AMS Expects that every Mathematician will do their ODEs: From the Battle of Trafalgar to Calculus (or Nelson to Newton)

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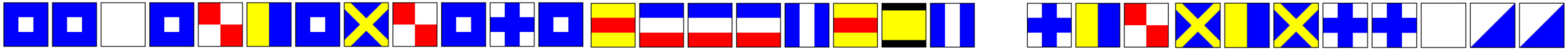
AMS Special Session on Modeling to Motivate the Teaching of Mathematics of Differential Equations
JMM, San Francisco, January 3, 2024



Goals of the talk:

- *Discuss opportunities to introduce students to mathematical modeling, especially with differential equations.*
- *Motivate the mathematics students see with engaging material.*
- *Highlight the importance of mathematical software and visual presentation.*

Classes used (other than differential equations!):

- Calculus II, Calculus III (haven’t done Calculus I in 15+ years).
- Probability
- Operations Research

Simpler related problems done in other classes (discrete predator – prey for example in Mathematic of the Pandemic, a 100 and 300 level class).

Application: Battle of Trafalgar

Modified from *Mathematics of Warfare by F. W. Lanchester*

Pre-requisites:

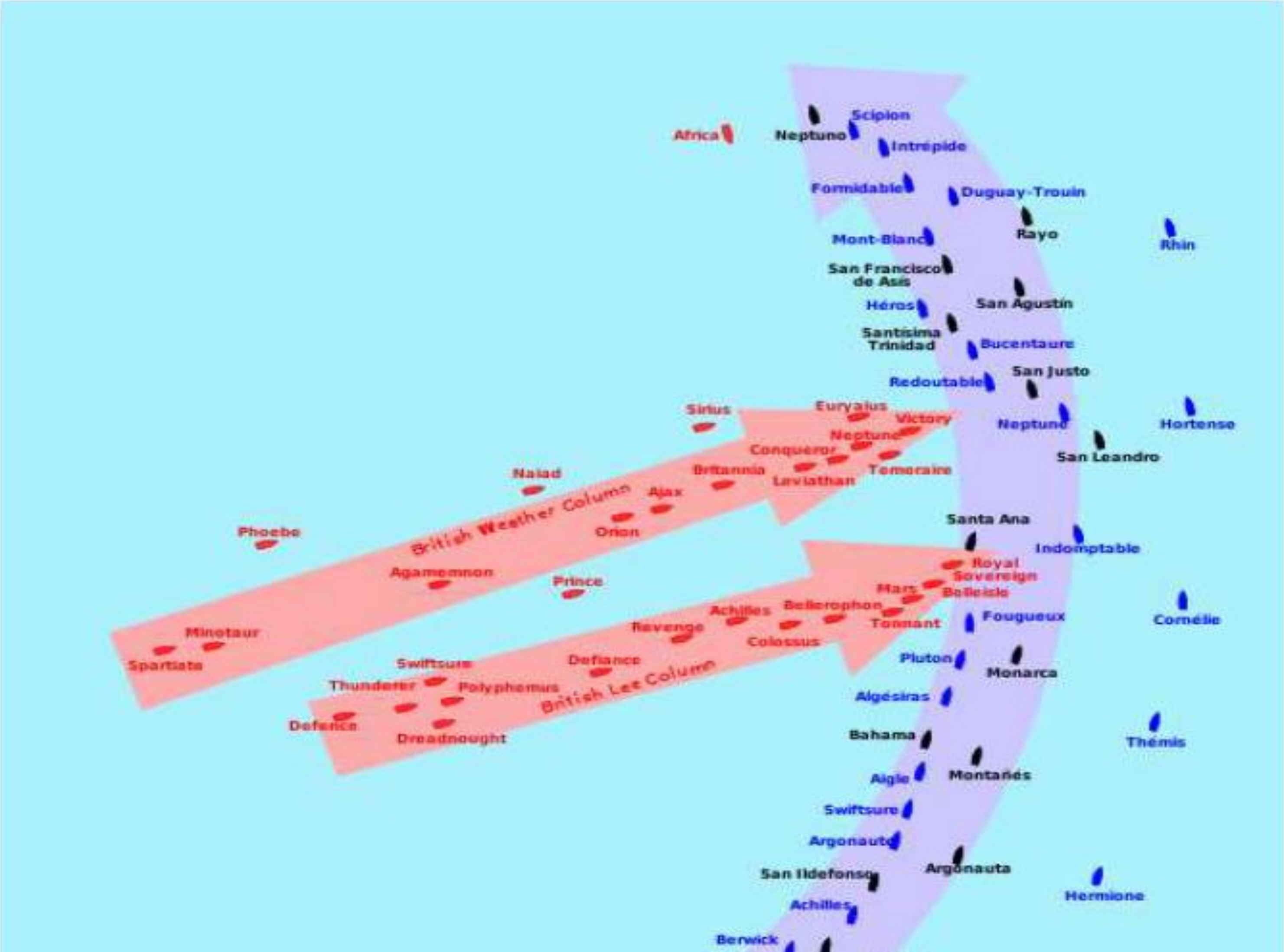
- ***Exponential function and its derivatives.***
- ***Often do discrete systems first (predator – prey).***
- ***Do not assume any historical knowledge!***

Battle of Trafalgar

Wikipedia: “The battle was the most decisive naval victory of the war. Twenty-seven British ships of the line led by Admiral Lord Nelson aboard HMS Victory defeated thirty-three French and Spanish ships of the line under French Admiral Pierre-Charles Villeneuve off the southwest coast of Spain, just west of Cape Trafalgar, in Caños de Meca.

