. As all the signs are even (odd), conditional only on GRH, we observe SO(even) (SO(odd)) symmetry.

6.1.3 $\mathcal{F} : y^2 = x^3 + tx^2 - (t+3)x + 1$ For this family (due to Washington)

$$
\begin{aligned}
c_4(t) &= 2^4(t^2 + 3t + 9) \\
\Delta(t) &= 2^4(t^2 + 3t + 9)^2 \\
j(E_t) &= 2^8(t^2 + 3t + 9).
\end{aligned} \tag{6.5}
$$

Washington ([Wa]) proved the rank is odd for $t^2 + 3t + 9$ square-free, assuming the finiteness of the Tate-Shafarevich group. Rizzo [Ri] proved the rank is odd for all t. While $j(E_t)$ is non-constant, $M(t) = 1$ ($M(t)$ is the product of all irreducible polynomials dividing $\Delta(t)$ but not $c_4(t)$). Thus, Helfgott's results on equidistribution of sign are not applicable.

For sieving convenience, we replace t with $12t+1$. Let $D(t) = 144t^2 + 60t + 13$. Tate's algorithm yields for $D(t)$ square-free, $C(t) = 2^3(144t^2 + 60t + 13)$.

$\delta_D = -2^4 3^5$, $k = 2$, $a_k = 2^4 3^2$ so $\mathcal{P} = \{2, 3\}$. $D(t)$ is a primitive integral polynomial. For $p \nmid 6$ the number of incongruent solutions of $D(t) \equiv 0 \bmod p^2$ equals the number of incongruent solutions

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of $D(t) \equiv 0 \bmod p$ (see [Nag]). As $\nu(2) = \nu(3) = 0$, by Theorem A.5, $c_{\mathcal{F}} > 0$.

Direct calculation gives

$$A_{1,\mathcal{F}}(p) = -p\left[1 + \left(\frac{-1}{p}\right)\right]. \tag{6.6}$$

Hence $A_{1,\mathcal{F}}(p)$ is $-2p$ for $p \equiv 1(4)$ and 0 for $p \equiv 3(4)$. By Theorem 2.2, the rank over $\mathbb{Q}(t)$ is 1.

As $j(E_t)$ is non-constant, by Michel's Theorem $A_{2,\mathcal{F}}(p) = p^2 + O(p^{\frac{3}{2}})$.

The conditions of Theorem 5.8 are satisfied with $r = 1$. We again verify the Katz-Sarnak predictions: there are two pieces to our densities. The first equals the contribution from 1 zero at the critical point; the second agrees with SO(odd) for small support.

6.2 Rational Families

We give two examples of rational families of elliptic curves over $\mathbb{Q}(t)$. See [Mil] for proofs, as well as a new method to generate rational families of moderate rank.

6.2.1 *Rank 1 Example*   Consider the rational family $y^2 = x^3 + 1 + tx^2$.

$$c_4(t) = 16t^2$$
$$\Delta(t) = -16(4t^3 + 27)$$
$$j(E_t) = -256\frac{t^6}{4t^3 + 27}$$
$$M(t) = 4t^3 + 27. \tag{6.7}$$

If we replace t with $6t + 1$, we can easily calculate the conductors for $D(t) = 4(6t + 1)^3 + 27$ square-free. In [Mil] we show $C(t) = 2^2\left(4(6t + 1)^3 + 27\right)$ for $D(t)$ square-free. By Hooley ([Ho], Theorem 3, page 69), as $D(t)$ is an irreducible polynomial of degree 3, $c_{\mathcal{F}} > 0$.

Direct calculations [Mil] gives $A_{1,\mathcal{F}}(p) = -p$, and a more involved calculation gives $A_{2,\mathcal{F}}(p) = p^2 - 3ph_{3,p}(2) - 1 + p\sum_{x(p)}\left(\frac{4x^3+1}{p}\right) = p^2 + O(p^{\frac{3}{2}})$, where $h_{3,p}(2)$ is one if 2 is a cube mod p and zero otherwise. Note this shows Michel's bound for $A_{2,\mathcal{F}}(p)$ is sharp.

As $j(E_t)$ and $M(t)$ are non-constant, we expect the signs to be equidistributed.

The Rational Surfaces Density Theorem is applicable, and we obtain orthogonal symmetry for the density of the non-family zeros.

6.2.2 *Rank 6 Example*   We give a more exotic example. See [Mil] for the details. Let

$$
\begin{aligned}
A &= 8916100448256000000 \\
B &= -811365140824616222208 \\
C &= 26497490347321493520384 \\
D &= -3431075943454448813363200 \\
a &= 16660111104 \\
b &= -1603174809600 \\
c &= 2149908480000
\end{aligned}
\tag{6.8}
$$

The rational family $y^2 = x^3t^2 + 2g(x)t - h(x)$, $g(x) = x^3 + ax^2 + bx + c$ and $h(x) = (A - 1)x^3 + Bx^2 + Cx + D$, has $A_{1,\mathcal{F}}(p) = -6p + O(1)$ for p large. Therefore, the family has rank 6 over $\mathbb{Q}(t)$.

Writing in Weierstrass normal form yields

$$
\begin{aligned}
y^2 &= x^3 + (2at - B)x^2 + (2bt - C)(t^2 + 2t - A + 1)x \\
&\quad + (2ct - D)(t^2 + 2t - A + 1)^2 \\
c_4(t) &= 2^{19}3^77^113^1(1475t^3 + \cdots - 7735999878503076170786750620939) \\
c_6(t) &= -2^{25}3^{11}(625t^5 + \cdots) \\
j(E_t) &= \frac{50141357421875t^9 + \cdots}{-1171875t^{10} + \cdots} \\
\Delta(t) &= -2^{44}3^{18}5^6(75t^{10} + \cdots).
\end{aligned}
\tag{6.9}
$$

This is a rational surface, $j(E_t)$ and $M(t)$ are non-constant. Thus, by the Rational Surfaces Density Theorem, we verify the Katz-Sarnak predictions for a family of rank 6 over $\mathbb{Q}(t)$!

