The Limiting Spectral Measure for the Ensemble of Generalized Real Symmetric Block m-Circulant Matrices

by

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ABSTRACT

Given an ensemble of $N \times N$ random matrices with independent entries chosen from a nice probability distribution, a natural question is whether the empirical spectral measures of typical matrices converge to some limiting measure as $N \to \infty$. It has been shown that the limiting spectral distribution for the ensemble of real symmetric matrices is a semi-circle, and that the distribution for real symmetric circulant matrices is a Gaussian. As a transition from the general real symmetric matrices to the highly structured circulant matrices, the ensemble of block m-circulant matrices with toroidal diagonals of period m exhibits an eigenvalue density as the product of a Gaussian and a certain even polynomial of degree $2m - 2$. This paper generalizes the m-circulant pattern and shows that the limiting spectral distribution is determined by the pattern of the elements within an m-period, depending on not only the frequency with which each element appears, but also the way the elements are arranged. For an arbitrary pattern, the empirical spectral measures converge to some nice probability distribution as $N \to \infty$.

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I welcome any comment on this thesis, and I am responsible for all errors.

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CONTENTS

1. INTRODUCTION

1.1. **History and Techniques.** [1]

In random matrix theory, we explore properties of matrices chosen according to some notion of randomness, which can range from taking the structurally independent entries as independent identically distributed random variables (i.i.d.r.v.'s) to studying subgroups of the classical compact groups under the Haar measure. While the subject dates from Wishart's [Wis] investigations in statistics in the 1920s, it was Wigner's work [Wig1, Wig2, Wig3, Wig4, Wig5] in the 1950s and Dyson's [Dy1, Dy2] contribution several years later that showed its incredible power and utility, as it was shown that random matrix ensembles successfully model the distribution of energy levels of heavy nuclei. The next milestone was established two decades later, when Montgomery [Mon] observed that the behavior of eigenvalues in certain random matrix ensembles correctly describes the statistical behavior of the zeros of the Riemann zeta function. The subject has continued expanding, with new applications emerging in a variety of fields including network theory [MNS], design of transportation systems [BBDS, KrSe], etc. [FM, Hay] provide a good review of the development of random matrix theory and the discovery of some of these applications.

One of the most studied ensembles is that of real symmetric matrices where the independent entries (the N entries on the main diagonal and the $\frac{N(N-1)}{2}$ entries in the upper right) are i.i.d.r.v.'s drawn from a fixed probability distribution p of real numbers with mean 0, variance 1, and finite higher moments. The remaining entries are chosen so that the matrix is symmetric. For such a matrix A,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji},$$

[1] Section 1 and Section 2 are standard set-up for studies on patterned matrices, and are largely paraphrased from [MMS, JMP, KKM] with permission.

we have

$$\text{Prob}(A) \;=\; \prod_{1\le i\le j\le N} p(a_{ij}), \quad \text{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) \;=\; \prod_{1\le i\le j\le N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij})dx_{ij} \tag{1.1}$$

We study the eigenvalues of A as we average over the matrix ensemble. Let $\delta(x - x_0)$ denote the shifted Delta functional (i.e, a unit point mass at x_0, satisfying $\int f(x)\delta(x - x_0)dx = f(x_0)$), and let $\{\lambda_1, \lambda_2, \dots, \lambda_N\}$ denote the eigenvalues of A. By the Central Limit Theorem, the correct scale to normalize the eigenvalues is on the order of $\sqrt{N}$.[2] To each A, we associate its empirical spacing measure:

$$\mu_{A,N}(x) \;=\; \frac{1}{N}\sum_{i=1}^{N} \delta\left(x - \frac{\lambda_i(A)}{\sqrt{N}}\right), \tag{1.2}$$

For the ensemble of real symmetric matrices, Wigner [Wig6] found the probability distribution of the normalized eigenvalues to be the semi-circle as $N \to \infty$. In other words, as $N \to \infty$, the empirical spacing measures of almost all A converge to the density of the semi-ellipse (with eigenvalues normalized by $\sqrt{N}$),

$$f_{\text{Wig}}(x) \;=\; \begin{cases} \frac{1}{\pi}\sqrt{1 - \left(\frac{x}{2}\right)^2}, & \text{if } |x| \le 2; \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

Note that, to obtain the standard semi-circle law, we need to normalize the eigenvalues by $2\sqrt{N}$ rather than $\sqrt{N}$. We may prove this result using Markov's Method of Moments, by which the convergence of measures follows from the convergence of the k^{th} moment averaged over the ensemble to the k^{th} moment of the semi-circle (see Theorem 1.3), with some control on the rate of convergence. While we would like to understand the behavior of eigenvalues, we only have information about matrix elements. Fortunately, the eigenvalues and the elements are connected through the eigenvalue trace lemma, which expresses powers of the eigenvalues in terms of the trace of powers of a matrix. This leads to

$$\text{k}^{\text{th}} \text{ moment of } \mu_{A,N}(x) \;=\; \frac{\sum_{i=1}^{N} \lambda_i(A)^k}{2^k N^{\frac{k}{2}+1}} \;=\; \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}, \tag{1.4}$$

[2] By the eigenvalue trace formula, $\sum_{i=1}^{N} \lambda_i^2 = \text{Trace}(A^2) = \sum_{i,j\le N} a_{ij}^2$. As each a_{ij} is draw from a distribution with mean 0 and variance 1, this sum is of order N^2, implying the average square of an eigenvalue is N.

and thus reduces the problem to analyzing the average of the trace of powers of the matrices.

1.2. **Ensembles of Patterned Matrices.** Since the eigenvalues of any real symmetric matrix are real, we may ask whether a limiting distribution exists for the density of normalized eigenvalues for subfamilies of real symmetric matrices. There are many interesting families to study, one of the earliest being that of d-regular graphs. Given any graph G, we form its adjacency matrix $A(G)$ by setting a_{ij} equal to the number of edges connecting vertices i and j. For undirected graphs, the resulting matrices are real symmetric, and we obtain a thin subfamily of all real symmetric matrices. In 1981, McKay [McK] proved that the limiting spectral measure for d-regular graphs exists, and as $N \to \infty$, for almost all such graphs G the associated measures $\mu_{A(G),N}(x)$ converge to Kesten's measure

$$f_{\text{Kesten},d}(x) \;=\; \begin{cases} \frac{d}{2\pi(d^2-x^2)}\sqrt{4(d-1)-x^2}, & |x| \le 2\sqrt{d-1}; \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

Note that, as $d \to \infty$, the measures converge to a scaled version of the semi-circle distribution, while the limiting distribution is a semi-circle for general real symmetric matrices.

This example is typical of what we study. Specifically, we investigate a thin subfamily whose limiting spectral measure exhibits different behavior from the semi-circle, but converges to the semi-circle as we fatten this subfamily to the full family of real symmetric matrices. Note that the elements of an adjacency matrix are not chosen independently: a d-regular graph is determined by choosing $\frac{dN}{2}$ out of $\frac{N(N-1)}{2}$ possible edges. Namely, for a fixed d, there are on the order of dN degrees of freedom for a d-regular graph. In comparison, there are on the order of N^2 degrees of freedom for $N \times N$ real symmetric matrices.

Numerous researchers have studied a myriad of real symmetric matrix subfamilies with special patterns. We concentrate on linked ensembles (see [BanBo]) that are closely related to our work. A linked ensemble of $N \times N$ matrices is specified by a link function

$$L_N : \{1, 2, ..., N\}^2 \to S \tag{1.6}$$

where S is some set. To each $s \in S$, we assign i.i.d.r.v. x_s from a fixed probability distribution p with mean 0, variance 1, and finite higher moments, and then set the

$(i,j)^{\text{th}}$ entry of the matrix $a_{i,j} := x_{L_N(i,j)}$.[3] For some linked ensembles, including those we will examine, it is convenient to specify the ensemble not by the link function, but by the equivalence relation $\sim$ the link function induces on $\{1,2,...,N\}^2$. In this case, a link function may be uncovered as the quotient map to the set of equivalence classes $\{1,2,...,N\}^2 \twoheadrightarrow \{1,2,...,N\}^2/\sim$. For example, the real symmetric ensemble is specified by the equivalence relation $(i,j) \sim (j,i)$, with a convenient link function $L(i,j) = (\min(i,j), \max(i,j))$.

An interesting subfamily is that of real symmetric Toeplitz matrices. A Toeplitz matrix is constant along its diagonals; thus an ensemble of real symmetric Toeplitz matrices has N degrees of freedom (or $N-1$, as without loss of generality, one may take all entries on the main diagonal to be zero). Despite some numerical evidence that, for large N, for almost all matrices the density of the normalized eigenvalues converge to the standard normal, Bose, Chatterjee and Gangopadhyay [BCG], Bryc, Dembo and Jiang [BDJ] and Hammond and Miller [HM] have shown that this is not the case. In particular, it is shown that the 4^{th} moment is $\frac{8}{3}$ rather than 3, which is the 4^{th} moment of the standard normal. The latter two papers derive many results about the higher moments of the limiting spectral distribution, including both the proof of existence and interpretations of the higher moments ([BDJ] view the moments as volumes of Eulerian solids, while [HM] interpret them in terms of solutions to systems of Diophantine equations). In addition, the analysis in [HM] shows that, though the moments grow significantly slower than the Gaussian's, they grow sufficiently fast to determine a universal distribution with unbounded support, and the deficit from the standard Gaussian's moments can be interpreted as obstructions to Diophantine equations.

[HM] proposed that, if the first row of a real symmetric Toeplitz matrix is a palindrome, then the obstructions to the Diophantine equations should vanish and the limiting spectral measure would be a Gaussian, and this conjecture was proved by Massey, Miller and Sinsheimber [MMS]. While the [MMS] approach involves an analysis of an associated

[3] For general linked ensembles, it is helpful to weight the random variables by how often they occur in the matrix: $a_{i,j} := c_N |L_N^{-1}(L_N(i,j))|^{-1} x_{L_N(i,j)}$. For the real symmetric ensemble, this corresponds to weighting the entries along the main diagonal by 2. For the ensembles we mention in this paper, this modification changes only lower order terms in the calculation of the limiting spectral measure.

system of Diophantine equations, by Cauchy's interlacing property we see that the problem is equivalent to determining the limiting spectral measure of real symmetric circulant matrices, which is a Gaussian. A symmetric circulant matrix is constant along diagonals and has first row $(x_0, x_1, x_2, \dots, x_2, x_1)$. Except for the main diagonal, a diagonal of length $N-k$ ($k \in \{1, 2, \dots, N-1\}$) in the upper right is paired with a diagonal of length k in the bottom left, and all entries along these two diagonals are equal. Therefore, a symmetric circulant matrix has the following structure:

$$\begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_2 & x_1 \\ x_1 & x_0 & x_1 & \cdots & x_3 & x_2 \\ x_2 & x_1 & x_0 & \cdots & x_4 & x_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_2 & x_3 & x_4 & \cdots & x_0 & x_1 \\ x_1 & x_2 & x_3 & \cdots & x_1 & x_0 \end{pmatrix}. \tag{1.7}$$

See Bose and Mitra [BM] for a proof that the limiting spectral distribution for real symmetric circulant matrices is normal. An advantage of working with circulant matrices is that explicit, tractable formulas exist for their eigenvalues.[4]

While there is a rapidly increasing amount of research on ensembles related to Toeplitz and circulant matrices [BasBo1, BasBo2, BanBo, BCG, BM, BDJ, HM, MMS], of particular interest to us are ensembles of patterned matrices with a variable parameter controlling the transition from the highly structured real symmetric circulant matrices to the general real symmetric matrices. Correspondingly, the limiting spectral measure of such ensembles is expected to be a transition from the Gaussian to the semi-circle. A few papers have taken a similar approach by studying the transition between related matrix families. For example, Kargin [Kar] studied banded Toeplitz matrices, Jackson, Miller and Pham [JMP] studied highly palindromic Toeplitz matrices whose first row has a fixed number of palindromes,

[4] The set of $N \times N$ circulant matrices is the group algebra of $\mathbb{Z}/N\mathbb{Z}$. The group algebra splits as a direct sum of irreducible representations, all of which are one-dimensional. Thus, the eigenvalues of the circulant matrix $\sum_{j=0}^{N-1} a_j z^j$, where z is the matrix with ones on the superdiagonal and the bottom left corner and zeros elsewhere, are $\sum_{j=0}^{N-1} a_j \chi(1)^j$ for each character χ of $\mathbb{Z}/N\mathbb{Z}$.

and in both cases the empirical measures of the ensembles of interest converge to that of the full Toeplitz ensemble, either as the band grows or the number of palindromes decreases.

Although it is often possible to infer the behavior of the limiting measure for special, patterned matrices, an explicit probability density for the limiting measure is seldom available because of computational complexity. Nevertheless, Koloğlu, Kopp, and Miller [KKM] derive a closed-form density for a family of transitional matrices between symmetric circulant matrices and general symmetric matrices: the real symmetric block m-circulant matrices. Recall that, in a symmetric circulant matrix, except for the main diagonal, a diagonal of length $N-k$ in the upper right is paired with a diagonal of length k in the bottom left, and all entries along these two diagonals are equal. [KKM] introduce a period parameter m and require each pair of diagonals to be periodic with m i.i.d.r.v.'s in the same order N/m times (always assuming $m|N$).[5] Since a circulant matrix is a Toeplitz matrix with additional circulant structure, it is helpful to define m-circulant matrices based on m-Toeplitz matrices.

Definition 1.1 (Block m-Circulant Matrices)**.** *Let $m|N$. An $N \times N$ block m-Toeplitz matrix is of the form*

$$\begin{pmatrix}
b_{1,0} & b_{1,1} & b_{1,2} & & & \cdots & & & b_{1,N-1} \\
b_{1,-1} & b_{2,0} & b_{2,1} & & & \cdots & & & b_{2,N-2} \\
b_{1,-2} & b_{2,-1} & b_{3,0} & & & \cdots & & & b_{3,N-3} \\
 & & & \ddots & & & & & \vdots \\
 & & & & b_{m,0} & b_{m,1} & b_{m,2} & \cdots & b_{m,N-m} \\
\vdots & \vdots & \vdots & & b_{m,-1} & b_{1,0} & b_{1,1} & \cdots & b_{1,N-m-1} \\
 & & & & b_{m,-2} & b_{1,-1} & b_{2,0} & \cdots & b_{2,N-m-2} \\
 & & & & \vdots & \vdots & \vdots & \ddots & \vdots \\
b_{1,1-N} & b_{2,2-N} & b_{3,3-N} & \cdots & b_{m,m-N} & b_{1,m-N+1} & b_{2,m-N+2} & \cdots & b_{m,0}
\end{pmatrix}.$$

An $N \times N$ block m-circulant matrix is one of the form above in which $b_{t,r} = b_{t,r'}$ whenever $r \equiv r' \pmod N$.

[5] There are a small number of exceptions due to the symmetry of the matrix, a fact that we will discuss shortly.

Based on the period m-circulant structure, [KKM] investigate real symmetric period m-circulant matrices, where all entries are real and a symmetric structure is imposed in addition to the m-circulant structure. In such matrices, there are m i.i.d.r.v.'s placed periodically on each pair of diagonals, one of length k in the upper right and another of length $N-k$ in the lower left, as well as on the main diagonal. Occasionally, the symmetry of the matrix forces additional entries on the paired diagonals of length $N/2$ to be equal. For simplicity, we will refer to real symmetric block m-circulant matrices as "m-circulant matrices" henceforth.