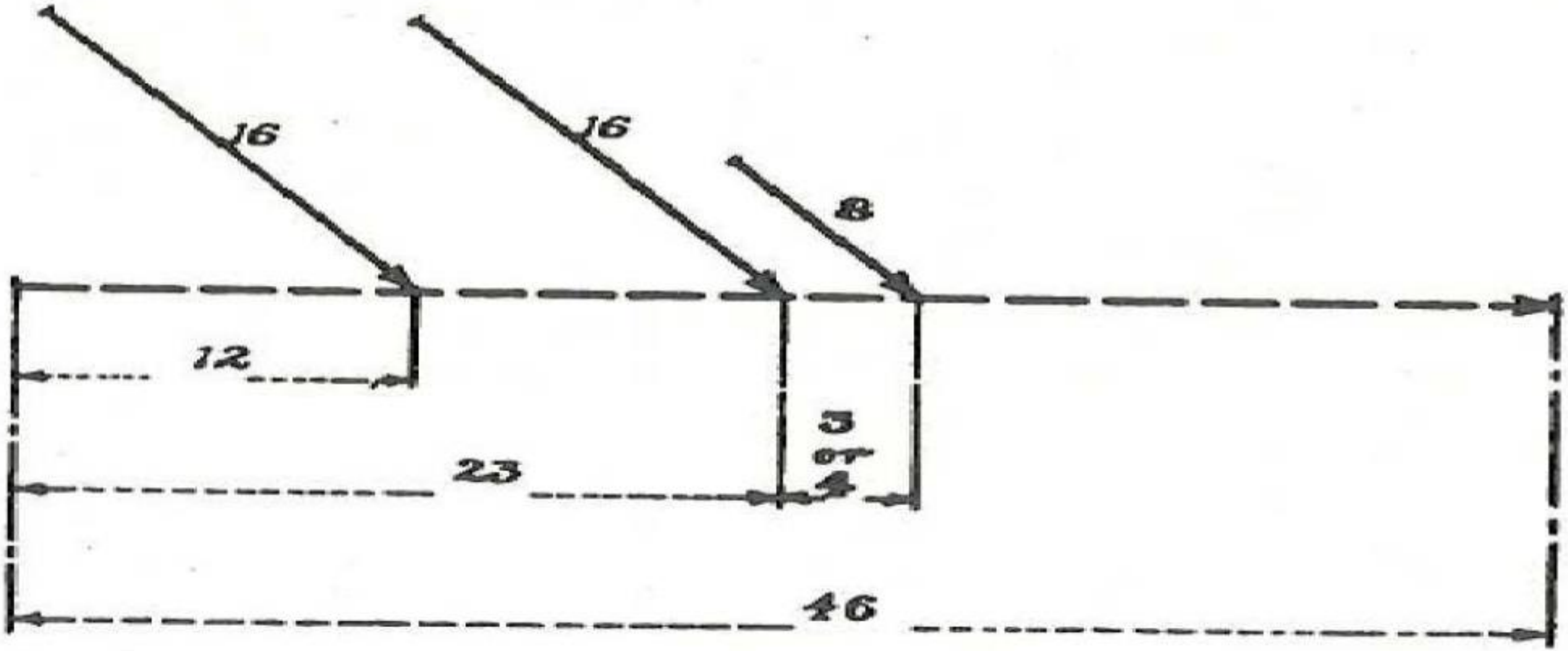
Trafalgar

Nelson outnumbered – how could he win?



Analysis of Nelson's Plan: I

Nelson assumed for the purpose of framing his plan of attack that his own force would consist of forty sail of the line, against forty-six of the com-



Analysis of Nelson's Plan: II

If for the purpose of comparison we suppose the total forces had engaged under the conditions described by Villeneuve as "the usage of former days," we have:—

Strength of combined fleet,		46^2	....	$= 2116$
"	British	" 40^2	....	$= 1600$
Balance in favour of enemy			....	516

Analysis of Nelson's Plan: III

Dealing with the position arithmetically, we have:—

Strength of British (in arbitrary n^2 units),

$$32^2 + 8^2 = 1088$$

And combined fleet,

$$23^2 + 23^2 = 1058$$

British advantage 30

Guidelines for Modeling:

- Model should capture key features of the system;
- Model should be mathematically tractable (solvable).

