Zeckendorf Decompositions Group
AIM 2014 REUF

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Fibonacci sequence and Zeckendorf's Theorem

Creating a sequence with bin size 1 and you can't decompose a number using more than one addend from adjacent bins.

[1] [2] [3] [5] [8] [13] [21] [34] [55] [89] [144] [233] [377] . . .

Example:

$$48 = 34 + 13 + 1$$

The 2 kid family from Kentucky sequence

Creating a sequence with bin size 2 and using the Kentucky Rule (i.e. you can't decompose a number using more than one summand from the same bin or addends from adjacent bins).

$$[1, 2]\ [3, 4]\ [5, 8]\ [11, 16]\ [21, 32]\ [43, 64]\ [85, 128]\ [171, 256] \dots$$

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Early patterns we noticed:

- $a_{2n} = a_{2n-1} + a_{2n-3} = 2^n$
- $a_{2n+1} = a_{2n} + a_{2n-3} = a_{2n-1} + 2a_{2n-3}$
- Looking at the OEIS we found this sequence of numbers counts the number of ways to tile a 3 by $(n-1)$ rectangle using only monominoes and dominoes.

Single Recurrence Relation and Uniqueness of Decomposition

$$[1, 2]\ [3, 4]\ [5, 8]\ [11, 16]\ [21, 32]\ [43, 64]\ [85, 128]\ [171, 256] \dots$$

$$a_m = a_{m-2} + 2a_{m-4}$$

Every whole number has a unique decomposition where the summands come from the 2 kid family Kentucky sequence.

Counting decompositions with k summands

- Let $p_{n,k}$ = the number of integers that can be decomposed with k summands in the interval $[0, a_{2n+1})$.
- Proved $p_{n,k} = \binom{2}{1}^k \binom{n-k+1}{k}$ using the recursive rule $p_{n,k} = 2p_{n-2,k-1} + p_{n-1,k}$ and induction.
- In fact we noticed that:

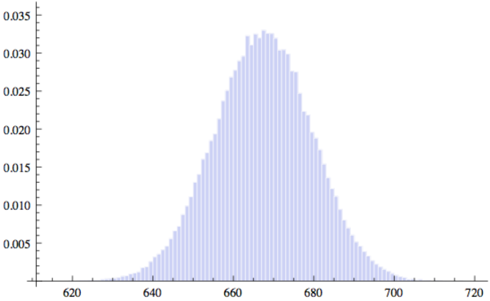
$$\sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} p_{n,k} = a_{2n+1}$$

The Distribution of the Number of Summands is Gaussian

- Numerical Evidence:
  Take a sample of 100,000 integers from $[0, 10^{600})$.
  Consider the number of summands.

  mean $=$ 666.838 (prediction $=$ 666.889)

  sd $=$ 12.156 (predicted $=$ 12.176)



Gaussian Distribution "Proof" Using Fibonacci Polynomials

- (Generating function) $F(x,y) = \sum_{n,k\geq 0} p_{n,k}x^n y^k$.

  We can compute $F(x,y)$ using *Fibonacci polynomials*:
  - $F_0(t) = 0, F_1(t) = 1, F_2(t) = t$, and for $n \geq 3$: $F_n(t) = tF_{n-1}(t) + F_{n-2}(t)$.
  - Known about $F_n(t)$:
    - $F_n(t) = \sum_{j=0}^{n} \binom{n-j-1}{j} t^{n-2j-1}$
    - $\sum_{n=0}^{\infty} F_n(t)w^n = \frac{w}{1-w^2-wt}$.

- Formula for $F(x,y)$ in terms of Fibonacci polynomials:

$$F(x,y) = \sum_{n=0}^{\infty} F_{n+2}\left(\tfrac{1}{\sqrt{2y}}\right)(\sqrt{2y})^{n+1}x^n$$

- For $n \geq 3$: $g_n(y) = \sum_{k=0}^{n} p_{n,k} y^k = F_{n+2}(\frac{1}{\sqrt{2y}})(\sqrt{2y})^{n+1}$.
- Let Y_n = random variable denoting the number of summands in the unique bin decomposition of an integer chosen uniformly from $[0, a_{2n+1})$. The **mean** of Y_n is

$$\mu_n = \sum_{i=0}^{n} iP(Y_n = i) = \sum_{i=0}^{n} i \frac{p_{n,i}}{\sum_{k=0}^{n} p_{n,k}} = \frac{g_n'(1)}{g_n(1)}.$$