7. Summary and Future Work

Our main result is that, modulo standard conjectures, the fluctuations of the non-family low lying zeros in one-parameter families of elliptic curves agree with the Katz-Sarnak conjectures. Further, a family of rank r over $\mathbb{Q}(t)$ has a density correction which equals the contribution of r zeros at the critical point, providing further evidence for the Birch and Swinnerton-Dyer conjecture.

We have found four families where the observed density agrees with the density of one (and only one) symmetry group. As expected, the first piece equals the contribution from r zeros at the critical point (where r is the geometric rank of the family), and the second equals SO(even) if all curves have even sign and SO(odd) if all curves have odd sign.

For these four families, we assumed only GRH. We are able to unconditionally handle the dependence of the conductors on t, the signs of the functional equations, and the error terms.

In general, the greatest difficulty is handling the variation in the conductors. Unlike other families investigated ([ILS], [Ru]), the conductors of elliptic curves vary wildly in a given family. If the discriminant $\Delta(t)$ has an irreducible factor of degree 4 or greater, either ABC or the Square-Free Sieve Conjecture must be assumed to perform the necessary sieving; if all irreducible factors are of degree at most 3, the sieving is unconditional.

The crucial observation is that, if we sieve to a positive percent subset where the conductors are monotone, then we can bound the error terms. Note the extreme delicacy of our arguments: for conductors of size $\log N$, we cannot bound the error terms if the conductors range from $\log N - \log c$ to $\log N + \log c$ for some constant c.

It was observed in [Mil] that in every family where $A_{2,\mathcal{F}}(p)$ can be directly calculated,

$$
A_{2,\mathcal{F}}(p) = p^2 + h(p) - m_{\mathcal{F}}p + O(1),
\tag{7.1}
$$

where $h(p)$ is of size $p^{\frac{3}{2}}$ and averages to zero, and $m_{\mathcal{F}}$ is a positive constant, often different for different families.

We have shown all rational families (with the same distribution of signs) have equal 1 and 2-level densities. We can, however, try to expand the densities in powers of $\frac{1}{\log N}$. The different $m_{\mathcal{F}}p$ terms will lead to potential corrections to the densities of size $\frac{1}{\log N}$, giving the exciting possibility of distinguishing different families by lower order corrections to the common densities.

Unfortunately, the size of the errors in the 1 and 2-level densities are $O\left(\frac{\log \log N}{\log N}\right)$; thus, a significantly more delicate analysis is needed before we can expand the densities.

Appendix A. Sieving Families of Elliptic Curves

Given a one-parameter family of elliptic curves E_t, we need to control the conductors $C(t)$ to determine the 1- and 2-level densities. Let the curves have discriminants $\Delta(t)$, and let $D(t)$ be the product of the irreducible polynomial factors of $\Delta(t)$.

$D(t)$ may always be divisible a fixed square; let B be the largest square dividing $D(t)$ for all t. We prove in Theorem B.2 that for a rational elliptic surface, by passing to a subsequence $\tau = c_1 t + c_0$, for $\frac{D(\tau)}{B}$ square-free, $C(t)$ is given by a polynomial in τ. Call such t (or $D(t)$ or τ) good.

In order to evaluate the sums of $\prod_i a_t^{r_i}(p_i)$, it is necessary to restrict t to arithmetic progressions; however, restricting to t good ($\frac{D(\tau)}{B}$ square-free) does not yield t in arithmetic progressions.

We overcome this difficulty by doing a partial sieve with good bounds on over-counting. For notational convenience, we consider the case where $B = 1$ below, and indicate how to modify for general B.

Let $S(t)$ be some quantity associated to our family which we desire to sum over $\mathcal{T}_{sqfree}$, where

$$\mathcal{T}_{sqfree} = \Big\{ t \in [N, 2N] : \ D(t) \text{ is sqfree} \Big\}$$

$$\mathcal{T}_N = \Big\{ t \in [N, 2N] : \ d^2 \nmid D(t) \text{ for } 2 \leqslant d \leqslant \log^l N \Big\}. \tag{A.1}$$

Clearly $\mathcal{T}_{sqfree} \subset \mathcal{T}_N$. We show $\mathcal{T}_N$ is a union of arithmetic progressions, and $|\mathcal{T}_N - \mathcal{T}_{sqfree}| = o(N)$.

The main obstruction is estimating the number of $t \in [N, 2N]$ with $D(t)$ divisible by the square of a prime $p \geqslant \log^l N$. If $k = \deg D(t)$,

$$\sum_{\substack{D(t) \ sqfree \\ t \in [N,2N]}} S(t) = \sum_{d=1}^{N^{k/2}} \mu(d) \sum_{\substack{D(t) \equiv 0 (d^2) \\ t \in [N,2N]}} S(t)$$

$$= \sum_{d=1}^{\log^l N} \mu(d) \sum_{\substack{D(t) \equiv 0 (d^2) \\ t \in [N,2N]}} S(t) + \sum_{d \geqslant \log^l N}^{N^{k/2}} \mu(d) \sum_{\substack{D(t) \equiv 0 (d^2) \\ t \in [N,2N]}} S(t). \tag{A.2}$$

For $k > 3$, the second piece is too difficult to estimate – there are too many d terms (d runs to $N^{k/2}$). If all the irreducible factors of $D(t)$ are of degree at most 3, the second piece is small. For factors of degree at most 2, this follows immediately, while for factors of degree 3 it follows from Hooley ([Ho]). For larger degrees, we need the ABC conjecture (or one of its consequences, the Square-Free Sieve conjecture).