The Square Law: I

Forces $r(t)$ and $b(t)$, effective fighting values N and M:

$$\begin{aligned} b'(t) &= -Nr(t) \\ r'(t) &= -Mb(t). \end{aligned}$$

Most important step – setting up the model!

Discuss why this is reasonable.

Discuss why tractable.

Ask about special case: Easy pairs (N, M)?

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If take derivatives again find

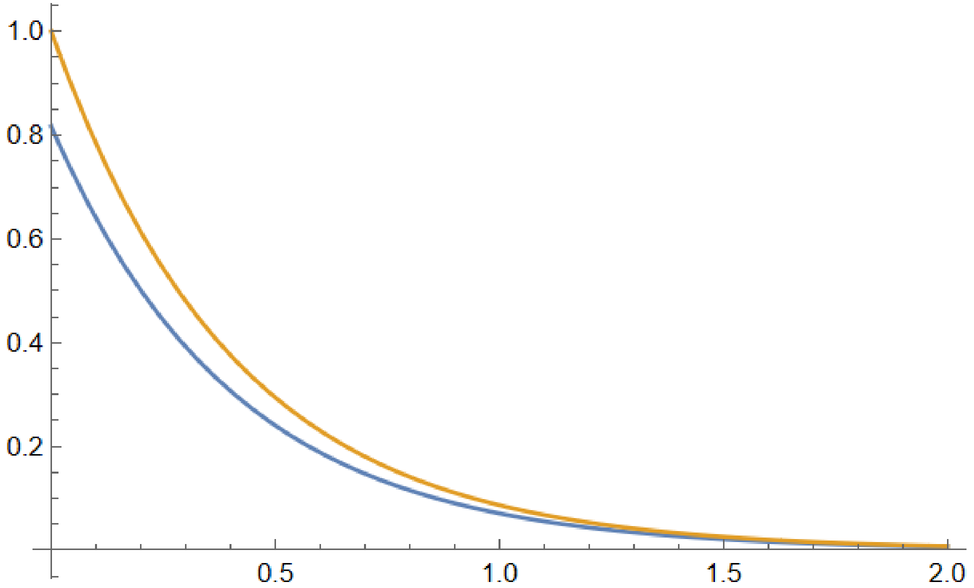
$$b''(t) = -Nr'(t) = NMb(t), \quad \text{yields}$$

$$b(t) = \beta_1 e^{\sqrt{NM}t} + \beta_2 e^{-\sqrt{NM}t}, \quad r(t) = \alpha_1 e^{\sqrt{NM}t} + \alpha_2 e^{-\sqrt{NM}t}.$$

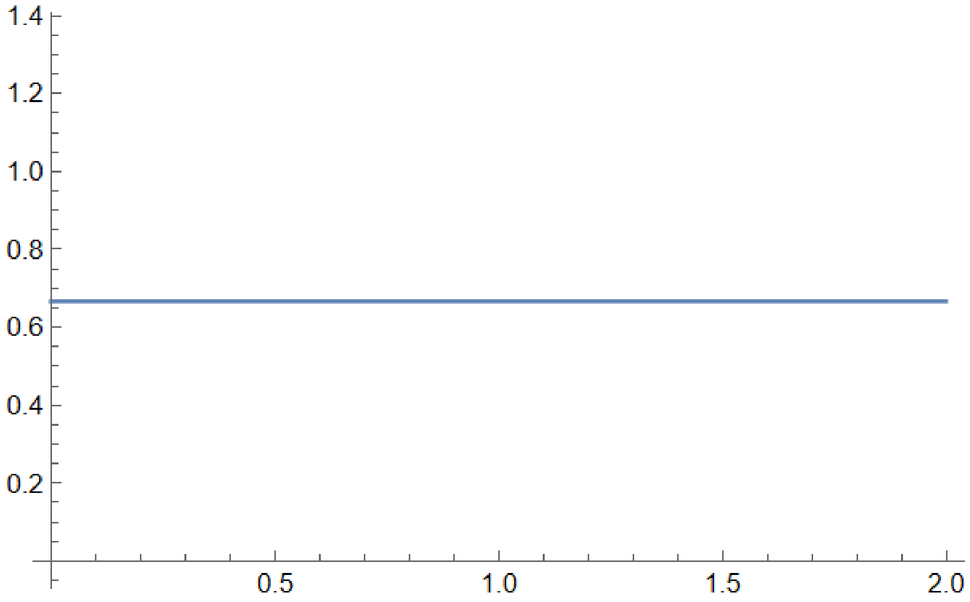
$b''(t)/b(t) = r''(t)/r(t)$ yields $Nr(t)^2 = Mb(t)^2$ (square law).

```
DSolve[{b'[t] == -2 r[t], r'[t] == -3 b[t], b[0] == Sqrt[2/3],
  r[0] == 1}, {b[t], r[t]}, t]
```

$$\left\{\left\{b[t] \to \sqrt{\frac{2}{3}}\, e^{-\sqrt{6}\, t},\; r[t] \to e^{-\sqrt{6}\, t}\right\}\right\}$$



Plot of r(t) and b(t)



$b(t)^2 / r(t)^2$

The n – square law in warfare

Modified from Lanchester's article

The differential equations are db / dt = –r and dr / dt = –b.