- For $n \geq 3$: $g_n(y) = \sum_{k=0}^{n} p_{n,k} y^k = F_{n+2}(\frac{1}{\sqrt{2y}})(\sqrt{2y})^{n+1}$.
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The **variance** of Y_n is

$$\begin{aligned}\sigma_n^2 = \sum_{i=0}^{\infty}(i-\mu_n)^2 P(Y_n = i) &= \sum_{i=0}^{n} i^2 \frac{p_{n,i}}{\sum_{k=0}^{n} p_{n,k}} - \mu_n^2 \\ &= \frac{\frac{d}{dy}(y g_n'(y)|_{y=1}}{g_n(1)} - \mu_n^2.\end{aligned}$$

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$$= \frac{\frac{d}{dy}(yg_n'(y)|_{y=1}}{g_n(1)} - \mu_n^2.$$

- To show Gaussian behavior we look at $g_n(e^{r_n})(r_n = t/\sigma_n)$ for some fixed value of t and see what happens as $n \to \infty$.

The Distribution of the Number of Summands is Gaussian

- A more general approach which generalizes to other scenarios.

  Let

$$F(x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{n^*} p_{n,k} x^n y^k.$$

  Use recurrence relation $p_{n,k} = 2p_{n-2,k-1} + p_{n-1,k}$,

$$F(x, y) = \frac{1 + 2xy}{1 - x - 2x^2 y}.$$

The Distribution of the Number of Summands is Gaussian

We'll need the coefficient of x^n: $g_n(y) = \sum_{k \geq 0} p_{n,k} y^k$.

Use partial fractions!

$$g_n(y) = \frac{1}{2^{n+1}\sqrt{1+8y}} \Big[4y(1+\sqrt{1+8y})^n - 4y(1-\sqrt{1+8y})^n + (1+\sqrt{1+8y})^{n+1} - (1-\sqrt{1+8y})^{n+1}\Big]$$

After this week (related to this work)

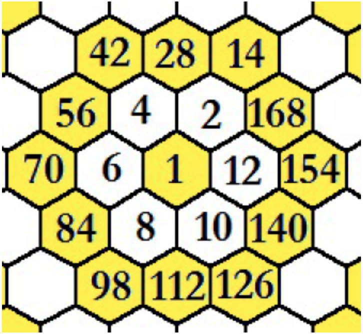
- Finish up the Kentucky sequence
  - Proof by Stirling's formula (undergraduate research)
  - Continue on the work with gap sizes in decompositions
- Investigate other similar sequences like the Tennessee sequence
- Generacci Sequences: vary the bin sizes and the restrict the selections of summands from a span of adjacent bins.

2-D analog

Problem: Describe a process by which one could tile the plane such that every positive integer has a unique decomposition as a sum of "non-adjacent" terms in a sequence.

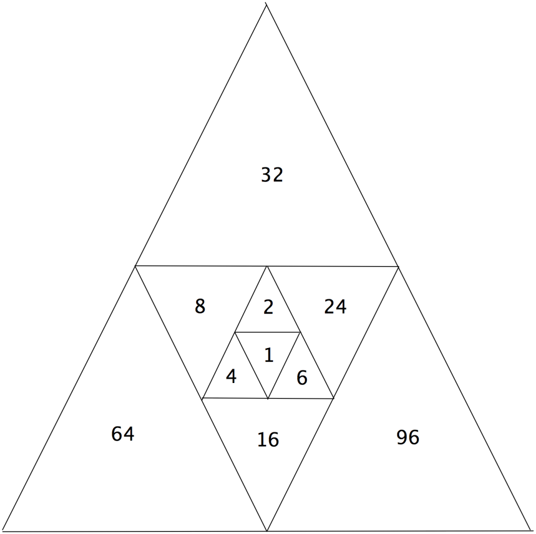
We answered this problem in a countable number of ways!