For example, an 8×8 and a 6×6 symmetric 2-circulant matrix are of the following forms, respectively,

$$\begin{pmatrix} c_0 & c_1 & c_2 & c_3 & c_4 & d_3 & c_2 & d_1 \\ c_1 & d_0 & d_1 & d_2 & d_3 & d_4 & c_3 & d_2 \\ c_2 & d_1 & c_0 & c_1 & c_2 & c_3 & c_4 & d_3 \\ c_3 & d_2 & c_1 & d_0 & d_1 & d_2 & d_3 & d_4 \\ c_4 & d_3 & c_2 & d_1 & c_0 & c_1 & c_2 & c_3 \\ d_3 & d_4 & c_3 & d_2 & c_1 & d_0 & d_1 & d_2 \\ c_2 & c_3 & c_4 & d_3 & c_2 & d_1 & c_0 & c_1 \\ d_1 & d_2 & d_3 & d_4 & c_3 & d_2 & c_1 & d_0 \end{pmatrix}; \quad \begin{pmatrix} c_0 & c_1 & c_2 & c_3 & c_2 & d_1 \\ c_1 & d_0 & d_1 & d_2 & c_3 & d_2 \\ c_2 & d_1 & c_0 & c_1 & c_2 & c_3 \\ c_3 & d_2 & c_1 & d_0 & d_1 & d_2 \\ c_2 & c_3 & c_2 & d_1 & c_0 & c_1 \\ d_1 & d_2 & c_3 & d_2 & c_1 & d_0 \end{pmatrix}. \tag{1.8}$$

Note that, for the 6×6 matrix, the symmetry allows only one, not two, i.i.d.r.v. on the paired diagonals of length $N/2$ (i.e., 3).

Clearly, if $m=1$, m-circulant matrices reduce to circulant matrices, and as $m\to N$, m-circulant matrices approach the full family of general symmetric matrices. Assuming m fixed and $N\to\infty$, for the ensembles of real symmetric m-circulant matrices, [KKM] derive explicit limiting spectral measures as the products of a Gaussian and a computable degree $2m-2$ polynomial. [KKM] thus quantify how the convergence of the limiting measure to the semi-circle depends on m.

Following [KKM], this paper generalizes the m-circulant structure: while [KKM] require m i.i.d.r.v.'s in a period of length m, this paper relaxes this requirement by allowing repeated r.v.'s in an m-period. In other words, there are fewer than m degrees of freedom in filling an m-period that determines a pair of diagonals. For simplicity of notation, we specify a "pattern" for an m-circulant structure by giving a few examples. The pattern $\{a, b\}$ indicates a 2-circulant structure, and in every 2-period there are 2 distinct r.v.'s. Different letters only indicate which r.v.'s in an m-period are distinct, and thus the pattern $\{a, b\}$ is equivalent to $\{b, a\}$, $\{a, b, c\}$ is equivalent to $\{c, a, b\}$, etc. For example, both the 8×8 and the 6×6 2-circulant matrix above follow the $\{a, b\}$ or $\{b, a\}$ pattern. The pattern $\{a, a, b, b\}$ indicates a 4-circulant structure, but in each 4-circulant period, there are only 2 distinct r.v.'s because the second element is forced to equal the first, and the fourth is forced to equal the third. Clearly, we may view the 2-circulant pattern $\{a, b\}$ as a 4-circulant pattern $\{a, b, a, b\}$, or any $2l$-circulant pattern for $l \in \{1, 2, \dots, \frac{N}{2}\}$.

Going through [KKM], it is a natural guess that the limiting spectral measure is determined solely by the frequency with which each letter appears. Namely, the limiting measure for the pattern $\{a, b\}$ should be the same as that for the pattern $\{a, a, b, b\}$ or $\{a, b, b, a\}$, the limiting measure for $\{a, b, c\}$ (3-circulant) should equal that for $\{a, a, b, b, c, c\}$ (6-circulant), etc. Figure 1 shows the histogram of numerical eigenvalues, together with the limiting spectral density,[6] for the pattern $\{a, b\}$. Figure 2 shows the distribution of numerical eigenvalues for $\{a, a, b, b\}$ and $\{a, b, b, a\}$, together with the spectral density for $\{a, b\}$. We see from these figures that the distribution of numerical eigenvalues for the three patterns are very similar, suggesting that the limiting spectral density for $\{a, b\}$ may apply to $\{a, a, b, b\}$ and $\{a, b, b, a\}$ as well. However, this paper shows that the limiting spectral measure is determined not only by the frequency of each element, but also by how the elements are arranged in an m-pattern. For example, the limiting measure for $\{a, b\}$ (or $\{a, b, a, b\}$) differs from that for $\{a, a, b, b\}$. We will first prove this claim about $\{a, b, a, b\}$ and $\{a, a, b, b\}$ in complete detail, and then comment on how one can easily generalize the argument to more complicated patterns. Although we have not managed to derive an

[6] According to [KKM], the limiting spectral density function $f_m(x)$ of the real symmetric block m-circulant ensemble equals

$$f_m(x) = \frac{e^{-mx^2/2}}{\sqrt{2\pi m}} \sum_{r=0}^{m-1} \frac{1}{(2r)!} \left(\sum_{s=0}^{m-r} \binom{m}{r+s+1} \frac{(2r+2s)!}{(r+s)!s!} \left(-\frac{1}{2}\right)^s \right) (mx^2)^r. \tag{1.9}$$

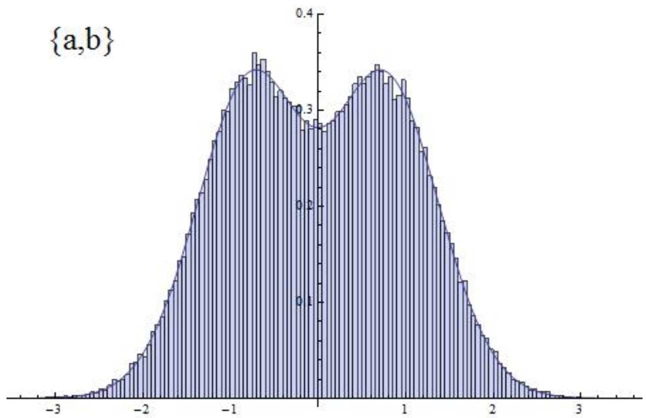

FIGURE 1. For the pattern $\{a, b\}$, histogram of numerical eigenvalues of 200 matrices of size 1200×1200, and plot of the limiting spectral density.

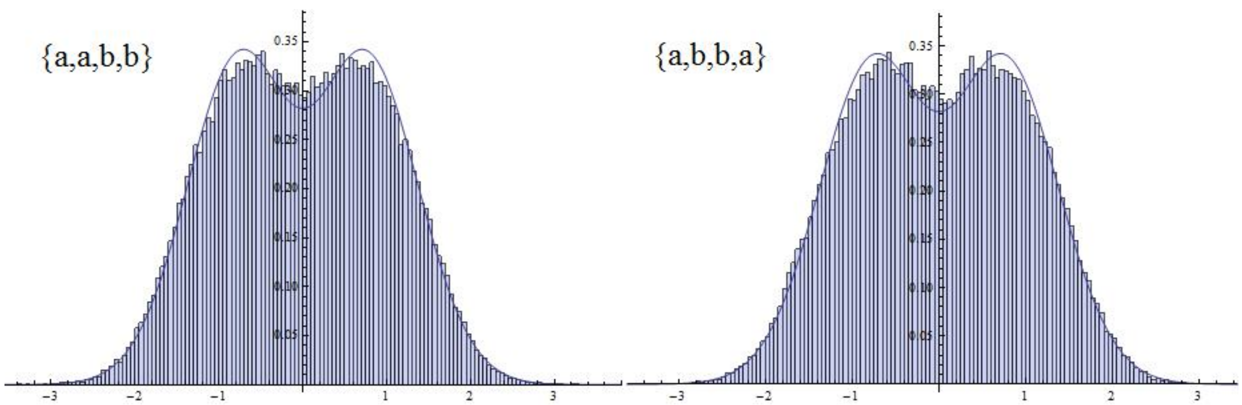

FIGURE 2. For the patterns $\{a, a, b, b\}$ (left) and $\{a, b, b, a\}$ (right) respectively, histogram of numerical eigenvalues of 200 matrices of size 1200×1200, and plot of the limiting spectral density for $\{a, b\}$.

explicit density for the generalized m-circulant patterns in which we allow repeated elements, e.g. $\{a, a, b, b\}$, we are able to show that, for any generalized pattern, the limiting measure exists and is finite, and the empirical measures of m-circulant matrices converge to the limiting measure.

1.3. **Results.** In this paper, we investigate the limiting spectral measure for ensembles of generalized real symmetric block m-circulant matrices. Prior to detailed results, we describe the probability spaces where the ensembles of m-circulant matrices are defined and state the various types of convergence that we will prove. The following set-up is standard in studies of patterned matrices, but are included for completeness.

1.3.1. *Related Definitions and Theorems.* Fix m and for each integer N, let $\Omega_{m,N}$ denote the set of $N \times N$ m-circulant matrices. Define an equivalence relation $\simeq$ on $\{1, 2, ..., N\}^2$: $(i, j) \simeq (i', j') \iff a_{ij} = a_{i'j'}$ in a patterned matrix. Consider the quotient map $\{1, 2, ..., N\}^2 \twoheadrightarrow \{1, 2, ..., N\}^2/\simeq$ that induces an injection $\mathbb{R}^{\{1,2,...,N\}^2/\simeq} \hookrightarrow \mathbb{R}^{N^2}$. The set $\mathbb{R}^{\{1,2,...,N\}^2/\simeq}$ has the structure of a probability space with the product measure of $p(x)dx$ by itself $|\mathbb{R}^{\{1,2,...,N\}^2/\simeq}|$ times, where dx is the Lebesgue measure. We thus define the probability space $(\Omega_{m,N}, \mathcal{F}_{m,N}, \mathbb{P}_{m,N})$ to be its image in $\mathbb{R}^{N^2} = M_{N^2}(\mathbb{R})$ under the injection, with the same probability distribution.

To each $A_N \in \Omega_{m,N}$, we attach a measure by placing a point mass of size $1/N$ at each normalized eigenvalue $\lambda_i(A_N)$:

$$\mu_{m,A_N}(x)dx \;=\; \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A_N)}{\sqrt{N}}\right)dx, \tag{1.10}$$

where $\delta(x)$ is the standard Dirac delta function; see Footnote 2 for an explanation why $\sqrt{N}$ is an appropriate normalization factor. We call μ_{m,A_N} the normalized spectral measure associated with A_N.

Definition 1.2 (Normalized empirical spectral distribution)**.** *Let $A_N \in \Omega_{m,N}$ have eigenvalues $\lambda_N \geq \cdots \geq \lambda_1$. The normalized empirical spectral distribution (the empirical distribution of normalized eigenvalues) $F_m^{A_N/\sqrt{N}}$ is given by*

$$F_m^{A_N/\sqrt{N}}(x) \;=\; \frac{\#\{i \leq N : \lambda_i/\sqrt{N} \leq x\}}{N}. \tag{1.11}$$

Since $F_m^{A_N/\sqrt{N}}(x) = \int_{-\infty}^{x}\mu_{m,A_N}(t)dt$, $F_m^{A_N/\sqrt{N}}$ is the cumulative distribution function associated to the measure μ_{n,A_N}. As $N \to \infty$, we study the behavior of a typical $F_m^{A_N/\sqrt{N}}$ as we vary A_N in the ensembles $\Omega_{m,N}$. Consider any probability space Ω_m with $\Omega_{m,N}$ as quotients (an easy example is the independent product). A series of papers [HM, MMS, JMP] concerning various Toeplitz ensembles fix Ω_m as the space of $\mathbb{N}$-indexed strings of real numbers picked independently from an underlying distribution p, with quotient maps to each $\Omega_{m,N}$ mapping a string to a matrix whose free parameters come from an initial segment of the right length. We follow this general approach. To each integer $k \geq 0$, we define the random variable $X_{k;m,N}$ on Ω_m by

$$X_{k;m,N}(A) \ = \ \int_{-\infty}^{\infty} x^k \, dF_m^{A_N/\sqrt{N}}(x), \tag{1.12}$$

which is the k^{th} moment of the measure μ_{m,A_N}.

Since we will show that the empirical measures of matrix ensembles converge to the limiting measure, we specify several types of converge we will be concerned with.

(1) (Weak convergence) For each k, $X_{k;m,N} \to X_{k,m}$ weakly if

$$\mathbb{P}_m(X_{k;m,N}(A) \leq x) \ \to \ \mathbb{P}(X_{k,m}(A) \leq x) \tag{1.13}$$

as $N \to \infty$ for all x at which $F_{X_{k,m}}(x) := \mathbb{P}(X_{k,m}(A) \leq x)$ is continuous.

(2) (Convergence in probability) For each k, $X_{k;m,N} \to X_{k,m}$ in probability if $\forall \epsilon > 0$,

$$\lim_{N\to\infty} \mathbb{P}_m(|X_{k;m,N}(A) - X_{k,m}(A)| > \epsilon) \ = \ 0. \tag{1.14}$$

(3) (Almost sure convergence) For each k, $X_{k;m,N} \to X_{k,m}$ almost surely if

$$\mathbb{P}_m\left(\{A \in \Omega_m : X_{k;m,N}(A) \to X_{k,m}(A) \text{ as } N \to \infty\}\right) \ = \ 1. \tag{1.15}$$

Alternative notations include *in distribution* for weak convergence, and *with probability 1* or *strongly* for almost sure convergence. Both almost sure convergence and convergence in probability imply weak convergence. We take $X_{k,m}$ as the random variable that is identically $M_{k,m}$, the limit of the average k^{th} moment (i.e., $\lim_{N\to\infty} M_{k,m;N}$). We often write $M_{k;m,N}$ as $M_{k;m}(N)$ to emphasize that k and m are fixed and N tends to infinity. We show below that the limits of the moments exist for every $k \in \mathbb{N}$ and these moments uniquely determine a probability distribution of eigenvalues for ensembles of m-circulant matrices. The method of studying the moments $M_{k;m}(N)$ in order to infer the behavior of the probability distribution $F_n^{A_N/\sqrt{N}}$ is based on the Moment Convergence Theorem (see [Ta] for example).

Theorem 1.3 (Moment Convergence Theorem). *Let $\{F_N(x)\}$ be a sequence of distribution functions such that the moments*

$$M_{k;N} \;=\; \int_{-\infty}^{\infty} x^k dF_N(x) \tag{1.16}$$

exist for all k. Let $\{M_k\}_{k=1}^{\infty}$ be a sequence of moments that uniquely determine a probability distribution, and denote the cumulative distribution function by Ψ. If $\lim_{N\to\infty} M_{k,N} = M_k$ then $\lim_{N\to\infty} F_N(x) = \Psi(x)$.