The solutions to the system is easily determined as the equations are `separable',
namely we find d^2 b / dt^2 = b and d^2 r / dr^2 = r.

Following Lancaster,
we call the forces blue and red (and not English and French, or ...). We
normalize so that there is initially one red unit,
and there are more blue units than red units.

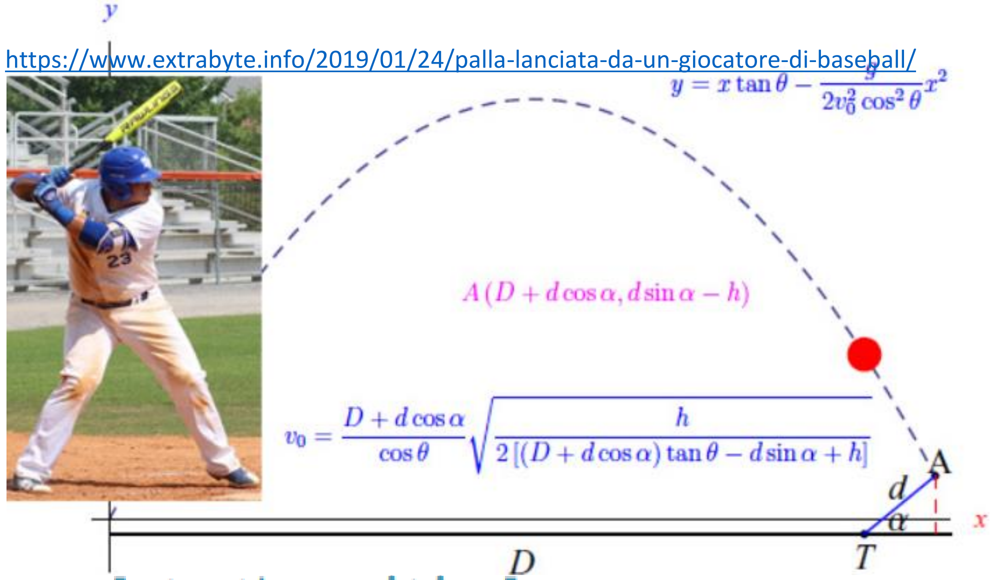
```
(* the lines below solve the equations and output the critical time where the red
 force is annihilated. *)
DSolve[{b'[t] == - r[t], r'[t] == - b[t], b[0] == B, r[0] == R}, {b, r}, t]
```

$$\left\{\left\{b \to \text{Function}\left[\{t\}, \frac{1}{2} e^{-t} \left(B + B e^{2t} + R - e^{2t} R\right)\right], r \to \text{Function}\left[\{t\}, -\frac{1}{2} e^{-t} \left(-B + B e^{2t} - R - e^{2t} R\right)\right]\right\}\right\}$$

```
Solve[-B + B e^(2 t) - R - e^(2 t) R == 0, t]
```

