Benford's Law and the $3x + 1$ Problem, or: Why the IRS cares about Discrete Dynamical Systems

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Introduction

Interesting Question

For a nice data set, such as the Fibonacci numbers, stock prices, street addresses of audience, ..., what percent of the leading digits are 1?

Plausible answers:

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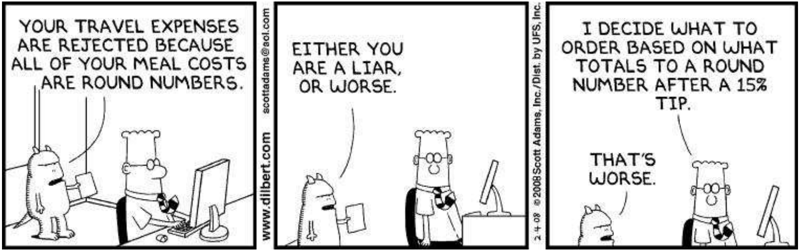
Plausible answers: 10%, 11%, about 30%.

Summary

- State Benford's Law.
- Discuss examples and applications.
- Sketch proofs.

Caveats!

- A math test indicating fraud is *not* proof of fraud: unlikely events, alternate reasons.



Benford's Law: Newcomb (1881), Benford (1938)

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For many data sets, probability of observing a first digit of d base B is $\log_B\left(\frac{d+1}{d}\right)$; base 10 about 30% are 1s.

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  - ◇ Long street $[1, L]$: $L = 199$ versus $L = 999$.
  - ◇ Oscillates between $1/9$ and $5/9$ with first digit 1.
  - ◇ Many streets of different sizes: close to Benford.

Examples

- recurrence relations
- special functions (such as $n!$)
- iterates of power, exponential, rational maps
- products of random variables
- L-functions, characteristic polynomials
- **iterates of the $3x + 1$ map**
- differences of order statistics
- hydrology and financial data
- many hierarchical Bayesian models

Applications

- analyzing round-off errors
- determining the optimal way to store numbers
- detecting tax and image fraud, and data integrity

General Theory

Significands

Significand: $x = S_{10}(x) \cdot 10^k$, k integer.

$S_{10}(x) = S_{10}(\widetilde{x})$ if and only if x and $\widetilde{x}$ have the same leading digits.

Key observation: $\log_{10}(x) = \log_{10}(\widetilde{x}) \bmod 1$ if and only if x and $\widetilde{x}$ have the same leading digits.

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Key observation: $\log_{10}(x) = \log_{10}(\widetilde{x})$ mod 1 if and only if x and $\widetilde{x}$ have the same leading digits.

◇ Thus often study $y = \log_{10} x$.

◇ Benefit: $e^{2\pi i (y \bmod 1)} = e^{2\pi i y}$.

Equidistribution and Benford's Law

Equidistribution

$\{y_n\}_{n=1}^{\infty}$ is equidistributed modulo 1 if probability $y_n \bmod 1 \in [a,b]$ tends to $b-a$:

$$\frac{\#\{n \le N : y_n \bmod 1 \in [a,b]\}}{N} \to b-a.$$

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- Thm: $\beta \notin \mathbb{Q}$, $n\beta$ is equidistributed mod 1.

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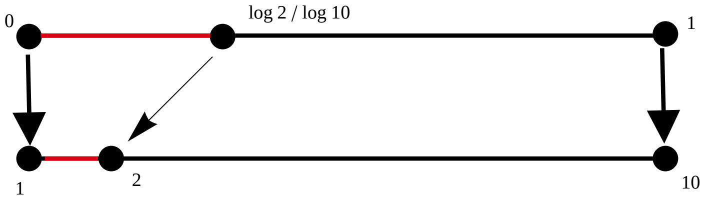
- Thm: $\beta \notin \mathbb{Q}$, $n\beta$ is equidistributed mod 1.
- Examples: $\log_{10} 2, \log_{10}\left(\frac{1+\sqrt{5}}{2}\right) \notin \mathbb{Q}$.

Logarithms and Benford's Law

Fundamental Equivalence

Data set $\{x_i\}$ is Benford base B if $\{y_i\}$ is equidistributed mod 1, where $y_i = \log_B x_i$.