A.1 Incongruent Solutions of Polynomials

Recall the following basic facts (see, for example, [Nag]) for an integral polynomial $D(t)$ of degree k and discriminant δ:

  i) Let p be a prime not dividing the coefficient of x^k. Then $D(t) \equiv 0 \bmod p$ has at most k incongruent solutions.

 ii) Let $D(t) \equiv 0 \bmod p_i^{\alpha_i}$ have ν_i incongruent solutions. If the primes are distinct, there are $\prod_{i=1}^r \nu_i$ incongruent solutions of $D(t) \equiv 0 \bmod \prod_{i=1}^r p_i^{\alpha_i}$.

iii) Suppose $p \nmid \delta$. Then the number of incongruent solutions of $D(t) \equiv 0 \bmod p$ equals the number of incongruent solutions of $D(t) \equiv 0 \bmod p^\alpha$.

DEFINITION A.1. *Let $\nu(d)$ be the number of incongruent solutions of $D(t) \equiv 0 \bmod d^2$.*

LEMMA A.2. *For d square-free, $\nu(d) \ll d^\epsilon$.*

The proof combines the above facts with the standard bound of the divisor function, $\tau(d) \ll d^\epsilon$.

A.2 Common Prime Divisors of Polynomials

LEMMA A.3. *Let $f(t)$ and $g(t)$ be integer polynomials with no non-constant factors over $\mathbb{Z}[t]$. Then $\exists c$ (independent of t) such that if p divides both $f(t)$ and $g(t)$, then $p|c$. In particular, $f(t)$ and $g(t)$ have no common large prime divisors.*

Proof: Euclid's algorithm.

A.3 Calculating $|\mathcal{T}_N|$

$$\sum_{t \in \mathcal{T}_N} 1 = \sum_{d=1}^{\log^l N} \mu(d) \sum_{\substack{D(t) \equiv 0 (d^2) \\ t \in [N, 2N]}} 1. \tag{A.3}$$

There are $\frac{N}{d^2}\nu(d) + O(\nu(d))$ solutions to $D(t) \equiv 0 \bmod d^2$ for $t \in [N, 2N]$. By Lemma A.2, $\nu(d) \ll d^\epsilon$ for square-free d. Thus

$$|\mathcal{T}_N| = \sum_{d=1}^{\log^l N} \mu(d) \left[\frac{N}{d^2}\nu(d) + O(\nu(d)) \right] = N \sum_{d=1}^{\log^l N} \frac{\mu(d)\nu(d)}{d^2} + O(\log^{l(1+\epsilon)} N). \tag{A.4}$$

As $\nu(d) \ll d^\epsilon$ for square-free d,

$$\left| \prod_{p < \log^l N} \left(1 - \frac{\nu(p)}{p^2} \right) - \sum_{d=1}^{\log^l N} \frac{\mu(d)\nu(d)}{d^2} \right| \ll \sum_{d=\log^l N}^{\infty} \frac{d^\epsilon}{d^2} \ll \frac{1}{\log^{l(1-\epsilon)} N}. \tag{A.5}$$

Therefore

$$|\mathcal{T}_N| = N \prod_{p < \log^l N} \left(1 - \frac{\nu(p)}{p^2} \right) + O\left(\frac{N}{\log^{l(1-\epsilon)} N} \right) + O(\log^{l(1-\epsilon)} N). \tag{A.6}$$

We may take the product over all primes with negligible cost as

$$1 - \prod_{p \geqslant \log^l N} \left(1 - \frac{\nu(p)}{p^2} \right) \ll \sum_{n \geqslant \log^l N} \frac{n^\epsilon}{n^2} \ll \frac{1}{\log^{l(1-\epsilon)} N}. \tag{A.7}$$

We have shown

LEMMA A.4. *$\mathcal{T}_N = \{t \in [N, 2N] : d^2 \nmid D(t) \text{ for } 2 \leqslant d \leqslant \log^l N\}$.*

$$|\mathcal{T}_N| = N \prod_p \left(1 - \frac{\nu(p)}{p^2} \right) + O\left(\frac{N}{\log^{l(1-\epsilon)} N} \right). \tag{A.8}$$

A.4 Estimating $\mathcal{T}_{sqfree}$

Assuming the ABC conjecture, Granville ([Gr], Theorem 1) proves the number of $t \in [N, 2N]$ such that $D(t)$ is square-free is

$$|\mathcal{T}_{sqfree}| = N \prod_p \left(1 - \frac{\nu(p)}{p^2}\right) + o(N). \tag{A.9}$$

Again, if the degree of $D(t)$ is at most 3, the ABC conjecture is not needed. The family has a positive percent of t giving $D(t)$ square-free (as we are assuming no square divides $D(t)$ for all t, no $\nu(p) = p^2$, hence the product can be bounded away from 0).

A.5 Evaluation of $|\mathcal{T}_N - \mathcal{T}_{sqfree}|$ and Applications

By Equations A.8 and A.9, as $\mathcal{T}_{sqfree} \subset \mathcal{T}_N$, we have $|\mathcal{T}_N - \mathcal{T}_{sqfree}| = o(N)$.