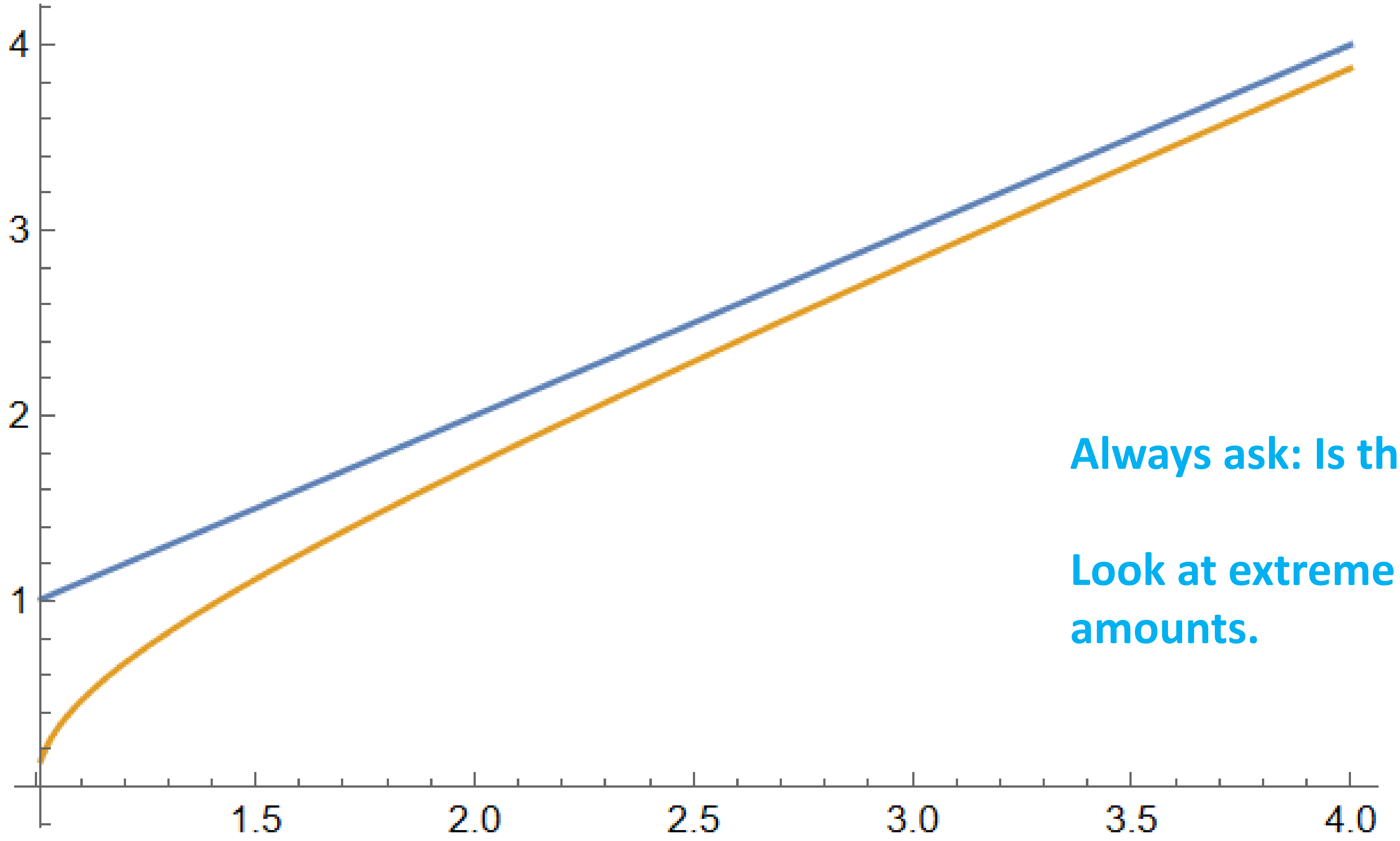
$$\left\{\left\{t \to \boxed{\frac{1}{2}\left(2\, i\, \pi\, c_1 + \text{Log}\left[\frac{B+R}{B-R}\right]\right) \text{ if } c_1 \in \mathbb{Z}}\right\}\right\}$$

https://www.extrabyte.info/2019/01/24/palla-lanciata-da-un-giocatore-di-baseball/




```
(* tcritical is where red is annihilated *)
tcritical[B_, R_] := Log[Sqrt[B + R]/Sqrt[B - R]];
(* adjusted time sets any time greater than the critical time equal to the critical
 time -- helps in plotting *)
adjustedtime[B_, R_, t_] := If[t ≤ tcritical[B, R], t, tcritical[B, R]];
(* btemp is how b evolves as long as there are units of r left. It is absurd to
 continue using this model after r is annhiliated,
as it would lead to negative red units creating blue units which create more negative
   blue units which.... *)
btemp[B_, R_, t_] := 1/2 e^-t (B + B e^(2 t) + R - e^(2 t) R);
b[B_, R_, t_] := btemp[B, R, adjustedtime[B, R, t]];
rtemp[B_, R_, t_] := -1/2 e^-t (-B + B e^(2 t) - R - e^(2 t) R);
r[B_, R_, t_] := rtemp[B, R, adjustedtime[B, R, t]];
```