Examples

- 2^n is Benford base 10 as $\log_{10} 2 \notin \mathbb{Q}$.

- Fibonacci numbers are Benford base 10.
  $a_{n+1} = a_n + a_{n-1}$.
  Binet: $a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$.

- Most linear recurrence relations Benford.
  $a_{n+1} = 2a_n$
  $a_{n+1} = 2a_n - a_{n-1}$
  Take $a_0 = a_1 = 1$ or $a_0 = 0$, $a_1 = 1$.

Digits of 2^n

First 60 values of 2^n (only displaying 30)

			digit	#	Obs Prob	Benf Prob
1	1024	1048576	1	18	.300	.301
2	2048	2097152	2	12	.200	.176
4	4096	4194304	3	6	.100	.125
8	8192	8388608	4	6	.100	.097
16	16384	16777216	5	6	.100	.079
32	32768	33554432	6	4	.067	.067
64	65536	67108864	7	2	.033	.058
128	131072	134217728	8	5	.083	.051
256	262144	268435456	9	1	.017	.046
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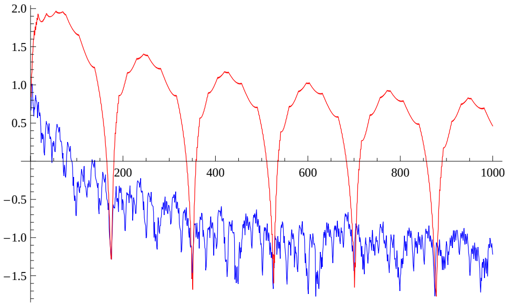
Logarithms and Benford's Law

χ^2 values for α^n, $1 \le n \le N$ (5% 15.5).

N	$\chi^2(\gamma)$	$\chi^2(e)$	$\chi^2(\pi)$
100	0.72	0.30	46.65
200	0.24	0.30	8.58
400	0.14	0.10	10.55
500	0.08	0.07	2.69
700	0.19	0.04	0.05
800	0.04	0.03	6.19
900	0.09	0.09	1.71
1000	0.02	0.06	2.90

Logarithms and Benford's Law: Base 10

$\log_{10}(\chi^2)$ vs N for π^n (red) and e^n (blue), $n \in \{1, \dots, N\}$. Note $\pi^{175} \approx 1.0028 \cdot 10^{87}$, (5% and 8 d.f., $\log_{10}(\chi^2) \approx .44$).



Applications

Detecting Fraud

Bank Fraud

- Audit of a bank revealed huge spike of numbers starting with 48 and 49, most due to one person.
- Write-off limit of $5,000. Officer had friends applying for credit cards, ran up balances just under $5,000 then he would write the debts off.

Benford Good Processes

Poisson Summation and Benford's Law: Definitions

- Feller, Pinkham (often exact processes)
- data $Y_{T,B} = \log_B \overrightarrow{X}_T$ (discrete/continuous):

$$\mathbb{P}(A) \;=\; \lim_{T\to\infty} \frac{\#\{n \in A : n \le T\}}{T}$$

- Poisson Summation Formula: f nice:

$$\sum_{\ell=-\infty}^{\infty} f(\ell) \;=\; \sum_{\ell=-\infty}^{\infty} \widehat{f}(\ell),$$

$$\text{Fourier transform } \widehat{f}(\xi) \;=\; \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx.$$

Benford Good Process

X_T is Benford Good if there is a nice f st

$$\mathrm{CDF}_{\vec{Y}_{T,B}}(y) = \int_{-\infty}^{y} \frac{1}{T} f\left(\frac{t}{T}\right) dt + E_T(y) := G_T(y)$$

and monotonically increasing h ($h(|T|) \to \infty$):

- Small tails: $G_T(\infty) - G_T(Th(T)) = o(1)$, $G_T(-Th(T)) - G_T(-\infty) = o(1)$.
- Decay of the Fourier Transform: $\sum_{\ell \neq 0} \left| \frac{\widehat{f}(T\ell)}{\ell} \right| = o(1)$.
- Small translated error: $\mathcal{E}(a, b, T)) = \sum_{|\ell| \leq Th(T)} [E_T(b+\ell) - E_T(a+\ell)] = o(1)$.