Regular Tilings



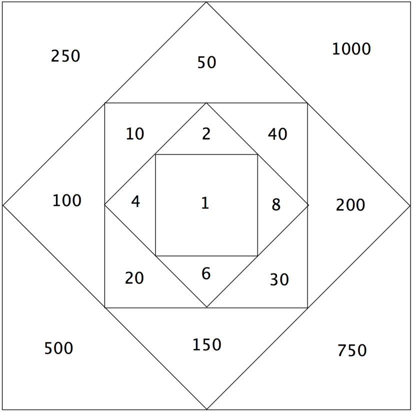
1, 2, 4, 6, 8, 10, 12, 14, 28, 42, 56, 70, 84, 98, 112, 126, 140, 154, 168, . . .

Circumscribing Triangles



1, 2, 4, 6, 8, 16, 24, 32, 64, 96, ...

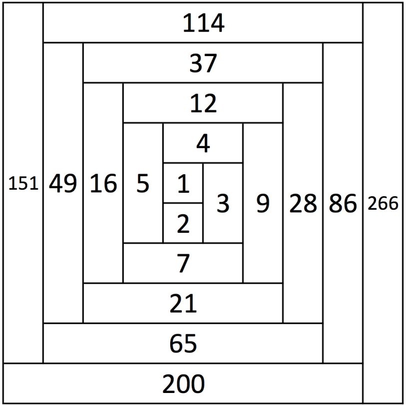
Circumscribing Squares



1, 2, 4, 6, 8, 10, 20, 30, 40, 50, 100, 150, 200, 250, 500, 750, 1000, ...

By circumscribing regular n-gons to cover the plane we arrive at a unique decompositions of each positive integer.

Fibonacci Quilt



- We noticed that for n large, $a_{n+1} = a_{n-1} + a_{n-2}$
- This is a recurrence relation where the first coefficient is zero.
- We do not have unique decomposition of all whole numbers.

Counting the number of decompositions

$d(m)$ = the number of FQ legal decompositions.

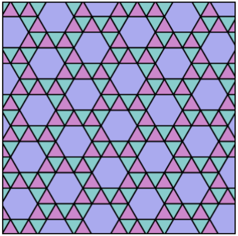
We proved that as m increases, $\max d(m) \to \infty$

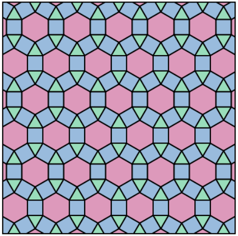
We still want to answer, for $m \in [0, a_n)$, what is the distribution of $d(m)$?

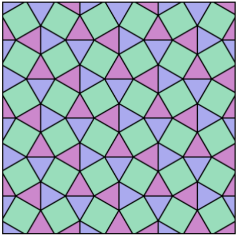
We can also ask questions about the different decompositions and the gaps.

Many avenues for undergraduate research!

Consider other tilings of the plane and develop a procedure under which every positive integer has a unique decomposition.



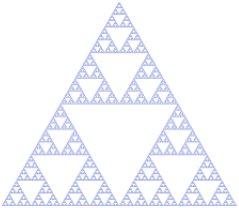




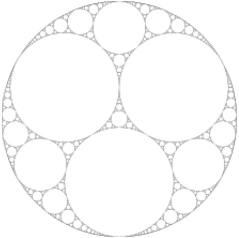
How about other 2-D objects?

Consider the same question regarding unique decomposition using:

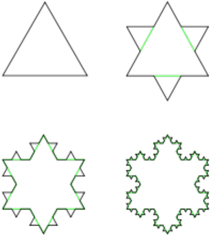
Sierpinski Triangle



Apollonian Gasket



Koch Snowflake



But that is not all...

Once procedures are developed which give rise to (unique) decompositions of positive integers we could ask some of the following questions:

- What can we say about the sequence of numbers we created? Namely those used to create the summands.
- What can we say about the distribution of the number of summands need to represent a positive integer? Is it Gaussian?
- What about gaps between summands?

One natural extension of all of these problems is to now consider 3-dimensional analog using polyhedra to pack 3-space!

Ideas?

Acknowledgements

Steve for taking time to show us some super cool math, sharing professional advice, and pushing us to work hard this week!

Huge thanks to AIM for bringing us together and providing the supportive environment to make this week and our future collaborations possible.