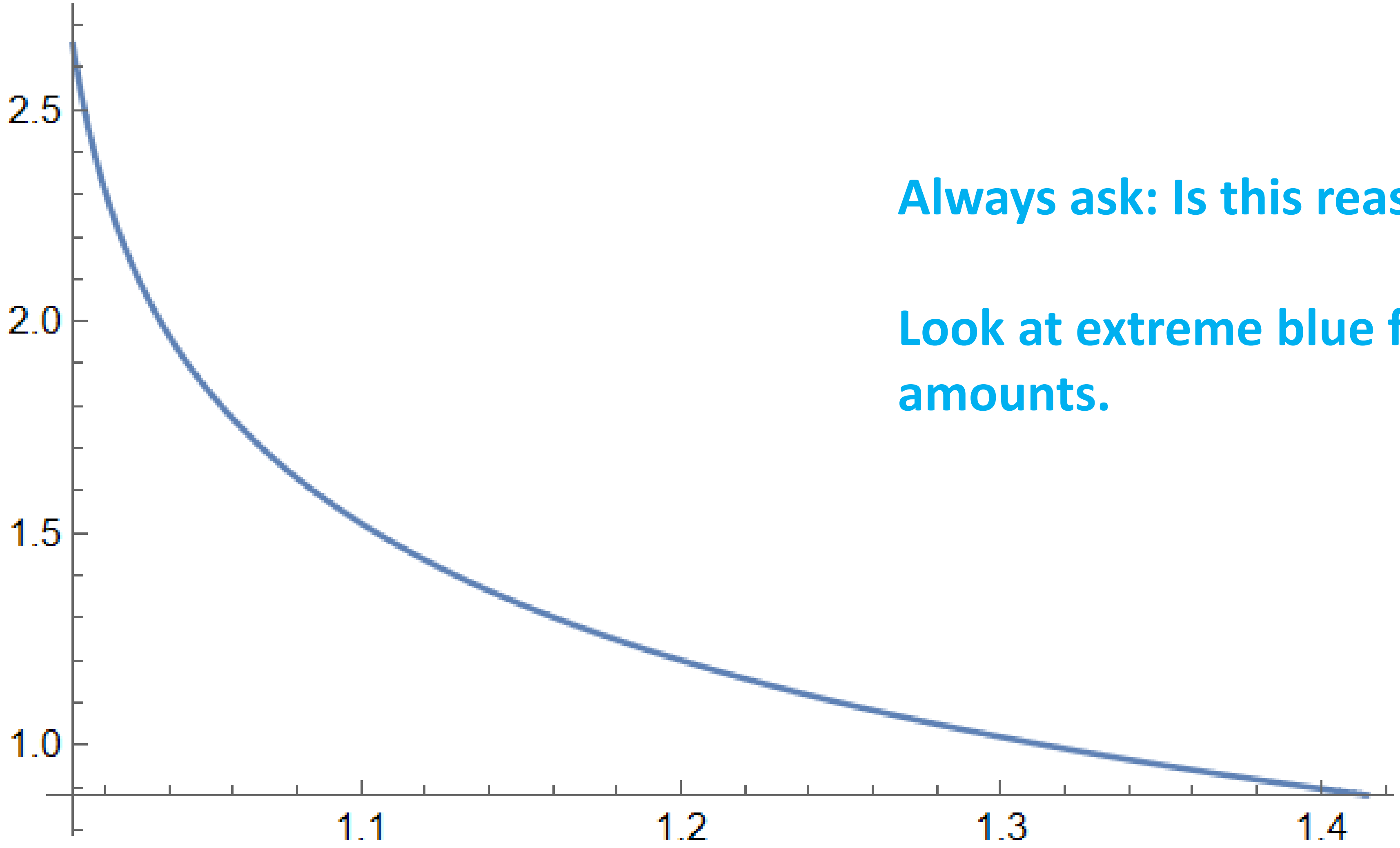
Below is a plot of how many blue survive
versus how many blue went to battle against a red force of 1 unit.



Always ask: Is this reasonable?

Look at extreme blue force amounts.

Below is a plot of how long it takes the blue force to wipe out the red force.



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Vertical axis is the size of the blue force, horizontal is the time to wipe out one red unit.