We have proved

$$\sum_{\substack{t \in [N,2N] \\ D(t) \, sqfree}} S(t) = \sum_{t \in \mathcal{T}_N} S(t) + O\left(\sum_{t \in \mathcal{T}} S(t)\right)$$

$$= \sum_{d=1}^{\log^l N} \mu(d) \sum_{\substack{D(t) \equiv 0(d^2) \\ t \in [N,2N]}} S(t) + O\left(\sum_{t \in \mathcal{T}} S(t)\right). \tag{A.10}$$

We use arithmetic progressions to handle the piece with $d \leqslant \log^l N$, and Cauchy-Schwartz to handle $t \in \mathcal{T}$.

$$\sum_{t \in \mathcal{T}} S(t) \ll \left(\sum_{t \in \mathcal{T}} S^2(t)\right)^{\frac{1}{2}} \left(\sum_{t \in \mathcal{T}} 1\right)^{\frac{1}{2}} \ll \left(\sum_{t \in [N,2N]} S^2(t)\right)^{\frac{1}{2}} o\left(\sqrt{N}\right). \tag{A.11}$$

If we can show $\sum_{t=N}^{2N} S^2(t) = O(N)$, then the error term is negligible as $N \to \infty$.

A.6 Conditions Implying $|\mathcal{F}| = c_{\mathcal{F}} N + o(N)$, $c_{\mathcal{F}} > 0$

Assume no square divides $D(t)$ for all t. The number of $t \in [N, 2N]$ with $D(t)$ not divisible by d^2, $d \leqslant \log^l N$, is $N \prod_p \left(1 - \frac{\nu(p)}{p^2}\right) + o(N)$. Let $D(t) = \prod_i D_i^{r_i}(t)$, $D_i(t)$ irreducible. By multiple applications of Lemma A.3, $\exists c$ such that $\forall t$, there is no prime $p > c$ which divides two of the $D_i(t)$. Thus, if $D(t)$ is divisible by p^2 for a large prime, one of the factors is divisible by p^2. As there are finitely many factors, it is sufficient to bound by $o(N)$ the number of $t \in [N, 2N]$ with $p^2|D(t)$ for a large prime for irreducible $D(t)$.

Let $|\mathcal{F}|$ equal the number of $t \in [N, 2N]$ with $D(t)$ square-free. Let $c_{\mathcal{F}} = \prod_{p \leqslant \log^l N} \left(1 - \frac{\nu(p)}{p^2}\right)$. We have seen extending the product to all primes costs $O(\frac{1}{\log^{l(1-\epsilon)} N})$. Thus, we need only bound $c_{\mathcal{F}}$ away from zero.

Let $D(t) = a_k t^k + \cdots + a_0$ with discriminant δ. For $p \nmid a_k \delta$, $\nu(p) \leqslant k$.

Let $\mathcal{P}$ be the set of primes dividing $a_k \delta$ and all primes at most $\sqrt{k}$. The contribution from $p \notin \mathcal{P}$ is bounded away from 0. Therefore, if $\nu(p) < p^2$ for $p|a_k\delta$ and $p \leqslant \sqrt{k}$, then $c_{\mathcal{F}} > 0$.

If $D(t)$ is divisible by a square for all t, the above arguments fail. Let P be the largest product of primes such that $\forall t$, $P^2|D(t)$. By changing variables $\tau \to P^m t + t_0$, for m sufficiently large,

$D(\tau)$ is divisible by fixed powers of $p|P$, depending only on $D(t_0)$. Thus, instead of sieving to $D(t)$ square-free, we sieve to $D(\tau)$ square-free except for primes dividing P.

Let δ_τ denote the new discriminant. As the discriminant is a product over the differences of the roots, t_0 does not change the discriminant, and P^m rescales by a power of P. Thus, $\delta_\tau = P^M \delta$. Further, the new leading coefficient is $P^{mk}a_k$. Thus, for $p \nmid P$, our previous arguments are still applicable, except we are no longer sieving over $p|P$. We have shown

THEOREM A.5 CONDITIONS ON $D(t)$ IMPLYING $|\mathcal{F}| = c_{\mathcal{F}}N + o(N)$. *Assume no square divides $D(t)$ for all t. Let $\mathcal{P}$ be the set of primes dividing $a_k\delta$ and all primes at most $\sqrt{k}$. If $\forall p \in \mathcal{P}$, $\nu(p) \leqslant p^2 - 1$, then $|\mathcal{F}| = c_{\mathcal{F}}N + o(N)$, $c_{\mathcal{F}} > 0$. If $\forall t$, $B^2|D(t)$ ($\exists p \in \mathcal{P}$, $\nu(p) = p^2$), let P be the product of all primes either in $\mathcal{P}$ or dividing B. By changing variables to $\tau = P^m t + t_0$ for m large and sieving to $D(\tau)$ square-free except for $p|P$ (where $\forall t$, the power of $p|P$ dividing $D(t)$ is constant), we again obtain $|\mathcal{F}| = c_{\mathcal{F}}N + o(N)$, $c_{\mathcal{F}} > 0$. In this case, $c_{\mathcal{F}}$ no longer includes factors from $p|P$.*

If all irreducible factors of $D(t)$ have degree at most 3, these results are unconditional; if there is an irreducible factor with degree at least 4 these results are conditional, and a consequence of the ABC or Square-Free Sieve conjecture.

Further, let $\mathcal{T} = \{t \in [N, 2N] : \exists d > \log^l N \text{ with } d^2|D(t)\}$. Then $\mathcal{T} = o(N)$.

Appendix B. Handling the Conductors $C(t)$

For many families of elliptic curves, by sieving to a positive percent subsequence of t we obtain a sub-family where the conductors are a monotone polynomial in t. In particular, we prove this for all rational surfaces.