While the analysis in [MMS] is simplified by the fact that the convergence is to the standard normal, similar arguments (see [JMP]) also hold in our case as the growth rate of the moments of the limiting spectral distribution implies that the moments uniquely determine a probability distribution. We now formally define the limiting spectral distribution.

Definition 1.4 (Limiting spectral distribution). *If as $N \to \infty$, $F_m^{A_N/\sqrt{N}}$ converges in some sense (for example, in probability or almost surely) to a distribution F_m, then F_m is the limiting spectral distribution of the ensemble.*

1.3.2. *Main Results.* With all related definitions and theorems clear, we now state our main results.

Theorem 1.5 (Limiting spectral distribution determined by the m-circulant pattern). *The limiting spectral distribution for ensembles of generalized real symmetric block m-circulant matrices is determined by not only the frequency at which each i.i.d.r.v. element appears, but also the way the elements are arranged, in an m-circulant pattern.*

Theorem 1.6 (Existence of the limiting spectral distribution and convergence of the empirical spectral meaure). *The limiting spectral distribution for ensembles of generalized real symmetric block m-circulant matrices exists for any m-circulant pattern. In addition, the empirical spectral measure of a typical real symmetric matrix following this m-circulant pattern converges to the limiting spectral distribution, both in probability and almost surely.*

2. The Trace Method and the Moments

In this section, we study the limiting spectral distribution for the ensemble of real symmetric m-criculant matrices using the method of moments. In particular, we investigate the traces of powers of a typical matrix in this ensemble.

2.1. **The Trace Method.** Recall that, for the eigenvalue density of a *particular* $N \times N$ matrix A, we define the empirical measure by

$$\mu_{A,N}(x)\,dx := \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{\sqrt{N}}\right)dx, \tag{2.1}$$

so integrating a real-valued function f against $\mu_{A,N}(x)\,dx$ gives $\sum_{i=1}^{N} f\left(\frac{\lambda_i(A)}{\sqrt{N}}\right)$. The normalization factor $\sqrt{N}$ may be justified as in [HM, MMS, Wig5] or by the calculations of the moments to follow. The n^{th} moment, given by integrating $f(x) = x^n$ against $\mu_{A,N}(x)\,dx$, is

$$M_{n;m}(A,N) \ = \ \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\lambda_i(A)}{\sqrt{N}}\right)^n \ = \ \frac{1}{N^{\frac{n}{2}+1}}\sum_{i=1}^{N}\lambda_i^n(A). \tag{2.2}$$

We then define

$$M_{n;m}(N) \ := \ \mathbb{E}(M_{n;m}(A,N)), \tag{2.3}$$

and set

$$M_{n;m} \ := \ \lim_{N\to\infty} M_{n;m}(N), \tag{2.4}$$

provided the limit exists. $\mathbb{E}(M_{n;m}(A,N))$ means the expected value of $M_{n;m}(A,N)$ for a random symmetric period m-circulant matrix $A \in \Omega_{m,N}$.

We use a standard method to compute the moments. By the eigenvalue trace lemma,

$$\mathrm{Tr}(A^n) \ = \ \sum_{i=1}^{N}\lambda_i^n, \tag{2.5}$$

and thus

$$M_{n;m}(A,N) \ = \ \frac{1}{N^{\frac{n}{2}+1}}\mathrm{Tr}(A^n). \tag{2.6}$$

Expanding $\mathrm{Tr}(A^n)$,

$$M_{n;m}(A,N) \ = \ \frac{1}{N^{\frac{n}{2}+1}}\sum_{1\le i_1,\dots,i_n\le N} a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ni_1}, \tag{2.7}$$

so by linearity of expectation,

$$M_{n;m}(N) \;=\; \frac{1}{N^{\frac{n}{2}+1}} \sum_{1\le i_1,\dots,i_n\le N} \mathbb{E}(a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ni_1}). \tag{2.8}$$

In an m-circulant matrix, we define an equivalence relation $\simeq$ on $\{1,2,...,N\}^2$: $(i,j)\simeq(i',j') \iff a_{ij} = a_{i'j'}$. For each term in the sum in (2.8), $\simeq$ induces an equivalence relation $\sim$ on $\{(1,2),(2,3),\dots,(n,1)\}$ by its action on $\{(i_1,i_2),(i_2,i_3),\dots,(i_n,i_1)\}$. Let $\eta(\sim)$ denote the number of n-tuples with $0 \le i_1,\dots,i_n \le N$ whose indices inherit $\sim$ from $\simeq$, then $\sim$ splits up $\{(1,2),(2,3),\dots,(n,1)\}$ into equivalence classes with sizes $d_1(\sim)$, $\dots, d_l(\sim)$. Since the entries of our random matrix are independent identically distributed,

$$\mathbb{E}(a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ni_1}) \;=\; m_{d_1(\sim)}\cdots m_{d_l(\sim)}, \tag{2.9}$$

where m_{d_k} is the k^{th} moment of the underlying distribution p. Thus, we may write

$$M_{n;m}(N) \;=\; \frac{1}{N^{\frac{n}{2}+1}} \sum_{\sim} \eta(\sim) m_{d_1(\sim)}\cdots m_{d_l(\sim)}. \tag{2.10}$$

As p has mean 0, $m_{d_1(\sim)}\cdots m_{d_l(\sim)} = 0$ unless $d_k > 1$ for every k, i.e. the entries of the matrix are matched at least in pairs. All the terms in the sum above vanish except for those that are matched at least in pairs under $\sim$.

To compute the moments, we need to find $\eta(\sim)$, the number of solutions to a system of Diophantine equations induced by $\simeq$ and involving the indices $\{i_1, i_2,\dots,i_n\}$. Given any two matrix elements $a_{i_si_{s+1}}$ and $a_{i_ti_{t+1}}$,[7] $a_{i_si_{s+1}} = a_{i_ti_{t+1}} \iff (s,s+1)\sim(t,t+1)$. The equivalence relation $\sim$ entails two conditions simultaneously for the circulant structure in general, whether in the [KKM] m-circulant matrices in which each element in an m-pattern is distinct, or in the generalized m-circulant matrices in which we allow repeated elements in an m-pattern. The first condition, which we call the "diagonal condition", requires $a_{i_si_{s+1}}$ and $a_{i_ti_{t+1}}$ to be on the diagonals that would allow $a_{i_si_{s+1}} = a_{i_ti_{t+1}}$ and is the same for the two families of circulant matrices.[8] The second condition, which we call the "modulo

[7] For simplicity of notation, we allow $a_{i_si_{s+1}}$ to denote any element in $\{a_{i_1i_2}, a_{i_2i_3},\dots,a_{i_{2k-1}i_{2k}}, a_{i_{2k}i_1}\}$, as in the sum in (2.8), and the same for $a_{i_ti_{t+1}}$.

[8] In fact, with slight modification, the diagonal condition for the circulant structure applies to the Toeplitz structure as well, see [HM, MMS, JMP].

condition", requires that $a_{i_s i_{s+1}}$ and $a_{i_t i_{t+1}}$ to be in the 'slots' in an m-period that would allow $a_{i_s i_{s+1}} = a_{i_t i_{t+1}}$, and this modulo condition differs between the [KKM] pattern and the generalized pattern. Before studying the relatively complicated modulo condition, we may make use of the diagonal condition and derive several useful results. For circulant structure in general, the diagonal condition entails

- $i_{s+1} - i_s \equiv i_{t+1} - i_t \pmod{N}$, or
- $i_{s+1} - i_s \equiv -(i_{t+1} - i_t) \pmod{N}$.

Given $i_\ell \in \{1, 2, \dots, N\}$, the diagonal condition may be specified as $i_s - i_{s+1} = -(i_t - i_{t+1})$, $i_s - i_{s+1} = -(i_t - i_{t+1}) + N$, or $i_s - i_{s+1} = -(i_t - i_{t+1}) - N$.

We temporarily ignore the modulo conditions and find that this system has at most $2^{n-l} N^{l+1}$ solutions. Specifically, we pick one difference $i_{s+1} - i_s$ freely from each congruence class of $\simeq$, and then we have at most 2 choices for the remaining differences. Next, we pick i_1 freely, and thus determine all the $i_s = i_1 + \sum_{s' < s} (i_{s'+1} - i_{s'})$. This method will not always produce an exact solution even without the modulo conditions, but it suffices to give an upper bound on the number of solutions.

When n is odd, say $n = 2k + 1$, then $l \leq k$. Thus $\frac{1}{N^{\frac{n}{2}+1}} \eta(\sim) \leq \frac{1}{N^{k+\frac{3}{2}}} 2^{n-l} N^{l+1} \leq \frac{1}{N^{k+\frac{3}{2}}} 2^{n-l} N^{k+1} = \frac{1}{\sqrt{N}} 2^{n-l} = O_n\left(\frac{1}{\sqrt{N}}\right)$. We then have

$$M_{2k+1;m}(N) \ = \ O_k\left(\frac{1}{\sqrt{N}}\right). \tag{2.11}$$

This implies that the odd moments vanish in the limit.

When n is even, say $n = 2k$, then l is at most k. If $l < k$, then $l \leq k - 1$, and similar to the case of odd moments above, $\frac{1}{N^{\frac{n}{2}+1}} \eta(\sim) \leq \frac{1}{N^{k+1}} 2^{n-l} N^{l+1} \leq \frac{1}{N^{k+1}} 2^{n-l} N^{k} = \frac{1}{N} 2^{n-l} = O_n\left(\frac{1}{N}\right)$. If $l = k$, then all the $d_j = 2$, and the entries are exactly matched in pairs. As p has variance 1 (i.e., $m_2 = 1$), the formula for the even moments (2.10) becomes

$$M_{2k;m}(N) \ = \ \frac{1}{N^{k+1}} \sum_{\sigma} \eta(\sigma) + O_k\left(\frac{1}{N}\right). \tag{2.12}$$

where the sum is over all pairings σ's on $\{(1,2), (2,3), \dots, (n,1)\}$, which we may consider as functions (specifically, involutions without fixed points) or equivalence relations. We have thus shown

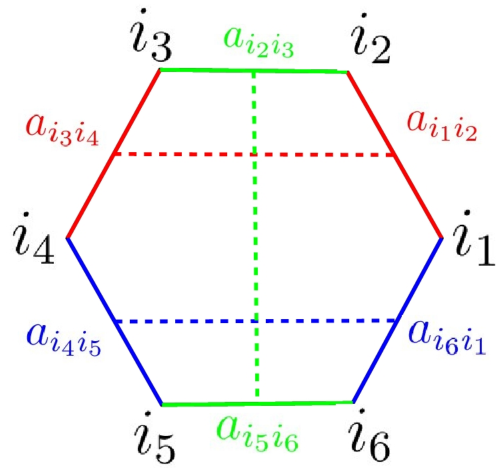

FIGURE 3. A 6-gon representing a possible way of pairing matrix entries in computing the 6^{th} moment.

Lemma 2.1. *For ensembles of real symmetric period m-circulant matrices,*

$$\begin{aligned} M_{2k+1;m}(N) &= O_k\left(\frac{1}{\sqrt{N}}\right); \\ M_{2k;m}(N) &= \frac{1}{N^{k+1}}\sum_{\sigma}\eta(\sigma)+O_k\left(\frac{1}{N}\right). \end{aligned} \tag{2.13}$$

In particular, all the odd moment averages vanish as $N \to \infty$.

2.2. **The Even Moments.** Now that we have shown the odd moments vanish like $\frac{1}{\sqrt{N}}$ as $N \to \infty$, we only need to focus on the $2k^{\text{th}}$ moments. Again, we will first exploit the diagonal condition to reduce the computational work. From Lemma 2.1, the only terms which contribute to the moments in the limit are those in which the $2k$ entries $a_{i_si_{s+1}}$'s are matched in k pairs. We may compare pairing the entries to pairing the edges of a $2k$-gon with vertices $1, 2, \dots, 2k$ and edges $(1, 2), (2, 3), \dots, (2k, 1)$. The vertices are labeled $i_1, \dots, i_{2k}$ and the edges are labeled $a_{i_1i_2}, \dots, a_{i_{2k}i_1}$, as in Figure 3. Note that this replicates the diagrams for pairings in [HM, MMS], where the $a_{i_si_{s+1}}$ are represented as vertices. To learn more about such an identification and its application in determining moments for random matrix ensembles, see [Fo] (Section 1.6) and [Zv].

Recall the diagonal condition: for a pair of matrix elements $a_{i_si_{s+1}}$ and $a_{i_ti_{t+1}}$,

- $i_{s+1} - i_s \equiv i_{t+1} - i_t \pmod{N}$, or

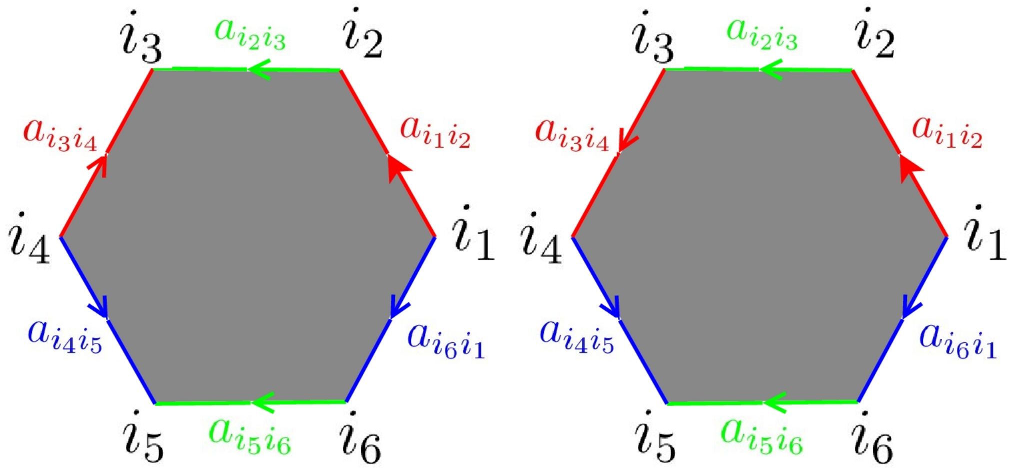

FIGURE 4. Some possible orientations of paired edges in a 6-gon.

- $i_{s+1} - i_s \equiv -(i_{t+1} - i_t) \pmod N$.

We may think of these two cases as pairing $(s, s+1)$ and $(t, t+1)$ in the same or opposite orientation, respectively. For example, in Figure 4 the hexagon on the left has all edges paired in opposite orientation, and the one on the right has all but the red edges paired in opposite orientation.

Now we dramatically reduce the number of pairings we need to consider by showing that the only pairings that contribute to the moments as $N \to \infty$ are those in which all edges are paired in opposite orientation. Topologically, these are exactly the pairings which give orientable surfaces [Hat, HarZa]. This result and its proof are based on their analogs in the Toeplitz cases (see [HM, MMS, JMP]) with minor modifications.

Lemma 2.2. *Consider a pairing σ with orientations ε_j's. If any ε_j is equal to 1, then the pairing contributes $O_k(1/N)$ to the $2k^{th}$ moment.*

Proof. The size of the contribution equals the number of solutions, divided by N^{k+1}, to the system of k Diophantine equations, each in the following form,

$$i_{s+1} - i_s \;\equiv\; \varepsilon_j(i_{\sigma(s)+1} - i_{\sigma(s)}) \pmod N, \tag{2.14}$$

in addition to some modulo constraints. We temporarily ignore the modulo constraints and bound the contribution from above by the number of solutions to the system of $\pmod N$

equations divided by N^{k+1}. As every $i_s \in \{1, 2, \ldots, N\}$, we consider the i_s's as elements of $\mathbb{Z}/N\mathbb{Z}$, and notate the $\pmod{N}$ congruences with equality.