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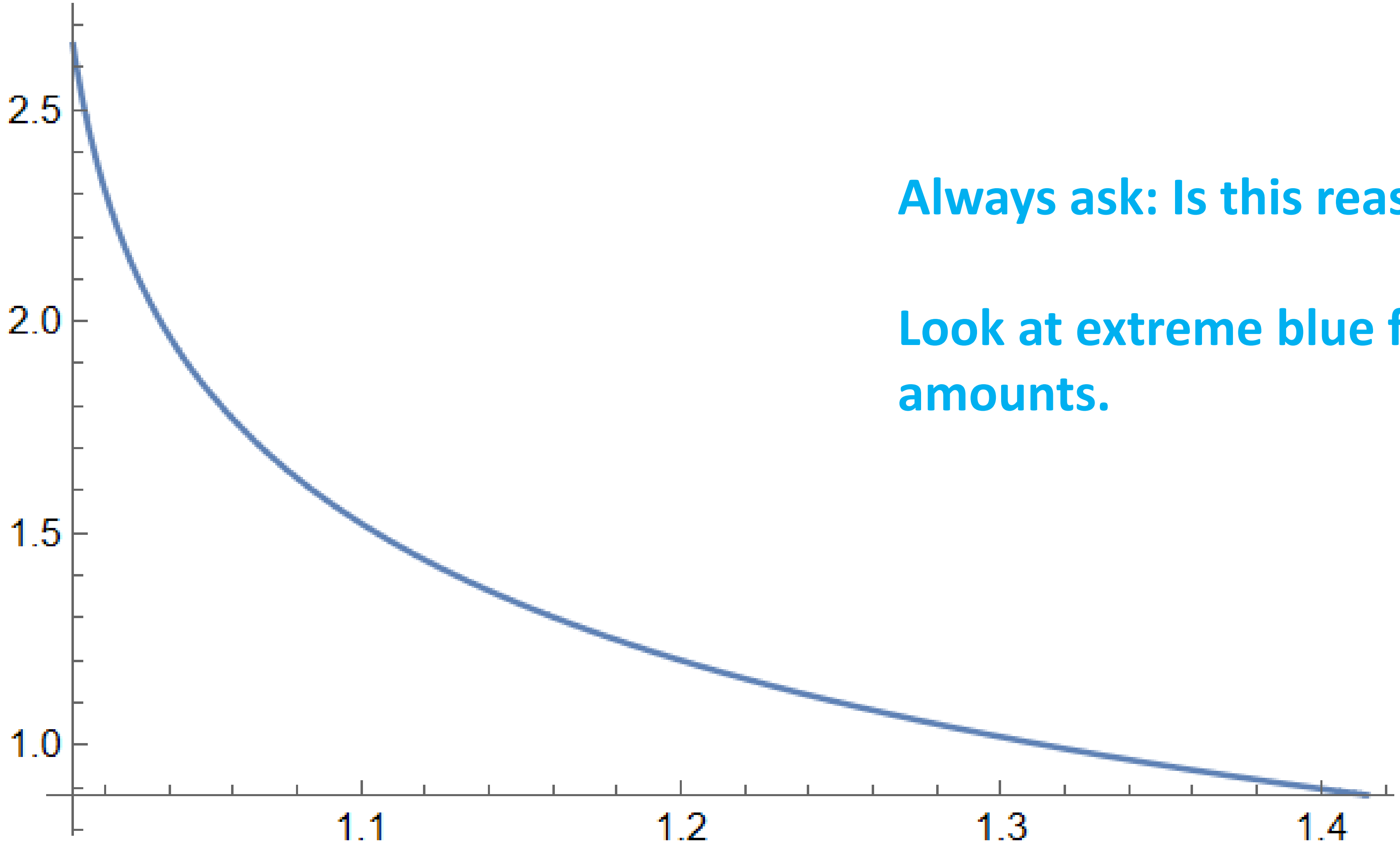

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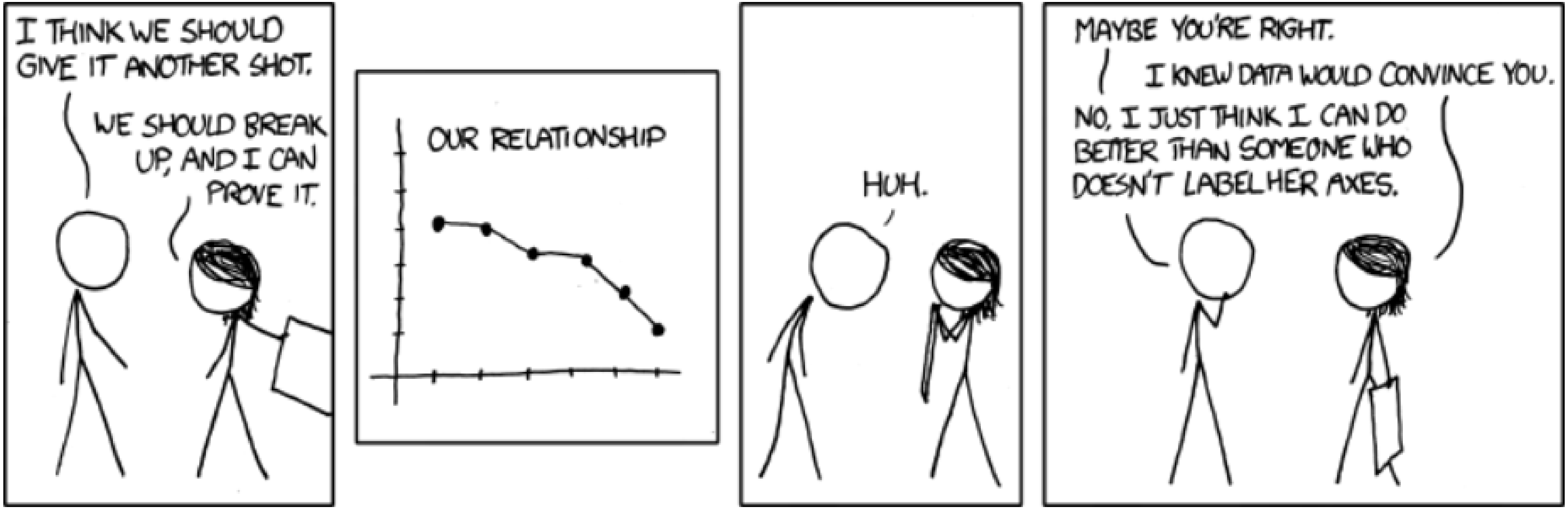
xkcd

A WEBCOMIC OF ROMANCE,
SARCASM, MATH, AND LANGUAGE.