Tate's algorithm (see [Cr], pages $49 - 52$) allows us to calculate the conductor $C(t)$ for an elliptic curve E_t over $\mathbb{Q}$:

$$C(t) = \prod_{p|\Delta(t)} p^{f_p(t)}, \tag{B.1}$$

where for $p > 3$, if the curve is minimal for p then $f_p(t) = 0$ if $p \nmid \Delta(t)$, 1 if $p|\Delta(t)$ and $p \nmid c_4(t)$, and 2 if $p|\Delta(t)$ and $p|c_4(t)$. If $p > 3$ and $p^{12} \nmid \Delta(t)$, then the equation is minimal at p. See [Si1].

Let $\Delta(t) = d\Delta_1(t)\Delta_2(t)$, where $\left(\Delta_2(t), c_4(t)\right) = 1$ and $\Delta_1(t)$ is the product of powers of irreducible polynomials dividing $\Delta(t)$ and $c_4(t)$. By possibly changing d, we may take $\Delta_i(t)$ primitive. Let $D_i(t)$ be the product of all irreducible polynomials dividing $\Delta_i(t)$, $D(t) = D_1(t)D_2(t)$.

For t with $D(t)$ square-free except for small primes, $C(t) = D_1^2(t)D_2(t)$ if $\Delta(t)$ has no irreducible polynomial factor occurring at least 12 times (except for corrections from the small primes). Hence, while $f_p(t)$ may vary, the product of $p^{f_p(t)}$, except for a finite set of primes, is well behaved.

Let

$$\mathcal{P}_0 = \{p : p \leqslant \deg \Delta(t)\} \cup \{p : p|cd\}, \quad P_0 = \prod_{p \in \mathcal{P}_0} p. \tag{B.2}$$

The idea is that while for such p, $f_p(t)$ may vary, by changing variables from t to $P_0^m t + t_1$ for some enormous m, for $p \in \mathcal{P}_0$, $f_p(P_0^m t + t_1) = f_p(t_1)$. Thus, for this subsequence and these primes, $f_p(t)$ is constant.

We need two preliminary results. First, given a finite set of primes $\mathcal{P}_0$, we may find an m and a t_1 such that for those primes, $f_p(P_0^m t + t_1)$ is constant. Second, Lemma A.3: given two polynomials

with no non-constant factors over $\mathbb{Q}$, there is a finite set of primes $\mathcal{P}_2$ such that if $\exists t$ such that $\exists p$ dividing both polynomials, then $p \in \mathcal{P}_2$.

B.1 $f_p(t)$, $p \in \mathcal{P}_0$

Consider the original family of elliptic curves

$$E_t : \ y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t). \tag{B.3}$$

Assume $\Delta(t)$ is not identically zero. Choose t_1 such that $\forall t \geqslant t_1$, $\Delta(t) \neq 0$. Apply Tate's algorithm to E_{t_1}. If the initial equation was non-minimal for p, we change coordinates by $T(0,0,0,p)$ (see [Cr]) and restart. After finitely many passes, Tate's algorithm terminates.

In determining $f_p(t_1)$, assume we passed through Tate's algorithm $L_{t_1}(p)$ times. For each prime p, after possibly many coordinate changes, one of the following conditions held: $p \nmid \Delta$, $p \nmid c_4$, $p^2 \nmid a_6$, $p^3 \nmid b_8$, $p^3 \nmid b_6$, $p \nmid w(a_2, a_4, a_6)$, $p \nmid xa_3^2(a_3) + 4xa_6(a_6)$, $p \nmid xa_4^2(a_4) - 4xa_2(a_2)xa_6(a_6)$, $p^4 \nmid a_4$, $p^6 \nmid a_6$, and every function is polynomial in the a_i's. Thus, after possibly many coordinate changes, some polynomial (with integer coefficients) of the a_i's is not divisible by either p, p^2, p^3, p^4 or p^6.

Consider $\tau = P_0^m t + t_1$. For m enormous, $f_p(\tau) = f_p(t_1)$ for $p \in \mathcal{P}_0$ because in Tate's algorithm, we only need the values modulo a power of p. We have

$$a_i(\tau) = a_i(P_0^m t + t_1) = P_0^m t \widehat{a}_i(P_0^m t) + a_i(t_1) = \widetilde{a}_i(t) + a_i(t_1). \tag{B.4}$$

If m is sufficiently large, we can ignore $\widetilde{a}_i(t)$ in all equivalence checks, as for these powers of p, $\widetilde{a}_i(t) \equiv 0$. Let

$$n_t(p) = \mathrm{ord}\Big(p, \Delta(t)\Big)$$
$$n = \max_{p \in \mathcal{P}_0} n_{t_1}(p)$$
$$L = \max_{p \in \mathcal{P}_0} L_{t_1}(p). \tag{B.5}$$

We prove $f_p(\tau) = f_p(t_1)$ for large m. How large must m be? Excluding lines $42 - 65$, on each pass through Tate's algorithm we sometimes divide our coefficients by powers of p: up to p^2 on lines 26 and 30, up to p^3 on line 34, up to p^4 on line 69, and p^{12} on line 80. Over-estimating, we divide by at most $p^{2 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + 1 \cdot 12} = p^{23}$.

For lines $42 - 65$, we have a loop which can be executed at most $n + 4$ times. We constantly divide by increasing powers of p; the largest power is the last time through the loop, which is at most $p^{2(n+6)}$. As we pass through this loop at most $n + 4$ times, we divide by at most $p^{2n^2 + 20n + 48}$.

Thus, on each pass we have divisions by at most $p^{2n^2 + 20n + 48 + 23}$. As we loop through the main part of Tate's algorithm at most L times, we have divisions by at most $p^{(2n^2 + 20n + 71)L}$. If $m > (2n^2 + 20n + 71)L$, then $\forall t$, none of the $\widetilde{a}_i(t) = P_0^m t \widehat{a}_i(t)$ terms affect any congruence. Significantly smaller choices of m work: many of the divisions (for example, from lines $42 - 65$) arise only once.