The pairing places the numbers $1, 2, \ldots, 2k$ into k equivalence classes of size two. We arbitrarily order the equivalence classes and pick an element from each to call s_j, and name the other element $t_j = \sigma(s_j)$. The $\mathbb{Z}/N\mathbb{Z}$ equations now look like

$$i_{s_j+1} - i_{s_j} \equiv \varepsilon_j(i_{t_j+1} - i_{t_j}) \pmod{N}. \tag{2.15}$$

We now define

$$\begin{aligned} x_j &:= i_{s_j+1} - i_{s_j} && (2.16) \\ y_j &:= i_{t_j+1} - i_{t_j}, && (2.17) \end{aligned}$$

and the $\mathbb{Z}/N\mathbb{Z}$ equations now look like $x_j = \varepsilon_j y_j$. Thus

$$0 = \sum_{s=1}^{2k}(i_{s+1} - i_s) = \sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j = \sum_{j=1}^{k}(\varepsilon_j + 1)y_j. \tag{2.18}$$

If any $\varepsilon_j = 1$, we will have a nontrivial relation among the y_j and lose a degree of freedom. We may choose $k-1$ of the y_j's freely (in $\mathbb{Z}/N\mathbb{Z}$), and then we have 1 or possibly 2 choices for the remaining y_j's (depending on the parity of N). The x_j's are now determined as well, i.e. $i_{s+1} - i_s$ is now determined for every s. If we choose i_1 freely, we will determine all the $i_s = i_1 + \sum_{s'<s}(i_{s'+1} - i_{s'})$. Thus, we have at most $N^{k-1} \cdot 2 \cdot N = 2N^k$ solutions to (2.14), and the contribution from a pairing with one $\varepsilon_j = 1$, or a positive orientation, is at most $O_k(\frac{2N^k}{N^{k+1}}) = O_k(\frac{1}{N})$. Note that the big-Oh constant depends on k because if some of the different pairs have the same value, we might not have k copies of the second moment of p but instead, say, four second moments and two eighth moments, and by assumption all these moments are finite. In any case, the contribution is trivially bounded above by $\max_{1\le\ell\le k}(1 + m_{2\ell})^k$, where $m_{2\ell}$ is the $2\ell^{\text{th}}$ moment of p. □

We have shown

$$M_{2k;m}(N) = \sum_{\sigma} w(\sigma) N^{-(k+1)} + O_k\left(\frac{1}{N}\right) \tag{2.19}$$

where $w(\sigma)$ denotes the number of solutions to

$$i_{j+1} - i_j \;\equiv\; -(i_{\sigma j+1} - i_{\sigma j})\,(\mathrm{mod}\ N) \tag{2.20}$$

plus some modulo constraints. We have significantly reduced the amount of computation needed to determine the moments by exploiting the diagonal condition, which applies to the circulant structure in general, i.e. both the [KKM] m-circulant matrices and the generalized m-circulant matrices where we allow repeated elements in an m-pattern.

3. Computation of Low Moments

In this section, we explore the modulo condition to compute some low moments, and show that the difference in the modulo condition between the [KKM] m-circulant matrices and the generalized m-circulant matrices leads to different values for moments, and hence to different limiting spectral distributions. This will complete the proof of Theorem 1.5: the limiting spectral distribution is dependent on the frequency of each element, as well as the way the elements are arranged, in an m-pattern.

3.1. **Zone-wise Locations and Pairing Conditions.** Since we have restricted the computation of moments to even moments, and have shown that the only configurations that contribute to the $2k^{\text{th}}$ moment are those in which the $2k$ matrix entries are matched in k pairs in opposite orientation, we are ready to compute the moments explicitly. We start by calculating the 2nd moment, which by (2.8) is $\frac{1}{N^2}\sum_{1\le i,j\le N} a_{ij}a_{ji}$. As long as the matrix is symmetric, $a_{ij} = a_{ji}$ and the 2nd moment is 1. We now describe the conditions for two entries $a_{i_s i_{s+1}}, a_{i_t i_{t+1}}$ to be paired, denoted as $a_{i_s i_{s+1}} = a_{i_t i_{t+1}} \iff (s, s+1) \sim (t, t+1)$, which we need to consider in detail for the computation of higher moments. To facilitate the practice of checking pairing conditions, we divide an $N \times N$ symmetric m-circulant matrix into 4 zones as in Figure 5, and then reduce an entry $a_{i_s i_{s+1}}$ in the matrix to its "basic form". Write $i_\ell = m\eta_\ell + \epsilon_\ell$, where $\eta_\ell \in \{1, 2, \dots, \frac{N}{m}\}$ and $\epsilon_\ell \in \{0, 1, \dots, m-1\}$, we have

(1) $0 \le i_{s+1} - i_s \le \frac{N}{2} - 1 \Rightarrow a_{i_s i_{s+1}} \in$ Zone 1 and $a_{i_s i_{s+1}} = a_{\epsilon_s, m(\eta_{s+1}-\eta_s)+\epsilon_{s+1}}$;

(2) $\frac{N}{2} \le i_{s+1} - i_s \le N - 1 \Rightarrow a_{i_s i_{s+1}} \in$ Zone 2 and $a_{i_s i_{s+1}} = a_{\epsilon_{s+1}, m(\eta_s + \frac{N}{m} - \eta_{s+1}) + \epsilon_s}$;

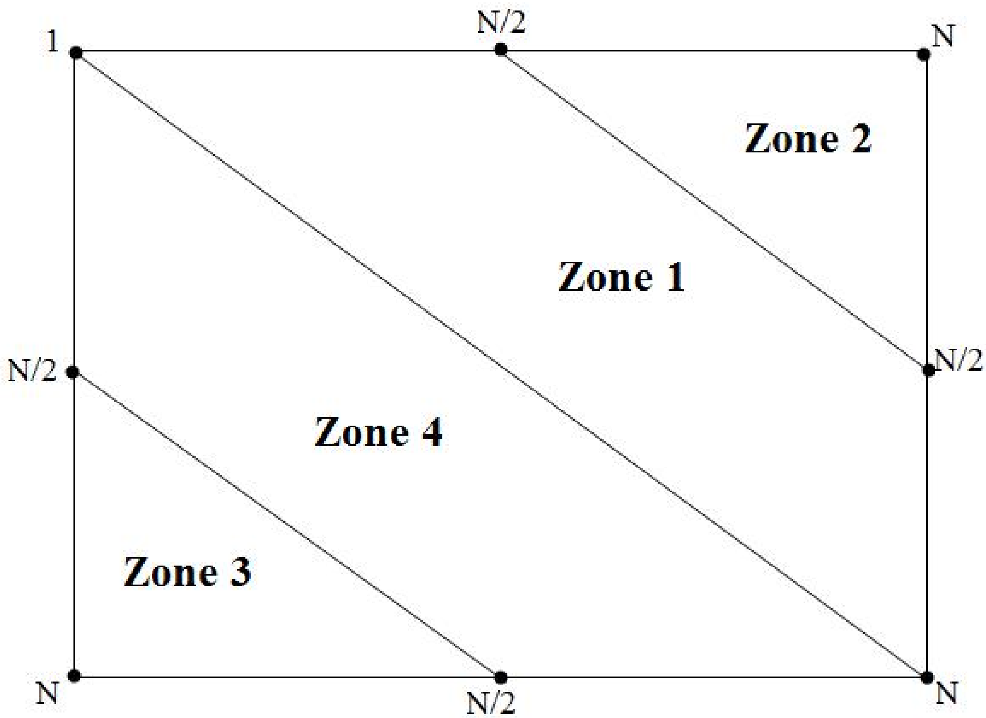

FIGURE 5. 4 zones in an $N \times N$ matrix.

(3) $\frac{N}{2} \leq i_s - i_{s+1} \leq N - 1 \Rightarrow a_{i_s i_{s+1}} \in$ Zone 3 and $a_{i_s i_{s+1}} = a_{\epsilon_s, m(\eta_{s+1} + \frac{N}{m} - \eta_s) + \epsilon_{s+1}}$;

(4) $0 \leq i_s - i_{s+1} \leq \frac{N}{2} - 1 \Rightarrow a_{i_s i_{s+1}} \in$ Zone 4 and $a_{i_s i_{s+1}} = a_{\epsilon_{s+1}, m(\eta_s - \eta_{s+1}) + \epsilon_s}$.

In short, $(i_{s+1} - i_s)$ determines which diagonal $a_{i_s i_{s+1}}$ is on. If $a_{i_s i_{s+1}}$ is in Zone 1 or 3 (Area I), ϵ_s determines the slot of $a_{i_s i_{s+1}}$ in an m-pattern; if $a_{i_s i_{s+1}}$ is in Zone 2 or 4 (Area II), ϵ_{s+1} determines the slot of $a_{i_s i_{s+1}}$ in an m-pattern.

Recall the two basic pairing conditions: the diagonal condition that we have explored before, and the modulo condition, for which we will define an equivalence relation $\mathcal{R}$. For a real symmetric m-circulant matrix following a generalized m-pattern and any two entries $a_{i_s i_{s+1}}, a_{i_t i_{t+1}}$ in the matrix, suppose that i_s and i_{t+1} are the indices that determine the slot of the respective entries. Then $i_s \mathcal{R} i_{t+1}$ if and only if $a_{i_s i_{s+1}}, a_{i_t i_{t+1}}$ are in certain slots in an m-pattern such that these two entries can be equal. For example, for the $\{a, b\}$ pattern, $i_s \mathcal{R} i_{t+1} \iff i_s \equiv i_{t+1} \pmod 2$; for the $\{a, a, b, b\}$ pattern, $i_s \mathcal{R} i_{t+1} \iff \mod(i_s, 4), \mod(i_{t+1}, 4) \in \{1, 2\}$ or $\mod(i_s, 4), \mod(i_{t+1}, 4) \in \{3, 0\}$.

We now formally define the two pairing conditions.

(1) (diagonal condition) $i_s - i_{s+1} \equiv -(i_t - i_{t+1}) \pmod N$.

(2) (modulo condition) $i_s \mathcal{R} i_{t+1}$ or $i_{s+1} \mathcal{R} i_t$, depending on which zone(s) $a_{i_s i_{s+1}}, a_{i_t i_{t+1}}$ are located in.

Since the diagonal condition implies a Diophantine equation for each of the k pairs of matrix entries, we only need to choose $k+1$ out of $2k$ i_ℓ's, and the remaining i_ℓ's are determined. This shows that, trivially, the number of non-trivial configurations is bounded above by N^{k+1}. In addition, the diagonal condition will always ensure that $a_{i_s i_{s+1}}$ and $a_{i_t i_{t+1}}$ are located in different areas. For instance, if $a_{i_s i_{s+1}} \in$ Zone 1 and $i_s - i_{s+1} = -(i_t - i_{t+1})$, then $a_{i_s i_{s+1}} \in$ Zone 4; if $a_{i_s i_{s+1}} \in$ Zone 1 and $i_s - i_{s+1} = -(i_t - i_{t+1}) - N$, then $a_{i_s i_{s+1}} \in$ Zone 2, etc. Thus, if i_s determines the slot for $a_{i_s i_{s+1}}$ in an m pattern, then i_{t+1} determines for $a_{i_t i_{t+1}}$; if i_{s+1} determines the slot for $a_{i_s i_{s+1}}$, then i_t determines for $a_{i_t i_{t+1}}$, and vice versa.

Considering the "basic" form of the entries, the two conditions above are equivalent to

(1) (diagonal condition)

$$(m\eta_s + \epsilon_s) - (m\eta_{s+1} + \epsilon_{s+1}) \equiv -(m\eta_t + \epsilon_t) + (m\eta_{t+1} + \epsilon_{t+1}) \pmod N \Rightarrow$$
$$m(\eta_s - \eta_{s+1} + \eta_t - \eta_{t+1}) + (\epsilon_s - \epsilon_{s+1} + \epsilon_t - \epsilon_{t+1}) = 0 \text{ or } \pm N.$$

(2) (modulo condition) $\epsilon_s \mathcal{R} \epsilon_{t+1}$ or $\epsilon_{s+1} \mathcal{R} \epsilon_t$.

Since $m|N$, this requires $m|(\epsilon_s - \epsilon_{s+1} + \epsilon_t - \epsilon_{t+1})$. Given the range of the η_ℓ's and ϵ_ℓ's, we have $\epsilon_s - \epsilon_{s+1} + \epsilon_t - \epsilon_{t+1} = 0$ or $\pm m$, which indicates that

$$\eta_s - \eta_{s+1} + \eta_t - \eta_{t+1} = 0, \pm 1, \frac{N}{m}, \frac{N}{m} \pm 1, -\frac{N}{m}, \text{ or } -\frac{N}{m} \pm 1. \tag{3.1}$$

As discussed before, if we allow repeated elements in an m-pattern, the equivalence relation $\mathcal{R}$ no longer necessitates a congruence relation as in the [KKM] pattern where each element is distinct. In the rest of this section, we will show that this difference in the modulo condition implies different moment values for two m-patterns even if these two patterns have the same frequency of each element. While the computation of high moments for general m-patterns appears intractable, fortunately we are able to illustrate how the difference

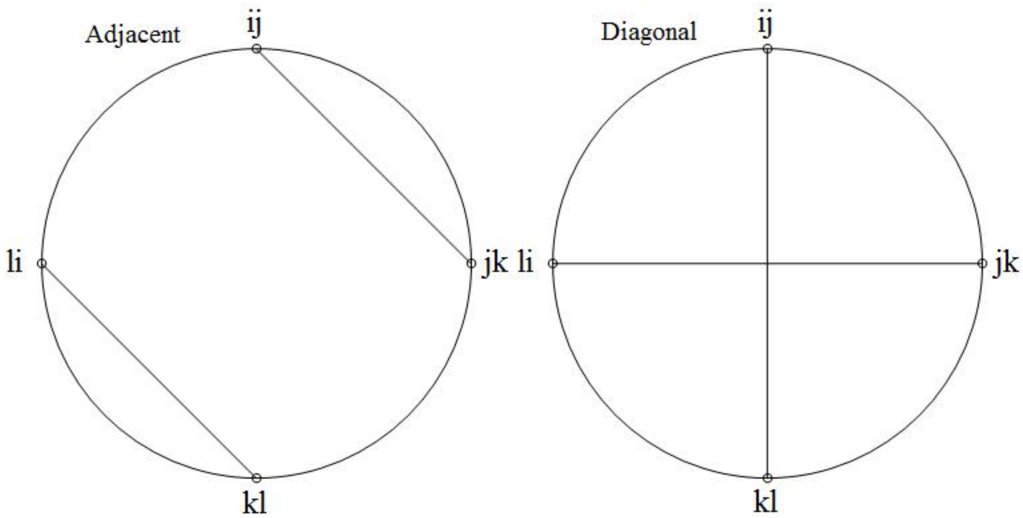

FIGURE 6. Matchings for the 4^{th} moment.

in the modulo condition affects moment values by comparing the low moments for two simple patterns $\{a, b, a, b\}$ and $\{a, a, b, b\}$.