Main Theorem

Theorem (Kontorovich and M–, 2005)

X_T converging to X as $T \to \infty$ (think spreading Gaussian). If X_T is Benford good, then X is Benford.

- Examples
  - ◇ L-functions
  - ◇ characteristic polynomials (RMT)
  - ◇ $3x + 1$ problem
  - ◇ geometric Brownian motion.

Sketch of the proof

- Structure Theorem:
  - ◇ main term is something nice spreading out
  - ◇ apply Poisson summation

- Control translated errors:
  - ◇ hardest step
  - ◇ techniques problem specific

Sketch of the proof (continued)

$$\sum_{\ell=-\infty}^{\infty} \mathbb{P}\left(a+\ell \leq \overrightarrow{Y}_{T,B} \leq b+\ell\right)$$

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$$= \widehat{f}(0)\cdot(b-a) + \sum_{\ell\neq 0} \widehat{f}(T\ell)\frac{e^{2\pi i b\ell} - e^{2\pi i a\ell}}{2\pi i\ell} + o(1).$$

The $3x + 1$ Problem and Benford's Law

$3x + 1$ Problem

- Kakutani (conspiracy), Erdös (not ready).
- x odd, $T(x) = \frac{3x+1}{2^k}$, $2^k || 3x + 1$.

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- $7 \to_1 11 \to_1 17 \to_2 13 \to_3 5 \to_4 1 \to_2 1$,
  2-path $(1, 1)$, 5-path $(1, 1, 2, 3, 4)$.
  m-path: $(k_1, \dots, k_m)$.

Heuristic Proof of $3x + 1$ Conjecture

$$a_{n+1} = T(a_n)$$

Heuristic Proof of $3x + 1$ Conjecture

$$\begin{aligned} a_{n+1} &= T(a_n) \\ \mathbb{E}[\log a_{n+1}] &\approx \sum_{k=1}^{\infty} \frac{1}{2^k} \log\left(\frac{3a_n}{2^k}\right) \end{aligned}$$

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Geometric Brownian Motion, drift $\log(3/4) < 1$.

Structure Theorem: Sinai, Kontorovich-Sinai

$$\mathbb{P}(A) = \lim_{N\to\infty} \frac{\#\{n\le N : n \equiv 1,5 \bmod 6, n \in A\}}{\#\{n \le N : n \equiv 1,5 \bmod 6\}}.$$

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$6 \cdot 2^{k_1+\cdots+k_m} p + q$.

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Theorem (Sinai, Kontorovich-Sinai)

k_i-values are i.i.d.r.v. (geometric, $1/2$):

$$\mathbb{P}\left(\frac{\log_2\left[\frac{x_m}{\left(\frac{3}{4}\right)^m x_0}\right]}{\sqrt{2m}} \le a\right) = \mathbb{P}\left(\frac{S_m - 2m}{\sqrt{2m}} \le a\right)$$

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$3x+1$ and Benford

Theorem (Kontorovich and M–, 2005)

As $m \to \infty$, $x_m/(3/4)^m x_0$ is Benford.

Theorem (Lagarias-Soundararajan 2006)

$X \geq 2^N$, for all but at most $c(B)N^{-1/36}X$ initial seeds the distribution of the first N iterates of the $3x+1$ map are within $2N^{-1/36}$ of the Benford probabilities.

Prereq: Irrationality Type

Irrationality type

α has irrationality type κ if κ is the supremum of all γ with

$$\underline{\lim}_{q\to\infty} q^{\gamma+1} \min_p \left|\alpha - \frac{p}{q}\right| = 0.$$

- Algebraic irrationals: type 1 (Roth's Thm).
- Theory of Linear Forms: $\log_B 2$ of finite type.

Prereq: Linear Forms

Theorem (Baker)

$\alpha_1, \dots, \alpha_n$ algebraic numbers height $A_j \geq 4$, $\beta_1, \dots, \beta_n \in \mathbb{Q}$ with height at most $B \geq 4$,

$$\Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n.$$

If $\Lambda \neq 0$ then $|\Lambda| > B^{-C\Omega \log \Omega'}$, with $d = [\mathbb{Q}(\alpha_i, \beta_j) : \mathbb{Q}]$, $C = (16nd)^{200n}$, $\Omega = \prod_j \log A_j$, $\Omega' = \Omega / \log A_n$.