And if you labeled your axes, I could tell you how much better. https://xkcd.com/833/

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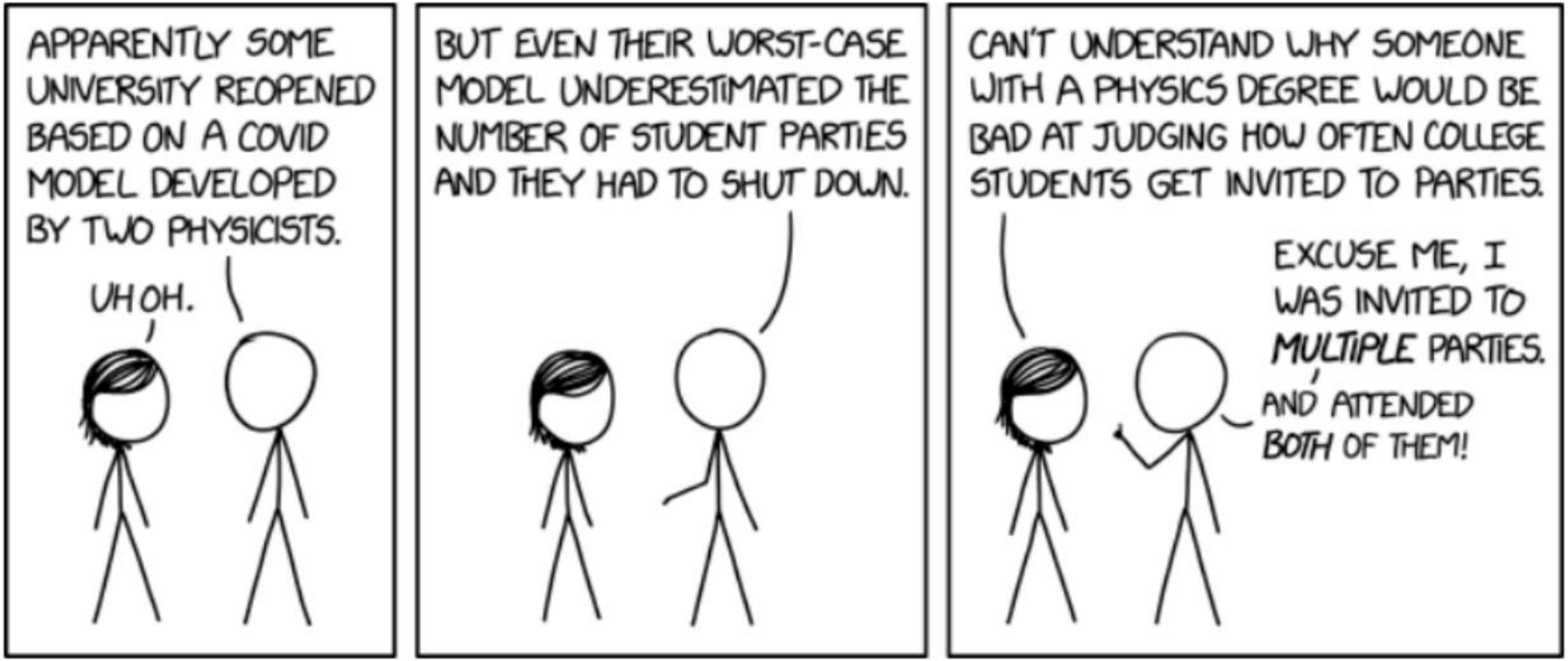

I admit this is an exaggeration, since I can think of at least three parties I was invited to while doing my degree, and I'm probably forgetting several more. https://xkcd.com/2355/

Additional Questions:
Engage students in extending….

Did special case $N = M$.

Have $b(t)^2 - r(t)^2 = \text{constant}$.

If a blue force of size B meets a superior red force of size 1, how should blue divide his forces and red's forces to maximize what he ends with?

Blue goes to βB and $(1-\beta)B$ to fight αR and $(1-\alpha) R$.

Assume $\beta B > \alpha R$.

The amount of blue remaining after the first battle is $(\beta^2 B^2 - \alpha^2 R^2)^{1/2}$. This force then meets the red force of size $((1 - \alpha)^2 R^2 - (1 - \beta)^2 B^2)^{1/2}$.

Assumes that the red and blue forces are split in two, and no matter which battle ends first, the winning group waits till the other battle finishes before joining their comrades at arms! Ridiculous assumption, better model is battles go simultaneously, and as soon as one finishes the survivors join the other battle.

Battle of Trafalgar

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AfterMATH of Battle of Trafalgar: English expectation

British: 0 of 27 ships, 1,666 dead or wounded.
Franco-Spanish: 22 of 33 ships, 13,781 captured, dead or wounded.

AfterMATH of Battle of Trafalgar: Issues & Remedies with Model

Biggest issue is deterministic.

Make fighting effectiveness random variables!

Leads to stochastic differential equations.

https://en.wikipedia.org/wiki/Stochastic_differential_equation

Have students come up with models for other systems…..

Thank you!

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