3.2. **The Fourth Moment.** Although we will show that the higher moments differ by the way the elements are arranged in an m-pattern, the 4^{th} moment is in fact independent of the arrangement of elements. We first show that the 4^{th} moment for any m-pattern is determined solely by the frequency at which each element appears, and then show that this lemma fails for the 6^{th} moment and higher.

Lemma 3.1. *For an ensemble of real symmetric period m-circulant matrices of size N, if within each m-pattern, we have n i.i.d.r.v. $\{\alpha_r\}_{r=1}^n$, each of which has a fixed number of occurrences ν_r such that $\sum_{r=1}^n \nu_r = m$, then the 4^{th} moment of the limiting spectral distribution is $2 + \sum_{r=1}^n (\frac{\nu_r}{m})^3$.*

By (2.8), we calculate $\frac{1}{N^{\frac{4}{2}+1}} \sum_{1 \leq i,j,k,l \leq N} a_{ij} a_{jk} a_{kl} a_{li}$ for the 4^{th} moment. There are 2 ways of matching the 4 entries in 2 pairs, as show in Figure 6:

(1) (adjacent, 2 variations) $a_{ij} = a_{jk}$ and $a_{kl} = a_{li}$;
(2) (diagonal, 1 variation) $a_{ij} = a_{kl}$ and $a_{jk} = a_{li}$.

Thus there are 3 matchings, with the two adjacent matchings contributing the same to the 4th moment. We first consider one of the adjacent matchings, $a_{ij} = a_{jk}$ and $a_{kl} = a_{li}$. The pairing conditions (3.1) in this case are:

(1) (diagonal condition) $i - j \equiv k - j \,(\mathrm{mod}\ N)$, $k - l \equiv i - l \,(\mathrm{mod}\ N)$;
(2) (modulo condition) $i\mathcal{R}k$ or $j\mathcal{R}j$, $k\mathcal{R}i$ or $l\mathcal{R}l$.

Since $1 \leq i, j, k, l \leq N$, the diagonal condition requires $i = k$, and then the modulo condition follows trivially, regardless of the m-pattern we study. Hence, we can choose j and l freely, each with N choices, i freely with N choices, and then k is fixed. This matching then contributes $\frac{N^3}{N^{\frac{4}{2}+1}} = 1$ (fully) to the 4th moment, so does the other adjacent matching.

We proceed to the diagonal matching, $a_{ij} = a_{kl}$ and $a_{jk} = a_{li}$. The pairing conditions (3.1) in this case are:

(1) (diagonal condition) $i - j \equiv l - k \,(\mathrm{mod}\ N)$, $j - k \equiv i - l \,(\mathrm{mod}\ N)$;
(2) (modulo condition) $i\mathcal{R}l$ or $j\mathcal{R}k$, $j\mathcal{R}i$ or $k\mathcal{R}l$.

The diagonal condition $j - k \equiv i - l \,(\mathrm{mod}\ N)$ is equivalent to $i - j \equiv l - k \,(\mathrm{mod}\ N)$, which entails

(1) $i + k = j + l$, or
(2) $i + k = j + l + N$, or
(3) $i + k = j + l - N$.

In any case, we only need to choose 3 indices out of i, j, l, k, and then the last one is fixed. In the following argument, without loss of generality, we choose (i, j, l) and thus fix k.

For a general m-pattern, we write $i = 4\eta_1 + \epsilon_1, j = 4\eta_2 + \epsilon_2, k = 4\eta_3 + \epsilon_3, l = 4\eta_4 + \epsilon_4$, where $\eta_1, \eta_2, \eta_3, \eta_4 \in \{0, 1, \dots, \frac{N}{m}\}$ and $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \{0, 1, \dots, m-1\}$. Before we consider the ϵ_ℓ's, we note that there exist Diophantine contraints. For example, if $i+k = j + l$, given that $1 \leq i, j, l \leq N$, $k = j + l - i$ also needs to satisfy $1 \leq k \leq N$. As a result, we need $0 \leq \eta_2 + \eta_4 - \eta_1 \leq \frac{N}{4}$. Note that, due to the ϵ_ℓ's, sometimes we may have $0 \leq \eta_2 + \eta_4 - \eta_1 \leq \frac{N}{4} + \varepsilon$, where the error term $\varepsilon \in (-\frac{m}{2}, \frac{m}{2})$ and only trivially affects the number of choices of (η_2, η_4, η_1) for a fixed m as $N \to \infty$.

We now explore the Diophantine constraints for each variation of the diagonal condition (3.2). The $i + k = j + l$ case is similar to that in [HM], where, in a Toeplitz matrix, the diagonal condition only entails $i + k = j + l$, and there are obstructions to the system of Diophantine equations following the diagonal condition. However, the circulant structure that adds $i + k = j + l + N$ and $i + k = j + l - N$ to the diagonal condition fully removes the Diophantine obstructions. This explains why the limiting spectral distribution for ensembles of circulant matrices has the moments of a Gaussian, while that for ensembles of Toeplitz matrices has smaller even moments. We now study the 3 possibilities of the diagonal condition for the circulant structure.

(1) $i + k = j + l$.

We first cite Lemma 2.5 in [HM] about the obstructions to Diophantine equations.

Lemma 3.2. *Let* $I_N = \{1, \dots, N\}$. *Then* $\#\{x, y, z \in I_N : 1 \leq x + y - z \leq N\} = \frac{2}{3}N^3 + \frac{1}{3}N$.

Proof. Let $x + y = S \in \{2, \dots, 2N\}$. For $2 \leq S \leq N$, there are $S - 1$ choices of z such that $1 \leq x + y - z \leq N$; for $S \geq N + 1$, there are $2N - S + 1$ choices. Similarly, the number of (x, y) with $x, y \in I_N$ and $x + y = S$ is $S - 1$ if $S \leq N + 1$ and $2N - S + 1$ otherwise. The number of triples (x, y, z) is thus

$$\sum_{S=2}^{N}(S-1)^2 + \sum_{S=N+1}^{2N}(2N - S + 1)^2 \;=\; \frac{2}{3}N^3 + \frac{1}{3}N. \tag{3.2}$$

□

Back to our case, let $M = \frac{N}{m}$, the number of possible combinations of (η_2, η_4, η_1) that allow $0 \leq \eta_3 \leq \frac{N}{4}$ is $\frac{2}{3}M^3 + \frac{1}{3}M$.[9] For each of η_2, η_4, η_1, we have m free choices of ϵ_ℓ, and thus the number of (i, j, l) is $m^3(\frac{2}{3}M^3 + \frac{1}{3}M) = \frac{2}{3}N^3 + O(N)$.

(2) $i + k = j + l + N$.

[9] In [HM], the related lemma is proven for $\eta_2, \eta_4, \eta_1 \in \mathbb{N}_+$, i.e. no cases where $\eta_2\eta_4\eta_1 = 0$. Thus we are supposed to start from $S = 0$ in (3.2). However, as $N \to \infty$, the error from this imprecision will diminish.

$1 \le k \le N$ requires $0 \le \eta_2 + \eta_4 - \eta_1 + \frac{N}{m} \le \frac{N}{m} \Rightarrow -\frac{N}{m} \le \eta_2 + \eta_4 - \eta_1 \le 0$. Similar to the $i + k = j + l$ case, we write $M = \frac{N}{m}$ and $S = \eta_2 + \eta_4$, and then $-\frac{N}{m} \le S - \eta_1 \le 0 \Rightarrow S \le \eta_1 \le M + S$ where obviously $S \le M$. We have $S + 1$ ways to choose (η_2, η_4) s.t. $\eta_2 + \eta_4 = S$, and $M - S + 1$ choices of η_1. The number of (i, j, l) is thus

$$m^3 \sum_{S=0}^{M} (S+1)(M-S+1) = m^3 \left(\frac{M^3}{6} + M^2 + \frac{5}{6}M \right) = \frac{N^3}{6} + O(N^2). \tag{3.3}$$

(3) $i + k = j + l - N$.

$1 \le k \le N$ requires $0 \le \eta_2 + \eta_4 - \eta_1 - \frac{N}{m} \le \frac{N}{m} \Rightarrow \frac{N}{m} \le \eta_2 + \eta_4 - \eta_1 \le \frac{2N}{m}$. Again, we write $M = \frac{N}{m}$ and $S = \eta_1 + \eta_4$, and then $M \le S - \eta_1 \le 2M \Rightarrow S - 2M \le \eta_1 \le S - M$ where obviously $S \ge M$. We have $2M - S + 1$ ways to choose (η_2, η_4) s.t. $\eta_2 + \eta_4 = S$, and $S - M + 1$ choices of η_1. The number of (i, j, l) is thus

$$m^3 \sum_{S=M}^{2M} (2M-S+1)(S-M+1) = m^3 \left(\frac{M^3}{6} + M^2 + \frac{5}{6}M \right) = \frac{N^3}{6} + O(N^2). \tag{3.4}$$

Therefore, with the additional diagonal conditions $i + k = j + l + N$ and $i + k = j + l - N$ induced by the circulant structure, the number of (i, j, l) is of the order $(\frac{2}{3} + \frac{1}{6} + \frac{1}{6})N^3 = N^3$, i.e. the circulant structure compensates for the obstructions to Diophantine equations in the Toeplitz case. Since the η_ℓ's do not matter for the modulo condition, to make a non-trivial configuration, we may choose three η_ℓ's freely, each with $\frac{N}{m}$ choices, and then choose some ϵ_ℓ's that satisfy the modulo condition, which we will study below.

For the modulo condition, it is necessary to figure out which zones the 4 entries are located in. Recall that the diagonal condition will always ensure that two paired entries are located in different areas. For the 4^{th} moment, each of the 3 variations of the diagonal condition is sufficient to ensure that any pair of entries involved are located in the right zones. We may check this rigorously by enumerating all possibilities of the zone-wise locations of the 4 entries, e.g. if $i + k = j + l + N$, then $a_{ij} \in$ Zone $1 \Rightarrow a_{kl} \in$ Zone

2.[10] As a result, for a pair of matrix elements in the diagonal matching, say $a_{ij} = a_{kl}$, if i determines the slot in an m-pattern for a_{ij} and thus matters for the modulo condition, then l determines the slot for a_{kl}; if j determines for a_{ij}, then k determines for a_{kl}, and vice versa.

With the zone-wise issues settled, we study how to obtain a non-trivial configuration for the 4th moment. Recall the modulo condition for the diagonal matching: $i\mathcal{R}l$ or $j\mathcal{R}k$, $j\mathcal{R}i$ or $k\mathcal{R}l$. This entails $2^2 = 4$ sets of equivalence relations:

$$i\mathcal{R}l\mathcal{R}j, i\mathcal{R}l\mathcal{R}k, j\mathcal{R}k\mathcal{R}i, j\mathcal{R}k\mathcal{R}l$$

Each set of equivalence relations appears with a certain probability, depending on the zone-wise locations of the 4 entries. For example, $i\mathcal{R}l\mathcal{R}j$ follows from $i\mathcal{R}l$ and $j\mathcal{R}i$, which requires both a_{ij} and a_{jk} $\in$ Area I. Regardless of the probability with which each set occurs, we choose one free index with N choices, and then the other two indices such that these 3 indices are related to each other under $\mathcal{R}$. The number of choices of the two indices after the free one is determined solely by the number of occurrences of the elements in an m-pattern.

We give a specific example of making a non-trivial configuration for the 4th for two simple patterns $\{a, b, a, b\}$ and $\{a, a, b, b\}$. Under the condition $i + k = j + l$, if $a_{ij} \in$ Zone 1 and $a_{jk} \in$ Zone 3, then $a_{kl} \in$ Zone 4 and $a_{li} \in$ Zone 2. We first select η_1, η_2, η_4 such that i, j, l and $k = j + l - i$ satisfy the zone-wise locations.[11] In this case, based on pairing conditions (3.1), pairing $a_{ij} = a_{kl}$ and $a_{jk} = a_{li}$ will require $\epsilon_1\mathcal{R}\epsilon_4$ and $\epsilon_2\mathcal{R}\epsilon_1$, or equivalently $\epsilon_1\mathcal{R}\epsilon_2\mathcal{R}\epsilon_4$. Without loss of generality, we can start with a free ϵ_1 with 4 choices, then there are 2 free choices for each of ϵ_2 and ϵ_4, and then we have a non-trivial configuration. We have similar stories under the other two variations of the diagonal condition and with other zone-wise locations of a_{ij} and a_{kl}. Therefore, we can choose 3 out of 4 η_ℓ's freely, each with $\frac{N}{4}$ choices, then one ϵ_ℓ with 4 choices, then another two ϵ_ℓ's each with 2 choices, and finally the last index is determined under the diagonal condition. As

[10] This enumeration is complicated since the zone where an entry a_{ij} is located imposes restrictions on the choice of i, j, e.g. when $a_{i,j} \in$ Zone 2, we have $i \geq \frac{N}{2}$ and $j \leq \frac{N}{2}$.

[11] It is noteworthy that the specific location of an element still depends on the ϵ_ℓ's, but as $N \to \infty$, the probability that the η_ℓ's alone determine the zone-wise locations of elements approaches 1, i.e. the probability that adding the ϵ_ℓ's changes the zone-wise location of an element approaches 0.

discussed before, such a choice of indices will always satisfy the zone-wise requirements and thus the ϵ-based pairing conditions. Thus there are $(\frac{N}{4})^3 \cdot 4 \cdot 2 \cdot 2 = \frac{N^3}{4}$ choices of (i, j, k, l) that will produce a non-trivial configuration. It follows that the contribution from the diagonal matching to the 4^{th} moment is $\frac{1}{N^3}(\frac{2}{3} + \frac{1}{6} + \frac{1}{6})\frac{N^3}{4} = \frac{1}{4}$.

The computation of the 4^{th} moment for the simple patterns $\{a, b, a, b\}$ and $\{a, a, b, b\}$ can be immediately generalized to the 4^{th} moment for other patterns. As emphasized before, both adjacent matchings contribute fully to the 4^{th} moment regardless of the m-pattern. For diagonal matching, the system of Diophantine equations induced by the diagonal condition are also independent of the m-pattern in question, and the way we count possible configurations can be easily generalized to an arbitary m-pattern. We have thus proved Lemma 3.1.

Note that Lemma 3.1 implies that the 4^{th} moment for any pattern depends solely on the frequency at which each element appears in an m-period. Besides the $\{a, a, b, b\}$ pattern that we have studied in depth, we may easily test two extreme cases. One case where $n = m$, i.e. each r.v. appears only once, represents the [KKM] m-circulant matrices for which the 4^{th} moment is $2+\frac{1}{m^2}$. Another case where $n = 1$ represents the circulant matrices for which the 4^{th} moment is 3. Numerical simulations for numerous patterns including $\{a, a, b\}$, $\{a, b, b\}$, $\{a, b, b, a\}$, $\{a, b, c, a, b, c\}$, $\{a, b, c, d, e, e, d, c, b, a\}$ etc. are consistent with Lemma 3.1 as well, see Table 1.