Gives $\log_{10} 2$ of finite type, with $\kappa < 1.2 \cdot 10^{602}$:

$$|\log_{10} 2 - p/q| = |q \log 4 - p \log 100| / 2q \log 10.$$

Prereq: Quantified Equidistribution

Theorem (Erdös-Turan)

$$D_N = \frac{\sup_{[a,b]} |N(b-a) - \#\{n \le N : x_n \in [a,b]\}|}{N}$$

There is a C such that for all m:

$$D_N \le C \cdot \left(\frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \right).$$

Prereq: Ideas for the proof of Erdös-Turan

Consider special case $x_n = n\alpha$, $\alpha \notin \mathbb{Q}$.

- Exponential sum $\le \frac{1}{|\sin(\pi h\alpha)|} \le \frac{1}{2||h\alpha||}$.
- Must control $\sum_{h=1}^{m} \frac{1}{h||h\alpha||}$, see irrationality type enter.
- type κ, $\sum_{h=1}^{m} \frac{1}{h||h\alpha||} = O\left(m^{\kappa-1+\epsilon}\right)$, take $m = \lfloor N^{1/\kappa} \rfloor$.

Sketch of the proof: Notation

- $[a, b]$ is an arbitrary sub-interval of $[0, 1]$.
- $c \in \left(0, \frac{1}{2}\right)$ and set $M = m^c$.
- $I_\ell \ := \ \{\ell M, \ell M+1, \dots, (\ell+1)M-1\}$.
- $C := \log_B 2$ an irrational number of type κ.
- $\eta(x)$ the density of the standard normal.
- S_m sum of m independent Geom(1/2) r.v.

Sketch of the proof

Step 1: Central Limit Theorem:

CLT: for any $k \in \mathbb{Z}$ have

$$\begin{aligned}\text{Prob}(C \cdot \overline{S}_m = C \cdot k) &= \text{Prob}\left(\frac{\overline{S}_m}{\sqrt{m}} = \frac{k}{\sqrt{m}}\right) \\ &= \frac{1}{\sqrt{m}}\,\eta\left(\frac{k}{\sqrt{m}}\right) + o\left(\frac{1}{\sqrt{m}}\right)\end{aligned}$$

Write $o\left(\frac{1}{\sqrt{m}}\right)$ as $O\left(\frac{1}{\sqrt{m}g(m)}\right)$ for some monotone increasing $g(m)$ which tends to infinity.

Sketch of the proof

Step 2: Variation in I_ℓ:

Let $k_1, k_2 \in I_\ell$. Then

$$\left|\frac{1}{\sqrt{m}}\,\eta\left(\frac{k_1}{\sqrt{m}}\right) - \frac{1}{\sqrt{m}}\,\eta\left(\frac{k_2}{\sqrt{m}}\right)\right|$$
$$\leq \ \frac{1}{\sqrt{m}}\,e^{-\ell^2M^2/2m}\cdot\left(1-\exp\left(-\frac{2\ell M^2+M^2}{2m}\right)\right).$$

Means that for the ℓ we must study, there is negligible variation in the Gaussian for $k \in I_\ell$.

Sketch of the proof

Step 3: Poisson Summation:

By Poisson Summation:

$$\frac{1}{\sigma}\sum_{n=-\infty}^{\infty} e^{-n^2\pi/\sigma^2} = \sum_{n=-\infty}^{\infty} e^{-n^2\pi\sigma^2}, \quad \sigma > 0.$$

Take $\sigma^2 = \frac{2m}{\pi M^2}$, and use this to calculate the main term.

Sketch of the proof

Step 4: Quantified Equidistribution:

For any $\epsilon > 0$, letting $\delta = 1 + \epsilon - \frac{1}{\kappa} < 1$ we have

$$\#\{k \in I_\ell : kC \bmod 1 \in [a, b]\} \ = \ M(b-a)+O(M^\delta).$$

The quantification of the equidistribution of $kC \bmod 1$ is the key ingredient in proving Benford behavior base B. Follows from the Erdös-Turan Theorem.