B.2 Rational Surfaces I

B.2.1 *Preliminaries* Recall an elliptic surface $y^2 = x^3 + A(t)x + B(t)$ is rational iff one of the following is true: (1) $0 < \max\{3\deg A(t), 2\deg B(t)\} < 12$; (2) $3\deg A(t) = 2\deg B(t) = 12$ and $\mathrm{ord}_{t=0} t^{12}\Delta(t^{-1}) = 0$. See [RSi], pages $46 - 47$ for more details.

Assume we are in case (1). No non-constant polynomial of degree 11 or more divides $\Delta(t)$; however, a twelfth or higher power of a prime might divide $\Delta(t)$. Let $k = \deg \Delta(t)$, and write

$$\Delta(t) = d\Delta_1(t)\Delta_2(t)$$
$$c_4(t) = c\gamma_1(t)\gamma_2(t)$$
$$\mathcal{P}_0 = \{p : p \leqslant \deg \Delta(t)\} \cup \{p : p|cd\}, \quad P_0 = \prod_{p \in \mathcal{P}_0} p. \tag{B.6}$$

where $\Delta_1(t)$ through $\gamma_2(t)$ are primitive polynomials, $\Delta_1(t)$ and $\gamma_1(t)$ are divisible by the same non-constant irreducible polynomials, and $\Delta_2(t)$ and $c_4(t)$ are not both divisible by any non-constant polynomial.

Let $D_i(t)$ be the product of all non-constant irreducible polynomials dividing $\Delta_i(t)$, and similarly for $c_i(t)$. Let $D(t) = D_1(t)D_2(t) = \alpha_\kappa t^\kappa + \cdots + \alpha_0$ ($\kappa \leqslant k$), $c(t) = c_1(t)c_2(t)$.

Apply Lemma A.3 to $c(t)$ and $D_2(t)$. Thus $\exists c'$ such that if $\exists t$ where p divides both polynomials, then $p|c'$. Let $\mathcal{P}_2$ be the prime divisors of c' not in $\mathcal{P}_0$ and let $\mathcal{P}_1$ be the prime divisors of $\alpha_\kappa \cdot$ Discriminant$(D(t))$ not in $\mathcal{P}_0$. Define

$$\mathcal{P} = \bigcup_{i=1}^{2} \mathcal{P}_i, \quad P = \prod_{p \in \mathcal{P}} p. \tag{B.7}$$

Note every prime in $\mathcal{P}$ is greater than k and not in $\mathcal{P}_0$.

As the product of primitive polynomials is primitive, $D(t)$ is primitive. For any prime, either $D(t) \bmod p$ is a constant not divisible by p or a non-constant polynomial of degree at most k. In the second case, as there are at most k roots to $D(t) \equiv 0 \bmod p$, we find that given a $p > k$, $\exists t_p$ such that $D(t_p) \not\equiv 0 \bmod p$. By the Chinese Remainder Theorem, $\exists t_0 \equiv t_p \bmod p$ for all $p \in \mathcal{P}$.

B.2.2 *Calculating the Conductor* $\quad \forall p \in \mathcal{P}$, $D(Pt + t_0) \equiv D(t_0) \not\equiv 0 \bmod p$. As $\mathcal{P}$ and $\mathcal{P}_0$ are disjoint, this implies that $D(Pt + t_0)$ is minimal for all $p \in \mathcal{P}$, as $\mathcal{P}_0$ contains the factors of d,2 and 3. Moreover, $f_p(Pt + t_0) = 0$ for $p \in \mathcal{P}$.

By changing variables again, from t to $P_0^m t + t_1$, we can determine the powers of $p \in \mathcal{P}_0$ in the conductor. Combining the two changes, we send t to $\tau = P(P_0^m t + t_1) + t_0$.

Originally we had $\Delta(t) = d\Delta_1(t)\Delta_2(t)$. Now we have $\Delta(\tau) = d\Delta_1(\tau)\Delta_2(\tau)$. It is possible that $D_1(\tau)D_2(\tau)$ is no longer primitive; however, if there is a common prime divisor p, p divides $\alpha_\kappa(P \cdot P_0^m)^\kappa$, implying $p \in \mathcal{P}_0 \sqcup \mathcal{P}$.

We sieve to $D(\tau)$ square-free for $p \notin \mathcal{P}_0 \sqcup \mathcal{P}$. As $\mathcal{P}_0 \sqcup \mathcal{P}$ contains all primes less than k, as well as the prime divisors of P_0, P, α_κ and Discriminant$(\Delta(t))$, we can perform the sieving. Note the discriminants of $\Delta(t)$ and $\Delta(\tau)$ differ by a power of $P \cdot P_0^m$. Thus, away from these primes, $D(\tau) \equiv 0 \bmod p^2$ has at most $k < p^2$ roots, and we may sieve to a positive percent of t. The sieving is unconditional if each irreducible factor of $D(\tau)$ is of degree at most 3.

$D(\tau)$ is divisible by fixed powers of primes in $\mathcal{P}_0$ and never divisible by primes in $\mathcal{P}$. Thus $\exists c_1$, c_2 with factors in $\mathcal{P}_0$ such that $\widetilde{D}(\tau) = \frac{D_1(\tau)}{c_1}\frac{D_2(\tau)}{c_2}$ is not divisible by any $p \in \mathcal{P}_0 \sqcup \mathcal{P}$. We sieve to $\widetilde{D}(\tau)$ square-free; for $p \notin \mathcal{P}_0 \sqcup \mathcal{P}$, this is the same as $D(\tau)$ not divisible by p^2.