3.3. **The Sixth Moment.** Although for an m-pattern with each element appearing at a fixed frequency, the 4^{th} moment is independent of how the elements are arranged within the pattern, the way the elements are arranged in an m-pattern does affect higher moments and thus the limiting spectral distribution. As we will show for the 6^{th} moment, for patterns with repeated elements, there exist "obstructions to modulo equations" that make trivial some non-trivial configurations for patterns without repeated elements. We illustrate this by explicitly computing the 6^{th} moment for the pattern $\{a, b, a, b\}$, and then showing why the 6^{th} moment for $\{a, a, b, b\}$ differs. It will then be clear that the modulo obstructions persist for more complicated patterns and higher moments.

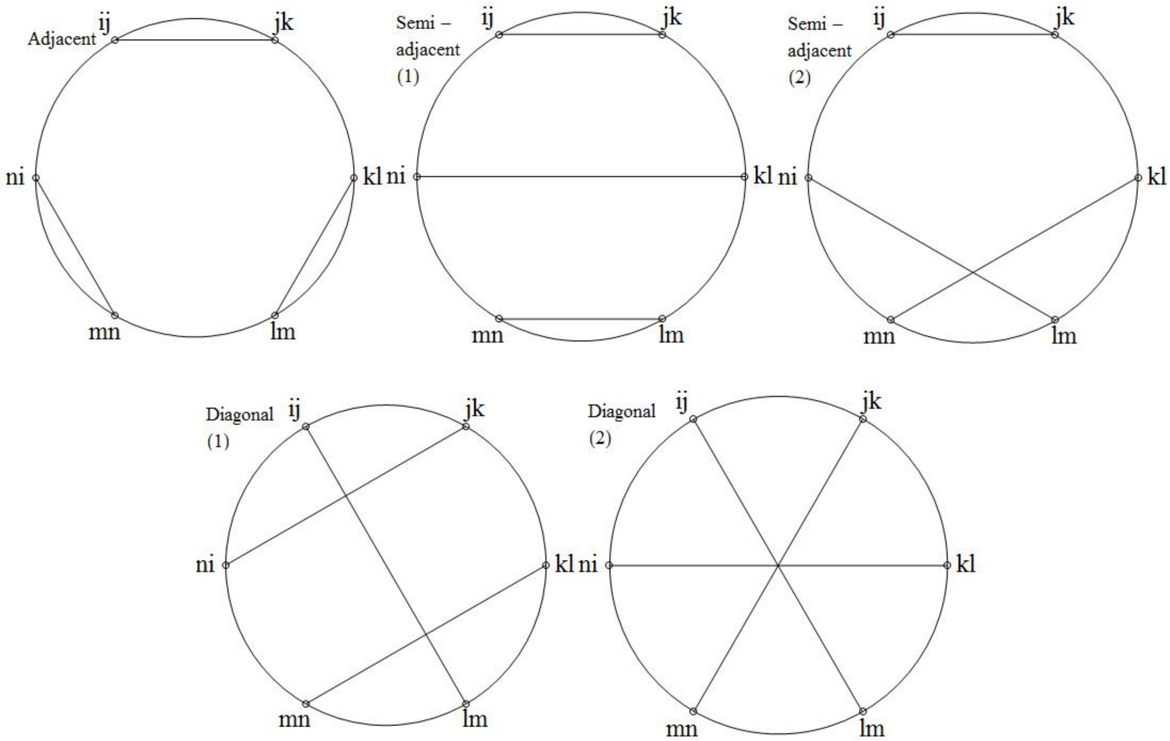

FIGURE 7. Matchings for the 6^{th} moment.

For the 6^{th} moment, we calculate $\frac{1}{N^{\frac{6}{2}+1}} \sum_{1 \leq i,j,k,l,m,n \leq N} a_{ij}a_{jk}a_{kl}a_{lm}a_{mn}a_{ni}$ by (2.8). There are $(6-1)!! = 15$ matchings, which can be classified into 5 types, so that the 6 entries are matched in 3 pairs, as illustrated in Figure 7:

(1) $a_{ij} = a_{jk}$, $a_{kl} = a_{lm}$, $a_{mn} = a_{ni}$ (adjacent, 2 variations).
(2) $a_{ij} = a_{jk}$, $a_{kl} = a_{ni}$, $a_{lm} = a_{mn}$ (semi-adjacent-1, 3 variations).
(3) $a_{ij} = a_{jk}$, $a_{kl} = a_{mn}$, $a_{lm} = a_{ni}$ (semi-adjaent-2, 6 variations).
(4) $a_{ij} = a_{lm}$, $a_{jk} = a_{ni}$, $a_{kl} = a_{mn}$ (diagonal-1, 3 variations).
(5) $a_{ij} = a_{lm}$, $a_{jk} = a_{mn}$, $a_{kl} = a_{ni}$ (diagonal-2, 1 variation).

Similar to the 4^{th} moment computation, we start with adjacent cases. For example, if $a_{ij} = a_{jk}$, then the two pairing conditions (3.1) require

(1) $i-j = k-j \Rightarrow i = k$. Given that $i, j, k, l \in \{1, 2, \dots, N\}$, neither $i-j = k-j+N$ nor $i-j = k-j-N$ is possible.

(2) $i \equiv k \pmod 2$ or $j \equiv j \pmod 2$, depending on the zone-wise location of a_{ij} and a_{jk}. Either follows trivially from the previous condition.

For Type 1 (adjacent), take $a_{ij} = a_{jk}$, $a_{kl} = a_{lm}$, $a_{mn} = a_{ni}$, the diagonal condition requires

$$i - j = k - j, k - l = m - l, m - n = i - n \Rightarrow i = m = k. \tag{3.5}$$

By the discussion on the adjacent case, the modulo condition is satisfied trivially. We can then freely choose i, j, l, n, each with N choices, and make a non-trivial configuration that contributes $\frac{N^4}{N^{\frac{6}{2}+1}} = 1$ (fully). Type 1 matchings thus contribute $2 \times 1 = 2$ (2 variations of Type 1) to the 6^{th} moment.

For Type 2 (semi-adjacent-1), take $a_{ij} = a_{jk}$, $a_{kl} = a_{ni}$, $a_{lm} = a_{mn}$, the adjacent case $a_{ij} = a_{jk}$ requires $i = k$ as discussed before. Thus the second pair $a_{kl} = a_{ni}$ is equivalent to $a_{kl} = a_{nk}$, which is again an adjacent case. The third pair $a_{lm} = a_{mn}$ is an adjacent case itself. Thus Type 2 is in fact equivalent to Type 1, and contributes $3 \times 1 = 3$ to the 6^{th} moment.

For Type 3 (semi-adjacent-2), the adjacent case $a_{ij} = a_{jk}$ requires $i = k$ as discussed before. Thus the third pair $a_{lm} = a_{ni}$ is equivalent to $a_{lm} = a_{nk}$, and the second and the third pair combined make the diagonal matching as in the 4^{th} moment computation. We have shown that this diagonal matching contributes $\frac{1}{4}$ to the 4^{th} moment (see Lemma 3.1). Note that j is free with N choices despite the restriction $i = k$. Thus this matching contributes $\frac{1}{4}$, and this type $6 \times \frac{1}{4} = \frac{3}{2}$, to the 6^{th} moment.

Note that Type 1 and 2 are independent of the m-circulant pattern along the diagonals in an m-circulant matrix, and Type 3 also applies to other variations of $\{a, b, a, b\}$ such as $\{a, a, b, b\}$ and $\{a, b, b, a\}$. Type 1 through 3 combined, we have $2 + 3 + \frac{3}{2} = 6.5$ in the 6^{th} moment.

We proceed to the diagonal matchings, for which we will discuss the modulo obstructions, and start with a simple case for Type 4. Take the matching $a_{ij} = a_{lm}$, $a_{jk} = a_{ni}$, $a_{kl} = a_{mn}$ as an example, the two pairing conditions (3.1) require:

(1) $i - j = m - l$, $j - k = i - n$, $k - l = n - m \Rightarrow i - j = m - l = n - k$,[12] which shows that we need to choose only 4 of the 6 indices, and the other 2 are determined.

(2) $i\mathcal{R}m$ or $j\mathcal{R}l$, $j\mathcal{R}i$ or $k\mathcal{R}n$, $k\mathcal{R}n$ or $l\mathcal{R}m$, depending on the zone-wise locations of the entries. For example, if $a_{ij} \in$ Zone 1, then $i - j = m - l \Rightarrow a_{lm} \in$ Zone 4. We have $2^3 = 8$ sets of equivalance relations, categorized as follows.

Category (1) (4 sets): $i\mathcal{R}m\mathcal{R}j, k\mathcal{R}n$; $j\mathcal{R}l\mathcal{R}m, k\mathcal{R}n$; $i\mathcal{R}m\mathcal{R}l, k\mathcal{R}n$; $j\mathcal{R}l\mathcal{R}i, k\mathcal{R}n$.

Category (2) (2 sets): $i\mathcal{R}m\mathcal{R}j\mathcal{R}l$; $j\mathcal{R}l\mathcal{R}i\mathcal{R}m$.

Category (3) (2 sets): $i\mathcal{R}m$, $k\mathcal{R}n$; $j\mathcal{R}l$, $k\mathcal{R}n$.

Each set of equivalance relations appears with a certain probability, and the probabilities of observing each $\mathcal{R}$ set sum up to 1. We show below that, regardless of the probability of observing each set, each set contributes $\frac{1}{4}$ to the 6^{th} moment, and thus the probability-weighted contribution is simply $\frac{1}{4}$.

For Cat.(1), the set of equivalence relations $i\mathcal{R}m\mathcal{R}j, k\mathcal{R}n$ requires $a_{ij}, a_{jk}, a_{kl} \in$ Zone 1 or 3. We can start with a free i with N choices, then select m, j, each with $\frac{N}{2}$ choices, such that $i\mathcal{R}j\mathcal{R}m$. Then we pick a k, and note that $i - j = n - k, i\mathcal{R}j \Rightarrow k\mathcal{R}n$. Recall that, for the pattern $\{a, b, a, b\}$, $i\mathcal{R}j$ indicates $2|(i - j)$, and it follows that $2|(n - k)$. In other words, we can freely choose a k with N choices, and the diagonal condition $i - j = n - k$ ensures that we have a good n. This set thus contributes $\frac{1}{N^4} \cdot (N \cdot \frac{N}{2} \cdot \frac{N}{2} \cdot N) = \frac{1}{4}$. The same analysis applies to the other 3 sets in Cat.(1).

For Cat.(2), take the set $i\mathcal{R}m\mathcal{R}j\mathcal{R}l$. We start with a free i, and then select m, j, each with $\frac{N}{2}$ choices, such that $i\mathcal{R}m\mathcal{R}j$. Note that, again, $i - j = m - l, i\mathcal{R}j \Rightarrow m\mathcal{R}l \Rightarrow i\mathcal{R}m\mathcal{R}j\mathcal{R}l$. This set thus contributes $\frac{1}{N^4} \cdot (N \cdot \frac{N}{2} \cdot \frac{N}{2} \cdot N) = \frac{1}{4}$. The same analysis applies to the other set in Cat.(2).

For Cat.(3), take the set $i\mathcal{R}m, k\mathcal{R}n$. We start with a free i, and then select m with $\frac{N}{2}$ free choices such that $i\mathcal{R}m$. Then we choose a free k with N choices and n with $\frac{N}{2}$ choices

[12] We temporarily ignore $i - j = m - l + N$ and $i - j = m - l - N$ for simplicity. In fact, as we show in the 4^{th} moment computation, the $i - j = m - l + N$ case and the $i - j = m - l + N$ case, each of which has $\frac{N^3}{6} + O(N^2)$ solutions, together make up for the obstructions to a Diophantine equation like $i - j = m - l$ that has $\frac{2N^3}{3} + O(N^2)$ solutions.

such that $k\mathcal{R}n$. This set thus contributes $\frac{1}{N^4} \cdot (N \cdot \frac{N}{2} \cdot N \cdot \frac{N}{2}) = \frac{1}{4}$. The same analysis applies to the other set of Cat.(3).

Since each individual set of equivalence relations in Cat.(1)-(3) contributes equally, the probability-weighted contribution to the 6th moment is $\frac{1}{4}$. Therefore, Type 4, with 3 variations, contributes $\frac{3}{4}$.

Similarly, the pairing conditions (3.1) entail the following for Type 5 (diagoal 2).

(1) $i - j = m - l = k - n$.

(2) 2 categories of equivalence relation set.

Cat.(1)(6 sets): $i\mathcal{R}m\mathcal{R}k, j\mathcal{R}n$; $j\mathcal{R}l\mathcal{R}n, k\mathcal{R}m$; $i\mathcal{R}m, j\mathcal{R}n\mathcal{R}l$; $j\mathcal{R}l, k\mathcal{R}m\mathcal{R}i$; $i\mathcal{R}m\mathcal{R}k, l\mathcal{R}n$; $j\mathcal{R}l\mathcal{R}n, k\mathcal{R}i$.

Cat.(2)(2 sets): $i\mathcal{R}m\mathcal{R}k$; $j\mathcal{R}l\mathcal{R}n$.

Replicating the analysis of Type 4, we find that, since each set of equivalence relations in Cat.(1) and Cat.(2) contributes $\frac{1}{4}$, the probability-weighted contribution must be $\frac{1}{4}$ as well. Since Type 5 has only 1 variation, Type 5 contributes $\frac{1}{4}$ to the 6th moment.

Therefore, the combined contribution from Type 4 and Type 5 is $\frac{3}{4} + \frac{1}{4} = 1$. The 6th moment for the pattern $\{a, b, a, b\}$ is then $6.5 + 1 = 7.5$, as evidenced by numerics and the explicit limiting density in [KKM].

Now we examine why the contribution from diagonal matchings for the pattern $\{a, a, b, b\}$ differs from that for $\{a, b, a, b\}$. As discussed before, Type 1 through 3 matchings, with a total contribution of 6.5, also apply to $\{a, a, b, b\}$. For Type 4 and 5, however, the combined contribution is less than 1. Recall a key argument in the analysis of Type 4 matching before: for the 2-circulant $\{a, b, a, b\}$ pattern, under $i - j = m - l = n - k$, $i\mathcal{R}j\mathcal{R}m$, and $k\mathcal{R}n$, if we choose a k freely, since $i\mathcal{R}j \iff i \equiv j \pmod 2$, then $n = i + k - j$ will satisfy $n \equiv k \pmod 2$ as well. Namely, $i - j = n - k, i\mathcal{R}j \Rightarrow n\mathcal{R}k$. However, for $\{a, a, b, b\}$, if we specify $\mathcal{R}$ as $s\mathcal{R}t \iff \mod(s, 4), \mod(t, 4) \in \{1, 2\}$ or $\mod(s, 4), \mod(t, 4) \in \{0, 3\}$, and choose $i\mathcal{R}j\mathcal{R}m$, it is possible that $n = i + k - j$ is not related to k under $\mathcal{R}$. For instance, when $\mod(i, 4) = 1$, $\mod(j, 4) = 2$,

$\mod (k,4) = 3$, we have $i\mathcal{R}j$, but $\mod (n,4) = 2$. Some configurations that are non-trivial for $\{a,b,a,b\}$ then become trivial for $\{a,a,b,b\}$, while all the non-trivial configurations for $\{a,a,b,b\}$ are still non-trivial for $\{a,b,a,b\}$. Thus, we expect the 6$^{\text{th}}$ moment for $\{a,a,b,b\}$ to be smaller than that for $\{a,b,a,b\}$, which is evidenced numerically in Table 1. We phrase such a loss of non-trivial configurations as due to "obstructions to modulo equations", or "modulo obstructions" for short, which should persist in higher moments for general m-circulant patterns with repeated elements.