Sketch of the proof

Step 5: Combining Pieces:

Must show as $m \to \infty$, for any $[a, b] \subset [0, 1]$,

$$P_m(a, b) \ := \ \text{Prob}(C\overline{S}_m \bmod 1 \in [a, b]) \ \longrightarrow \ b-a.$$

We have

$$\begin{aligned} P_m(a, b) = & \sum_{|\ell| \le \frac{\sqrt{m}h(m)}{M}} \text{Prob}(\overline{S}_m = k \in I_\ell : kC \bmod 1 \in [a, b]) \\ & + \sum_{|\ell| > \frac{\sqrt{m}h(m)}{M}} \text{Prob}(\overline{S}_m = k \in I_\ell : kC \bmod 1 \in [a, b]) \\ = & \sum_{|\ell| \le \frac{\sqrt{m}h(m)}{M}} \text{Prob}(\overline{S}_m = k \in I_\ell : kC \bmod 1 \in [a, b]) + o(1). \end{aligned}$$

$3x+1$ Data: random 10,000 digit number, $2^k || 3x+1$

80,514 iterations ($(4/3)^n = a_0$ predicts 80,319); $\chi^2 = 13.5$ (5% 15.5).

Digit	Number	Observed	Benford
1	24251	0.301	0.301
2	14156	0.176	0.176
3	10227	0.127	0.125
4	7931	0.099	0.097
5	6359	0.079	0.079
6	5372	0.067	0.067
7	4476	0.056	0.058
8	4092	0.051	0.051
9	3650	0.045	0.046

$3x + 1$ Data: random 10,000 digit number, $2|3x + 1$

241,344 iterations, $\chi^2 = 11.4$ (5% 15.5).

Digit	Number	Observed	Benford
1	72924	0.302	0.301
2	42357	0.176	0.176
3	30201	0.125	0.125
4	23507	0.097	0.097
5	18928	0.078	0.079
6	16296	0.068	0.067
7	13702	0.057	0.058
8	12356	0.051	0.051
9	11073	0.046	0.046

$5x+1$ Data: random 10,000 digit number, $2^k || 5x+1$

27,004 iterations, $\chi^2 = 1.8$ (5% 15.5).

Digit	Number	Observed	Benford
1	8154	0.302	0.301
2	4770	0.177	0.176
3	3405	0.126	0.125
4	2634	0.098	0.097
5	2105	0.078	0.079
6	1787	0.066	0.067
7	1568	0.058	0.058
8	1357	0.050	0.051
9	1224	0.045	0.046

$5x+1$ Data: random 10,000 digit number, $2|5x+1$

241,344 iterations, $\chi^2 = 3 \cdot 10^{-4}$ (5% 15.5).

Digit	Number	Observed	Benford
1	72652	0.301	0.301
2	42499	0.176	0.176
3	30153	0.125	0.125
4	23388	0.097	0.097
5	19110	0.079	0.079
6	16159	0.067	0.067
7	13995	0.058	0.058
8	12345	0.051	0.051
9	11043	0.046	0.046

Conclusions

Conclusions and Future Investigations

- Different systems exhibit Benford behavior.
- Ingredients of proofs (logarithms, equidistribution).
- Applications to fraud detection / data integrity.
- Current studies include dependent random variables (partitions and amalgamations) and quantifying convergence to Benford.

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L-functions and Random Matrix Theory

Good L-functions ($\zeta(s)$, full level cusp form)

L-function is good if:

- Euler product:

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} \prod_{j=1}^{d} \left(1 - \alpha_{f,j}(p) p^{-s}\right)^{-1}.$$

- meromorphic continuation to $\mathbb{C}$, of finite order, at most finitely many poles (all on the line $\mathrm{Re}(s) = 1$).
- Functional equation: $\omega \in \mathbb{R}$, $G(s)$ prod Γ-fns:

$$e^{\mathrm{i}\omega} G(s) L(s,f) = e^{-\mathrm{i}\omega} \overline{G(1-\overline{s}) L(1-\overline{s})}.$$

Good L-functions

- For some $\aleph > 0$, $c \in \mathbb{C}$, $x \geq 2$:

$$\sum_{p \leq x} \frac{|a_f(p)|^2}{p} = \aleph \log\log x + c + O\left(\frac{1}{\log x}\right).$$