We need to determine $f_p(\tau)$ for $p \in \mathcal{P}_0$, $p \in \mathcal{P}$, and $p \notin \mathcal{P}_0 \sqcup \mathcal{P}$.

By our previous arguments, if m is sufficiently large, $f_p(\tau) = f_p(Pt_1 + t_0)$ for $p \in \mathcal{P}_0$.

If $p \in \mathcal{P}$ then $p \notin \mathcal{P}_0$. Mod p, $\Delta(\tau) = \Delta\Big(P(P_0 t + t_1) + t_0\Big) \equiv \Delta(t_0) \not\equiv 0$. Thus, for these p, $f_p(\tau) = 0$.

Assume $p \notin \mathcal{P}_0 \sqcup \mathcal{P}$. The leading term of $dD(\tau)$ is $d\alpha_\kappa(P \cdot P_0^m)^\kappa$. By construction, p does not

divide the leading coefficient of $\Delta(\tau)$, as $\mathcal{P}_0 \sqcup \mathcal{P}$ contains the prime divisors of d, α_k, P and P_0. If we sieve to $\widetilde{D}(\tau)$ square-free for $p \notin \mathcal{P}_0 \sqcup \mathcal{P}$, then as the degree of $\Delta(\tau)$ is at most 10, the curve is minimal for such p. Thus, $f_p(\tau)$ is 1 if $p|D_2(\tau)$ and 2 if $p|D_1(\tau)$.

Thus, we have shown

THEOREM B.1. *All quantities as above, for $\widetilde{D}(\tau)$ square-free, the conductors are*

$$C(\tau) = \prod_{p \in \mathcal{P}_0} p^{f_p} \cdot \left(\frac{|D_1(\tau)|}{c_1} \right)^2 \frac{|D_2(\tau)|}{c_2}. \tag{B.8}$$

For sufficiently large τ, $C(\tau)$ is a monotone increasing polynomial (we may drop the absolute values), and a positive percent of τ yield $\widetilde{D}(\tau)$ square-free.

B.3 Rational Surfaces II

We consider what could go wrong in our proof if we are in case (2), where $3\deg A(t) = 2\deg B(t) = 12$ and $\mathrm{ord}_{t=0} t^{12}\Delta(t^{-1}) = 0$.

Thus, $\Delta(t)$ is a degree twelve polynomial, and we need to worry about minimality issues. As before, we have

$$\Delta(t) = -2^4 \left(2^2 A^3(t) + 3^3 B^2(t) \right) = d\Delta_1(t)\Delta_2(t)$$
$$c_4(t) = c\gamma_1(t)\gamma_2(t)$$
$$\mathcal{P}_0 = \{p : p \leqslant \deg \Delta(t)\} \cup \{p : p|cd\}, \quad P_0 = \prod_{p \in \mathcal{P}_0} p. \tag{B.9}$$

There are three cases:

− $\Delta(t)$ not divisible by a twelfth power;
− $(\alpha t + \beta)^{12}|\Delta(t)$, $(\alpha t + \beta)\nmid c_4(t)$;
− $(\alpha t + \beta)^{12}|\Delta(t)$, $(\alpha t + \beta)|c_4(t)$.

These cases are handled in a similar fashion as before; see [Mil] for the calculations.

B.4 Generalizations

The previous arguments are applicable to any family where $\deg \Delta(t) \leqslant 12$ (which can include some non-rational families). It is straightforward to generalize these arguments for all families.

B.5 Summary

We summarize our sieving and conductor results:

THEOREM B.2 CONDUCTORS AND CARDINALITIES FOR FAMILIES. *For a one-parameter family with $\deg \Delta(t) \leqslant 12$, which includes all rational families, by sieving to a positive percent subsequence we obtain a family with conductors given by a monotone polynomial; further, by Theorem A.5, after changing variables to $\tau = P^m t + t_0$, a positive percent of $t \in [N, N]$ give $D(\tau)$ square-free except for primes $p|P$, where the power of such p dividing $D(\tau)$ is independent of t. If all the irreducible factors of $\Delta(t)$ are degree 3 or less, the sieving is unconditional; for degree 4 and higher, the sieving is a consequence of the ABC or Square-Free Sieve conjecture.*

Appendix C. Sums of Test Functions at Primes

We list several standard sums of test functions over primes. $\widehat{F}$, $\widehat{f_i}$ are even Schwartz functions with compact support, $\varphi(m)$ is the Euler phi-function.

All statements below are straightforward applications of partial summation and RH (or GRH for Dirichlet L-functions if $m \neq 1$) to handle the prime sums (see, for example, [Mil]); weaker error terms are obtainable by the Prime Number Theorem.

LEMMA C.1 SUM OF $\widehat{F}$ OVER PRIMES.

$$\frac{1}{\log N} \sum_{p \equiv b(m)} \frac{\log p}{p} \widehat{F}\Big(a \frac{\log p}{\log N}\Big) = \frac{1}{2a\varphi(m)} F(0) + O\Big(\frac{1}{\log N}\Big). \tag{C.1}$$

Setting $m = 1$ and $a = 1, 2$ yields

COROLLARY C.2. $\frac{1}{\log N} \sum_p \frac{\log p}{p} \widehat{F}\Big(\frac{\log p}{\log N}\Big) = \frac{1}{2} F(0) + O\Big(\frac{1}{\log N}\Big).$

COROLLARY C.3. $\frac{1}{\log N} \sum_p \frac{\log p}{p} \widehat{F}\Big(2\frac{\log p}{\log N}\Big) = \frac{1}{4} F(0) + O\Big(\frac{1}{\log N}\Big).$