Based on the brute-force computation above, we may also bound the even moments for generalized m-circulant patterns. It is clear that a lower bound is the moment for the m-circulant pattern of the same period length and in which each element is distinct. For example, in terms of high ($2k^{\text{th}}$, $k \geq 2$) moments, $\{a,a,b,b\} > \{a,b,c,d\}$ (both of length 4). In the computation of high moments, a pattern with repeated elements has all the non-trivial configurations that an all-distinct pattern of the same length can have, and gains extra non-trivial configurations due to the repeated elements.

An easy upper bound is the moment of the standard Gaussian, which is the limiting spectral distribution for the ensemble of circulant matrices. We may also easily find a sharper upper bound for a family of simple m-circulant patterns in which each element appears at the same frequency, e.g. $\{a,b,c,c,b,a\}$, $\{a,a,b,c,b,c\}$, etc. For this family, an upper bound will be associated with a pattern where each element only appears once. For example, in terms of high moments, $\{a,b,c\} > \{a,b,c,c,b,a\}$. We may take $\{a,b,c\}$ as $\{a,b,c,a,b,c\}$, and note that, although in $\{a,b,c,a,b,c\}$, the probability of choosing each letter is the same as in $\{a,b,c,c,b,a\}$, the former pattern is free of modulo obstructions that exist for the latter.

For a more general m-circulant pattern, however, a sharper upper bound is not easily attainable. For instance, it is not clear whether $\{a,a,b,c\} > \{a,b,c\}$. Some numerical evidence suggests that a pattern in which $\gcd(\nu_1, \nu_2 \dots \nu_\ell) = 1$, where ν_ℓ is the number of occurrences of an element in an m-period, has larger high moments than those with the same frequency of each element but $\gcd(\nu_1, \nu_2 \dots \nu_\ell) \geq 2$. For example, $\{a,b,c,c\} > \{a,a,b,b,c,c,c,c\}$.

Obviously, the accounting above will become significantly more involved for more complicated patterns or higher moments, but the basic ideas will remain the same. We also foresee that as the moments get higher, the number of configurations that contribute trivially will increase so quickly that the higher moments get increasingly farther below the standard Gaussian moments. This is also evidenced by simulations (see Table 1).

4. Existence and Convergence of High Moments

Although it is impractical to find every moment for a general m-circulant pattern using brute-force computation, we are still able to prove that, for any m-circulant pattern, every moment exists and is finite, and thus there exists a limiting spectral distribution. In addition, the empirical spectral measure of a typical real symmetric m-circulant matrix converges to this limiting measure, and we will show both convergence in probability and almost sure convergence.

4.1. Existence.

We have shown that all the odd moments vanish as $N \to \infty$, and thus we focus on the even moments. We need to prove the following theorem.

Theorem 4.1. $\lim_{N\to\infty} M_{2k}(N)$ *exists and is finite* $\forall k \in \mathbb{N}_+$.

Proof. First off, it is trivial that $M_{2k}(N)$ is finite. As discussed before, it is bounded below by the $2k^{\text{th}}$ moment for the ensemble of m-circulant matrices where, in the m-pattern, each element is distinct, and [KKM] have found an explicit density for such an m-circulant pattern. It is bounded above by the $2k^{\text{th}}$ moment for the ensemble of circulant matrices, and we know that the limiting spectral distribution for this matrix ensemble is a Gaussian.

We now show that $\lim_{N\to\infty} M_{2k}(N)$ exists. To calcuate $M_{2k}(N)$, we match $2k$ elements from the matrix, $\{a_{i_1 i_2}, a_{i_2 i_3}, \dots, a_{i_{2k} i_1}\}$, in k pairs, where $i_\ell \in \{1, 2, \dots, N\}$ and this will give $(2k-1)!!$ matchings. For each matching, there are a certain number of configurations, and most of such configurations do not contribute to the moments as $N \to \infty$.

For the [KKM] m-circulant pattern, the equivalence relation $\mathcal{R}$ implies that $\epsilon_s \mathcal{R} \epsilon_{t+1} \Leftrightarrow \epsilon_s = \epsilon_{t+1}$, and since $m | (\epsilon_s - \epsilon_{s+1} + \epsilon_t - \epsilon_{t+1})$, we have $\epsilon_{s+1} = \epsilon_t$ as well (see (3.1)).[13] Thus

[13] This explains why, for an m-pattern without repeated elements, the zone-wise locations of matrix entries do not matter in making a non-trivial configuration.

$\eta_s - \eta_{s+1} + \eta_t - \eta_{t+1} = 0$ or $\pm \frac{N}{m}$, three equations that have $(\frac{N}{m})^3 + O((\frac{N}{m})^2)$ solutions in total, as we have shown in the 4^{th} moment computation.

However, if there are repeated elements in an m-period, then $\epsilon_s \mathcal{R} \epsilon_{t+1}$ no longer necessitates $\epsilon_s = \epsilon_{t+1}$, and it is possible that $(\epsilon_s - \epsilon_{s+1} + \epsilon_t - \epsilon_{t+1}) = \pm m$. Thus, the zone-wise locations of elements matter in making non-trivial configurations. Recall that the zone-wise location (see (3.1)) of an element $a_{i_s i_{s+1}}$ is determined by $(i_{s+1} - i_s)$: if $a_{i_s i_{s+1}}$ is in Zone 1 or 3 (Area I), ϵ_s determines the slot of $a_{i_s i_{s+1}}$ in an m-period; if $a_{i_s i_{s+1}}$ is in Zone 2 or 4 (Area II), ϵ_{s+1} determines the slot of $a_{i_s i_{s+1}}$ in an m-period. In addition, the diagonal condition will always ensure that two paired entries $a_{i_s i_{s+1}}$ and $a_{i_t i_{t+1}}$ are located in different areas.

Recall that for any matching $\mathcal{M}$, the k pairs of matrix elements, each pair in the form of $a_{i_s i_{s+1}} = a_{i_t i_{t+1}}$, are fixed. For any $\mathcal{M}$, to make a non-trivial configuration, we first choose an ϵ vector of length $2k$. If we choose all the ϵ_ℓ's freely, there are m^{2k} possible choices for an ϵ vector, most of which do not meet the modulo condition, and trivially, m^{2k} is an upper bound for the number of valid ϵ vectors. It is noteworthy that out of the $2k$ ϵ_ℓ's of an ϵ vector, only some of the ϵ_ℓ's will matter for the modulo condition. Which ϵ_ℓ's in fact matter depends on how we pair the $2k$ matrix entries $a_{i_s i_{s+1}}$'s and the zone-wise locations of the paired $a_{i_s i_{s+1}}$'s, which we cannot determine without fixing the η_ℓ's (and thus the i_ℓ's).

However, for any matching, the way we pair the $2k$ matrix entries into k pairs is fixed, and for each fixed pair $a_{i_s i_{s+1}} = a_{i_t i_{t+1}}$, two ϵ_ℓ's will matter for the modulo condition: either $\epsilon_s \mathcal{R} \epsilon_{t+1}$ or $\epsilon_{s+1} \mathcal{R} \epsilon_t$. Thus there are 2^k ways to choose k pairs of ϵ_ℓ's for each matching. For each way of fixing the k pairs of ϵ_ℓ's, we examine each ϵ pair, say $(\epsilon_{\ell_1}, \epsilon_{\ell_2})$, and there are a certain number of choices of $(\epsilon_{\ell_1}, \epsilon_{\ell_2})$ such that $\epsilon_{\ell_1} \mathcal{R} \epsilon_{\ell_2}$. Continuing in this way, for each ϵ pair, we choose two ϵ_ℓ's that satisfy the equivalence relation $\mathcal{R}$. Note that an ϵ_ℓ may matter twice, once, or never for the modulo condition depending on the zone-wise locations of the $a_{i_s i_{s+1}}$'s. We then choose the other ϵ_ℓ's that do not matter for the modulo condition such that for each pair of $a_{i_s i_{s+1}} = a_{i_t i_{t+1}}$, we have $\epsilon_s - \epsilon_{s+1} + \epsilon_t - \epsilon_{t+1} = 0$ or $\pm m$, and finally we have a valid ϵ vector. The number of valid ϵ vectors will be determined by m, k, and the pattern of an m-period, but will be independent of N since the system of k equivalence relations for the modulo condition does not involve N.

With a valid ϵ vector, we have fixed the zone-wise locations of the $2k$ matrix elements by fixing the ϵ_ℓ's that matter for the modulo condition. We now turn to the diagonal condition and study the η_ℓ's. With k equations in the form of

$$m(\eta_s - \eta_{s+1} + \eta_t - \eta_{t+1}) + (\epsilon_s - \epsilon_{s+1} + \epsilon_t - \epsilon_{t+1}) = 0 \text{ or } \pm N, \tag{4.1}$$

and $(\epsilon_s - \epsilon_{s+1} + \epsilon_t - \epsilon_{t+1})$ known in each of the k equations, we in fact have k equations in the form of

$$\eta_s - \eta_{s+1} + \eta_t - \eta_{t+1} = \gamma, \tag{4.2}$$

where $\gamma \in \{0, \pm 1, \frac{N}{m}, \frac{N}{m} \pm 1, -\frac{N}{m}, -\frac{N}{m} \pm 1\}$. This gives us $k+1$ degrees of freedom in choosing the η_ℓ's, and trivially, we can have at most $(\frac{N}{m})^{k+1}$ vectors of η_ℓ's. Since the ϵ vector is fixed, for one equation $\eta_s - \eta_{s+1} + \eta_t - \eta_{t+1} = \gamma$, there are only 3 choices of γ. With k equations in this form, we have at most 3^k systems of η equations. Note that not all of the η vectors satisfying an η equation system derived from the diagonal condition will help make a non-trivial configuration, since the η_ℓ's need to be chosen such that the $a_{i_s i_{s+1}}$'s will satisfy the zone-wise locations in order to be coherent with the pre-determined ϵ vector. For example, if in a pair of matrix entries $a_{i_s i_{s+1}} = a_{i_t i_{t+1}}$ where $\epsilon_s \mathcal{R} \epsilon_{t+1}$, even though the η_ℓ's are chosen such that $\eta_s - \eta_{s+1} + \eta_t - \eta_{t+1} = \gamma$, it is possible that $a_{i_s i_{s+1}}, a_{i_t i_{t+1}}$ are located in certain zones such that we need $\epsilon_{s+1} \mathcal{R} \epsilon_t$ to ensure a non-trivial configuration.

The following steps mirror those in [HM], suggested by David Farmer. Denote an η equation system by $\mathcal{S}$. For any $\mathcal{S}$ we have k equations with $\eta_1, \eta_2, \ldots, \eta_{2k} \in \{1, 2, \ldots, \frac{N}{m}\}$. Let $z_\ell = \frac{\eta_\ell}{N/m} \in \{\frac{m}{N}, \frac{2m}{N}, \ldots, 1\}$. Without the zone-wise concerns discussed before, the system of k equations would have $k+1$ degrees of freedom and determine a nice region in the $(k+1)$-dimensional unit cube. Taking into account the zone-wise concerns, however, we will still have $k+1$ degrees of freedom. For example, for a pair of matrix elements $a_{i_s i_{s+1}} = a_{i_t i_{t+1}}$, the system $\mathcal{S}$ requires $\eta_s - \eta_{s+1} + \eta_t - \eta_{t+1} = \gamma$. If we need $\epsilon_s \mathcal{R} \epsilon_{t+1}$ to make a non-trivial configuration, say $a_{i_s i_{s+1}} \in$ Zone 1, then we will obtain an additional equation $0 \leq i_{s+1} - i_s \leq \frac{N}{2} - 1 \Rightarrow 0 \leq (\eta_{s+1} - \eta_s) + \epsilon_{s+1} - \epsilon_s \leq \frac{N}{2} - 1$ with $(\epsilon_{s+1} - \epsilon_s) \in \{-m+1, -m+2, \ldots, 0, 1, \ldots, m-2, m-1\}$. Based on the region determined by

$\eta_s - \eta_{s+1} + \eta_t - \eta_{t+1} = \gamma$, this additional zone-related restriction will only allow a slice of the region for us to choose valid η_ℓ's. With k zone-wise restrictions, only a proportion of the original region in the unit cube will be preserved for the choice of the η vector. Nevertheless, the "width" of each slice is of order $\frac{N}{2}$, and we still have $k+1$ degrees of freedom.

Therefore, with m fixed and as $N \to \infty$, we obtain to first order the volume of this region, which is finite. Unfolding back to the η_ℓ's, we obtain $M_{2k}(\mathcal{S})(\frac{N}{m})^{k+1} + O_k((\frac{N}{m})^k)$, where $M_{2k}(\mathcal{S})$ is the volume associated with this η system. Summing over all η systems, we obtain the number of non-trivial configurations for the $2k^{\text{th}}$ moment from this particular ϵ vector. Next, within a given matching $\mathcal{M}$, we sum over all valid ϵ vectors, the number of which is independent of N as we have shown before. In the end, we sum over the $(2k-1)!!$ matchings to obtain $M_{2k}N^{k+1}+O_k(N^k)$, and the $2k^{\text{th}}$ moment is simply $\frac{M_{2k}N^{k+1}+O_k(N^k)}{N^{k+1}} = M_{2k} + O(\frac{1}{N})$. □

4.2. **Convergence.** Having established that the limit of any high-order moment exists and is finite as $N \to \infty$, we proceed to showing the convergence in probability and almost sure convergence (both of which imply weak convergence, see (1.3.1)) of empirical moment values to the corresponding limit. We will thus complete the proof that the empirical spectral measure for the ensemble of m-circulant matrices converges to some nice probability distribution.

4.2.1. *Convergence in Probability.* We start with the proof of convergence in probability, for which the arguments in [HM] are sufficiently general to be immediately applicable. We thus only sketch the proof below. Let A be an infinite sequence of real numbers drawn i.i.d.r.v. from a nice probability distribution p and A_N be the associated $N \times N$ matrix. Let $X_{m,N}(A)$ denote the m^{th} moment of the measure associated with A_N and M_m be the limit of the m^{th} moment, which is finite as we have shown. Setting $X_m(A) = M_m$, we have $X_{m,N} \to X_m$ in probability if $\forall \epsilon > 0$

$$\lim_{N\to\infty} \mathbb{P}_{\mathbb{N}}(\{A \in \Omega_{\mathbb{N}} : |X_{m,N}(A) - X_m(A)| > \epsilon\}) = 0. \tag{4.3}$$

By the triangle inequality,

$$|X_m(A_N) - X_m(A)| \leq |X_m(A_N) - M_m(N)| + |M_m(N) - M_m|. \tag{4.4}$$

By Chebyshev's inequality,

$$\mathbb{P}_{\mathbb{N}}(\{A \in \Omega_{\mathbb{N}} : |X_{m,N}(A) - M_m(N)| > \epsilon\}) \leq \frac{\text{Var}[M_m(A_N)]}{\epsilon^2} = \frac{\mathbb{E}[M_m(A_N)^2] - \mathbb{E}[M_m(A_N)]^2}{\epsilon^2}. \tag{4.5}$$

Since $M_m(N) - M_m \to 0$ as $N \to \infty$, it suffices to show that $\forall m$,

$$\lim_{N\to\infty} (\mathbb{E}[M_m(A_N)^2] - \mathbb{E}[M_m(A_N)]^2) = 0 \tag{4.6}$$

and then apply the Moment Convergence Theorem (Theorem (1.3)).