- The $\alpha_{f,j}(p)$ are (Ramanujan-Petersson) tempered: $|\alpha_{f,j}(p)| \leq 1$.
- $N(\sigma, T) = \#\{\rho : L(\rho, f) = 0, \mathrm{Re}(\rho) \geq \sigma, \mathrm{Im}(\rho) \in [0, T]\}$. $\exists \beta > 0$

$$N(\sigma, T) = O\left(T^{1-\beta\left(\sigma-\frac{1}{2}\right)} \log T\right).$$

Log-Normal Law (Hejhal, Laurinčikas, Selberg)

Log-Normal Law

$$\frac{\mu\left(\{t \in [T, 2T] : \log|L(\sigma + it, f)| \in [a, b]\}\right)}{T} =$$

$$\frac{1}{\sqrt{\psi(\sigma, T)}} \int_a^b e^{-\pi u^2/\psi(\sigma,T)} du + \text{Error}$$

$$\psi(\sigma, T) = \aleph \log\left[\min\left(\log T, \frac{1}{\sigma - \frac{1}{2}}\right)\right] + O(1)$$

$$\frac{1}{2} \le \sigma \le \frac{1}{2} + \frac{1}{\log^\delta T}, \quad \delta \in (0, 1).$$

Values of L-functions and Benford's Law

Theorem (Kontorovich and M–, 2005)

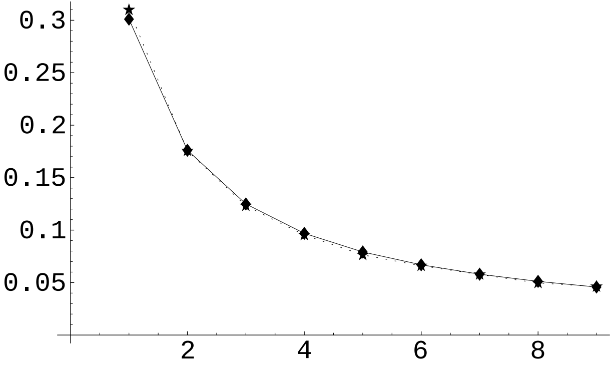
$L(s,f)$ a good L-function, as $T \to \infty$, $L(\sigma_T + it, f)$ is Benford.

Ingredients

- Approximate $\log L(\sigma_T + it, f)$ with $\sum_{n \le x} \frac{c(n)\Lambda(n)}{\log n} \frac{1}{n^{\sigma_T + it}}$.
- study moments $\int_T^{2T} |\cdot|$, $k \le \log^{1-\delta} T$.
- Montgomery-Vaughan: $\int_T^{2T} \sum a_n n^{-it} \overline{\sum b_m m^{-it}} dt = H \sum a_n \overline{b}_n + O(1)\sqrt{\sum n|a_n|^2 \sum n|b_n|^2}$.

Riemann Zeta Function

$$\left|\zeta\left(\frac{1}{2} + i\frac{k}{4}\right)\right|, k \in \{0, 1, \ldots, 65535\}.$$



Random Matrices: Preliminaries

- $N \times N$ unitary matrices U (Haar measure):

$$p_N(U) = \frac{1}{(2\pi)^N N!} \prod_{1 \le j < m \le N} \left| e^{i\theta_j} - e^{i\theta_m} \right|.$$

- characteristic polynomial:

$$Z(U,\theta) = \det(I - Ue^{-i\theta}) = \prod \left(1 - e^{i(\theta_n - \theta)}\right).$$

- $\rho_N(x)$ the probability density for $\log |Z(U,\theta)|$:

$$\widetilde{\rho}_N(x) \ = \ \sqrt{Q_2(N)}\, \rho_N(\sqrt{Q_2(N)}\, x),$$

variance $Q_2(N) \sim (\log N)/2$.

Random Matrices and Benford's Law

Theorem (Kontorovich and M–, 2005)

As $N \to \infty$, the distribution of digits of the absolute values of the characteristic polynomials of $N \times N$ unitary matrices (with respect to Haar measure) converges to the Benford probabilities.

- Key Ingredient: Keating-Snaith:

$$\widetilde{\rho}_N(x)dx \;=\; \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}dx + O\left((\log N)^{-3/2}dx\right).$$