LEMMA C.4.

$$4 \sum_p \frac{\log^2 p}{\log^2 M} \frac{1}{p} \widehat{f_1} \widehat{f_2}\Big(\frac{\log p}{\log M}\Big) = 2 \int_{-\infty}^{\infty} |u| \widehat{f_1}(u) \widehat{f_2}(u) du + O\Big(\frac{1}{\log M}\Big). \tag{C.2}$$

For $p \equiv b(m)$ we have

LEMMA C.5.

$$4 \sum_{p \equiv b(m)} \frac{\log^2 p}{\log^2 M} \frac{1}{p} \widehat{f_1} \widehat{f_2}\Big(\frac{\log p}{\log M}\Big) = \frac{2}{\varphi(m)} \int_{-\infty}^{\infty} |u| \widehat{f_1}(u) \widehat{f_2}(u) du + O\Big(\frac{1}{\log M}\Big). \tag{C.3}$$

LEMMA C.6. *Let $\mathcal{E}$ have rank r over $\mathbb{Q}(t)$ and assume Tate's conjecture for $\mathcal{E}$ (known if $\mathcal{E}$ is a rational surface). Then*

$$2 \sum_p \frac{\log p}{\log X} \frac{1}{p} \widehat{f}\Big(\frac{\log p}{\log X}\Big) \frac{-A_{1,\mathcal{F}}(p)}{p} = r f(0) + o(1). \tag{C.4}$$

Finally, we constantly encounter sums such as

$$\sum_p \frac{\log p}{\log C(t)} \frac{1}{p^r} \widehat{f}\Big(r \frac{\log p}{\log C(t)}\Big) a_t^r(p), \tag{C.5}$$

where $r \in \{1, 2\}$ and $\log C(t)$ is $k \log N + o(\log N)$.

By Hasse, $a_t^r(p) \leqslant (2\sqrt{p})^r$. The contribution S_l from $p \leqslant \log^l N$ is

$$S_l \ll \frac{1}{\log N} \sum_{p \leqslant \log^l N} \frac{\log p}{p^{r/2}}. \tag{C.6}$$

Clearly the larger contribution is from $r = 1$. By the Prime Number Theorem, $\sum_{p \leqslant x} \log p \ll x$. By partial summation, $\sum_{p \leqslant x} \frac{\log p}{\sqrt{p}} \ll \sqrt{x}$. Thus

$$S_l \ll \frac{\sqrt{\log^l N}}{\log N}. \tag{C.7}$$

We have shown

SOME

LEMMA C.7 REMOVING SMALL PRIMES. *The sums over primes $p \leqslant \log^l N$ in the Explicit Formula contribute $O(\log^{\frac{l}{2}-1} N)$. For $l < 2$, this is negligible.*

Appendix D. Handling the Error Terms in the 2-Level Density

Following Rudnick-Sarnak [RS] and Rubinstein [Ru], we handle the error terms in the 2-level density, assuming we are able to prove the 1-level density theorem with error terms. By the Explicit Formula (Equation 2.3)

$$\sum_{j_i} F_i\Big(\frac{\log N_E}{2\pi}\gamma_E^{(j_i)}\Big) = \mathbf{Good}_i + O\Big((\log N_E)^{-\frac{1}{2}}\Big), \tag{D.1}$$

where $\mathbf{Good}_i$ is the good part of the Explicit Formula, involving $\widehat{F}(0)$, $F(0)$, and sums of $a_E(p)$ and $a_E^2(p)$ for primes $p > \log N$.

Multiplying and summing over i yields

$$\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\prod_{i=1}^{2}\Bigg[\sum_{j_i} F_i\Big(\frac{\log N_E}{2\pi}\gamma_E^{(j_i)}\Big) + O\Big((\log N_E)^{-\frac{1}{2}}\Big)\Bigg] = \frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\prod_{i=1}^{2}\mathbf{Good}_i. \tag{D.2}$$

Multiplying out the LHS yields terms like

$$O\Bigg[\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}(\log N_E)^{-\frac{2-k}{2}}\prod_{m=1}^{k}\sum_{j_{m_i}} F_i\Big(\frac{\log N_E}{2\pi}\gamma_E^{(j_{m_i})}\Big)\Bigg]. \tag{D.3}$$

If each function F_i were positive, we could insert absolute values and move $\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}$ past the $\log^{-\frac{2-k}{2}} N_E$ factor. We assume our family has been sieved, so that the conductors satisfy $\log N_E = c\log N + o(\log N)$.

There are three terms. If $k = 0$ there is clearly no net contribution. For $k = 1$ we have a 1-level density, which is finite by assumption. No error hits the $k = 2$ piece (this is the piece we want to calculate!). Only the $k = 1$ piece is troublesome for F_i not positive.

If F_i is not positive, we increase the above by replacing F_i with a positive function g_i such that g_i is an even Schwartz function whose Fourier Transform is supported in the same interval as that of F_i and $g_i(x) \geqslant |F_i(x)|$. As the g_i satisfy the necessary conditions, we may apply the 1-Level Density Theorem to the g_i's, obtaining a bounded quantity. Hitting this with $(\log N_E)^{-\frac{1}{2}}$, we see there is negligible contribution.

For a construction of g_i, see Rubinstein [Ru], pages $40-41$ or Rudnick-Sarnak [RS], pages $302-304$.

We have shown:

THEOREM D.1 HANDLING THE ERROR TERMS. *If we are able to do the 1-level density calculations, then we may ignore the error terms in the 2-level density.*

Note: the error need not be $O(\log^{-\frac{1}{2}} N)$; $o(1)$ also works.

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