By (2.8), we have

$$\mathbb{E}[M_m(A_N)^2] = \frac{1}{N^{m+2}} \sum_{1\leq i_1,\dots,i_m\leq N} \times \sum_{1\leq j_1,\dots,j_m\leq N} \mathbb{E}[a_{i_1,i_2}\cdots a_{i_m,i_1} a_{j_1,j_2}\cdots a_{j_m,j_1}], \tag{4.7}$$

$$\mathbb{E}[M_m(A_N)]^2 = \frac{1}{N^{m+2}} \sum_{1\leq i_1,\dots,i_m\leq N} \mathbb{E}[a_{i_1,i_2}\cdots a_{i_m,i_1}] \times \sum_{1\leq j_1,\dots,j_m\leq N} \mathbb{E}[a_{j_1,j_2}\cdots a_{j_m,j_1}]. \tag{4.8}$$

There are two possibilities of the contribution from the i configurations and the j configurations. If in an i configuration $a_{i_1,i_2}\cdots a_{i_m,i_1}$, any $a_{i_s i_{s+1}}$ is not equal to any $a_{j_t j_{t+1}}$ in a j configuration $a_{j_1,j_2}\cdots a_{j_m,j_1}$, then these two configurations contribute equally to $\mathbb{E}[M_m(A_N)^2]$ and $\mathbb{E}[M_m(A_N)]^2$. We now estimate the difference for the crossover cases, where we have at least one pair of equal entries $a_{i_s i_{s+1}} = a_{j_t j_{t+1}}$. We adopt the method of counting degrees of freedom in [HM] and show that the contribution from crossovers is $O_m(\frac{1}{N})$ to both $\mathbb{E}[M_m(A_N)^2]$ and $\mathbb{E}[M_m(A_N)]^2$. Basically, one crossover is associated with the loss of at least one degree of freedom, since $a_{i_s i_{s+1}} = a_{j_t j_{t+1}}$ imposes a diagonal condition and a modulo condition (see (3.1)) on the 4 indices involved (i_s, i_{s+1}, j_t, and j_{t+1}). In comparison with the analogous proof in [HM], all the steps follow trivially except the changes in the $O_k(\frac{1}{N})$ constants, which do not alter the result that the contribution from crossovers diminishes as $N \to \infty$.

4.2.2. *Almost Sure Convergence.* Again, the proof in [HM] that shows almost sure convergence for the ensemble of Toeplitz matrices by counting degrees of freedom can be readily applied here, and thus we only sketch the proof. To show almost sure convergence for the ensemble of m-circulant matrices, we show that $\forall m \in \mathbb{N}$,

$$X_{m;N}(A) \to X_m(A) = M_m \text{ almost surely,} \tag{4.9}$$

and then apply the Moment Convergence Theorem (1.3). The key step in proving this is showing that

$$\lim_{N\to\infty} \mathbb{E}[|M_m(A_N) - \mathbb{E}[M_m(A_N)]|^4] = O_m(\frac{1}{N^2}). \tag{4.10}$$

The proof is completed in three steps. Recall the triangle inequality (4.4), as $\lim_{N\to\infty} |M_m(N) - M_m| = 0$, we only need to show the $M_m(A_N) - M_m(N) \to 0$ for almost all A_N.

By counting degrees of freedom, we can show that $|M_m(A_N) - M_m(N)|^4 = O_m(\frac{1}{N^2})$. We then cite Chebychev's inequality: for any random variable X with mean 0 and finite ℓ^{th} moment,

$$\text{Prob}(|X| \geq \epsilon) \leq \frac{\mathbb{E}[|X|^\ell]}{\epsilon^\ell}. \tag{4.11}$$

For our proof, we let $\ell = 4$ and obtain

$$\mathbb{P}_{\mathbb{N}}(|X_{m;N}(A) - X_m(A)| \geq \epsilon) \leq \frac{\mathbb{E}[|M_m(A_N) - M_m(N)|^4]}{\epsilon^4} \leq \frac{C_m}{N^2\epsilon^4}, \tag{4.12}$$

where C_m is some constant. Fix a large c, write $B_N^{(c,m)} = \{A \in T_{\mathbb{N}} : |M_m(A_N) - M_m(N)| \geq c\}$, and we see that $\text{Prob}(B_N^{(c,m)}) \leq \frac{C_m c^4}{N^2}$. Applying the Borel-Cantelli lemma,[14] and letting $c \to \infty$, we find that for any fixed m, as $N \to \infty$, $M_m(A_N) \to M_m$ with probability 1, completing the proof of almost sure convergence.

5. Future Research

Based on the generalized m-circulant pattern that we have studied in this paper, we propose several tentative directions for future research.

[14] Let B_i be a sequence of events with $\sum_i \text{Prob}(B_i) < \infty$, and let $B = \left\{\omega : \omega \in \bigcap_{j=1}^\infty \bigcup_{i=j}^\infty B_i\right\}$, then $\text{Prob}(B) = 0$.

5.1. **Deviation from the Base Pattern.** Clearly, we may view a generalized m-circulant pattern as a deviation from an m-circulant pattern where each element is distinct and which we refer to as the associated "base pattern". For example, $\{a, a, b, b\}$ deviates from its base pattern $\{a, b, c, d\}$ in that, in the computation of high moments, $\{a, a, b, b\}$ has all the non-trivial configurations for $\{a, b, c, d\}$, and gains additional non-trivial configurations due to the repeated elements. Given that [KKM] has derived an explicit eigenvalue density for any all-distinct m-circulant pattern by computing all the associated high moments, we are tempted to search for an explicit density for an arbitrary generalized m-circulant pattern. To this end, it may be helpful to quantify the deviation of a generalized pattern from its base pattern so that the [KKM] method of computing high moments can be adjusted with regard to this deviation.

Although it is tricky to quantify this deviation for general m-circulant patterns, we may start from three basic properties of a pattern: the number of free elements, the number of occurrences of each free element, and the location of each element. Table 1 suggests that any of these three properties affects the high moment values and thus the limiting spectral measure for an m-circulant pattern. In general, the smaller the number of free elements, the more likely two matrix entries are matched to be part of a non-trivial configuration, and thus the larger the high moment values. For example, the high moment values for $\{a, a, a, b, b, b\}$ (2 free elements) are much greater than those for $\{a, b, c, a, b, c\}$ (3 free elements). Comparing the high moment values for patterns in Table 1.2, where there are 2 free elements in every pattern, we clearly see that the number of occurrences of each element matters for the moment values. Table 1.3 provides empirical evidence that the high moments depend on the location of each element: in every pattern we have 3 free elements, each of which appears exactly twice, but the high moment values differ significantly across the patterns. This suggests that the way we quantify the deviation of a generalized m-circulant pattern from its base pattern should depend on the three basic properties of a pattern.

TABLE 1. **Numerical Moment Values for Several m-Circulant Patterns**

1.1: {a,b} and its variations.

Pattern k-th moment	{abab} (using pdf.)	{abab}	{aabb}	{abba}	Standard Gaussian (using pdf.)
k = 1	0.0000	0.0010	-0.0006	0.0015	0
2	1.0000	1.0016	1.0014	0.9972	1
3	0.0000	0.0010	0.0000	0.0056	0
4	2.2500	2.2583	2.2541	2.2405	3
5	0.0000	-0.0205	0.0098	0.0239	0
6	7.5000	7.5577	7.3212	7.2938	15
7	0.0000	-0.2779	0.0940	0.1249	0
8	32.8125	33.2506	30.4822	30.5631	105
9	0.0000	-2.9417	0.8285	0.8584	0
10	177.1880	180.8270	153.9530	155.6930	945

1.2: more variations of {a,b}.

Pattern k-th moment	{ababab} (using pdf.)	{aaabbb}	{aaaabbbb}	{aaaaabbbbb}	{aababb}
k=2	1.0000	1.0008	1.0001	0.9984	0.9996
4	2.2500	2.2541	2.2441	2.2449	2.2502
6	7.5000	7.3011	7.2098	7.2551	7.2319
8	32.8125	30.3744	29.5004	30.0127	29.5378
10	177.1880	155.0380	145.8240	150.7220	145.4910

1.3: {a,b,c} and its variations.

Pattern k-th moment	{abcabc} (using pdf.)	{abccba}	{aabbcc}	{abbcca}	{aabcbc}
k=2	1.0000	1.0005	1.0006	0.9983	1.0013
4	2.1111	2.1122	2.1153	2.1047	2.1161
6	6.1111	6.0248	6.0540	6.0083	6.0235
8	22.0370	20.9398	21.2004	20.9908	20.8411
10	94.6296	85.0241	87.0857	85.9902	84.2097

Notes: 1. Matrix size 4000 (Table 1.1) or 3600 (Tables 1.2 and 1.3), 200 simulations for each pattern.
2. All moment values are computed numerically unless noted "using pdf.", which shows that the moment value is calculated using an explicit probability density.
3. For simplicity, odd moments, which vanish in the limit, are not computed numerically in Table 1.2 and 1.3.

REFERENCES

[BBDS] J. Baik, A. Borodin, P. Deift, and T. Suidan, *A Model for the Bus System in Cuernevaca (Mexico)*, Math. Phys. 2005, 1-9. Online available at http://arxiv.org/abs/math/0510414.

[BasBo1] A. Basak and A. Bose, *Limiting spectral distribution of some band matrices*, to appear in Periodica Mathematica Hungarica.

[BasBo2] A. Basak and A. Bose, *Balanced random Toeplitz and Hankel matrices*, Electronic Communications in Probability, 15, 134-148.

[BanBo] S. Banerjee and A. Bose, *Noncrossing partitions, Catalan words and the Semicircle Law*, Technical Report R7/2010, `www.isical.ac.in/~statmath` (14 pages), submitted for publication.

[BCG] A. Bose, S. Chatterjee, and S. Gangopadhyay, *Limiting spectral distributions of large dimensional random matrices*, J. Indian Statist. Assoc. (2003), **41**, 221–259.

[BM] A. Bose and J. Mitra, *Limiting spectral distribution of a special circulant*, Statist. Probab. Lett. **60** (2002), no. 1, 111–120.

[BDJ] W. Bryc, A. Dembo, and T. Jiang, *Spectral Measure of Large Random Hankel, Markov, and Toeplitz Matrices*, Annals of Probability **34** (2006), no. 1, 1–38.

[Dy1] F. Dyson, *Statistical theory of the energy levels of complex systems: I, II, III*, J. Mathematical Phys. **3** (1962) 140–156, 157–165, 166–175.

[Dy2] F. Dyson, *The threefold way. Algebraic structure of symmetry groups and ensembles in quantum mechanics*, J. Mathematical Phys., **3** (1962) 1199–1215.

[FM] F. W. K. Firk and S. J. Miller, *Nuclei, Primes and the Random Matrix Connection*, Symmetry **1** (2009), 64–105; doi:10.3390/sym1010064.

[Fo] P. J. Forrester, *Log-Gases and Random Matrices*, London Mathematical Society Monographs **34**, Princeton University Press, Princeton, NJ 2010.
Series, and Products, Academic Press, New York, 1965.

[HM] C. Hammond and S. J. Miller, *Eigenvalue spacing distribution for the ensemble of real symmetric Toeplitz matrices*, Journal of Theoretical Probability **18** (2005), no. 3, 537–566.

[HarZa] J. Harer and D. Zagier, *The Euler characteristic of the moduli space of curves*, Invent. Math. **85** (1986), 457–485.

[Hat] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[Hay] B. Hayes, *The spectrum of Riemannium*, American Scientist **91** (2003), no. 4, 296–300.

[JMP] S. Jackson, S. J. Miller, and V. Pham, *Distribution of Eigenvalues of Highly Palindromic Toeplitz Matrices*, `http://arxiv.org/abs/1003.2010`.

[Kar] V. Kargin, *Spectrum of random Toeplitz matrices with band structure*, Elect. Comm. in Probab. **14** (2009), 412–421.

[KKM] M. Kologlu, G. S. Kopp, and S. J. Miller, *The Limiting Spectral Measure for the Ensemble of Symmetric Period m-Circulant Matrices*, `http://arxiv.org/abs/1008.4812`.

[KrSe] M. Krbalek and P. Seba, *The statistical properties of the city transport in Cuernavaca (Mexico) and Random matrix ensembles*, J. Phys. A: Math. Gen 2000, 33, L229L234.

[MMS] A. Massey, S. J. Miller, J. Sinsheimer, *Distribution of eigenvalues of real symmetric palindromic Toeplitz matrices and circulant matrices*, Journal of Theoretical Probability **20** (2007), no. 3, 637-662.

[MNS] S. J. Miller, T. Novikoff and, A. Sabelli, *The distribution of the second largest eigenvalue in families of random regular graphs*, Experimental Mathematics **17** (2008), no. 2, 231–244.

[McK] B. McKay, *The expected eigenvalue distribution of a large regular graph*, Linear Algebra Appl. **40** (1981), 203–216.

[Mon] H. Montgomery, *The pair correlation of zeros of the zeta function*. Pages 181–193 in *Analytic Number Theory*, Proceedings of Symposia in Pure Mathematics, vol. 24, AMS, Providence, RI, 1973.

[Ta] L. Takacs, *A Moment Convergence Theorem*, The American Mathematical Monthly **98** (Oct., 1991), no. 8, 742–746.

[Wig1] E. Wigner, *On the statistical distribution of the widths and spacings of nuclear resonance levels*, Proc. Cambridge Philo. Soc. **47** (1951), 790–798.

[Wig2] E. Wigner, *Characteristic vectors of bordered matrices with infinite dimensions*, Ann. of Math. **2** (1955), no. 62, 548–564.

[Wig3] E. Wigner, *Statistical Properties of real symmetric matrices*. Pages 174–184 in *Canadian Mathematical Congress Proceedings*, University of Toronto Press, Toronto, 1957.

[Wig4] E. Wigner, *Characteristic vectors of bordered matrices with infinite dimensions. II*, Ann. of Math. Ser. 2 **65** (1957), 203–207.

[Wig5] E. Wigner, *On the distribution of the roots of certain symmetric matrices*, Ann. of Math. Ser. 2 **67** (1958), 325–327.

[Wig6] E. Wigner, *Random Matrices in Physics*, SIAM Review, Vol. 9, No. 1 (Jan., 1967), pp. 1-23.

[Wis] J. Wishart, *The generalized product moment distribution in samples from a normal multivariate population*, Biometrika **20 A** (1928), 32–52.

[Zv] A. Zvonkin, *Matrix integrals and map enumeration: An accessible introduction*, Mathematical and Computer Modelling **26** (1997), no. 8-10, 281–304.