Why the IRS cares about the Riemann Zeta Function and Number Theory (and why you should too!)

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Interesting Question

Motivating Question: For a nice data set, such as the Fibonacci numbers, stock prices, street addresses of college employees and students, ..., what percent of the leading digits are 1?

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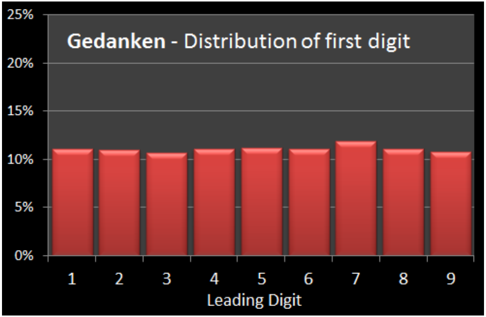

Natural guess: 10% (but immediately correct to 11%!).

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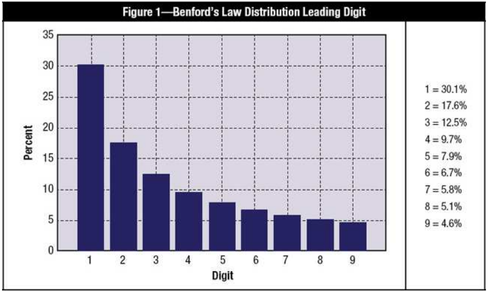
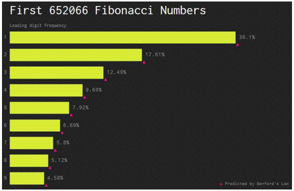


Answer: Benford's law!

Examples with First Digit Bias

Fibonacci numbers



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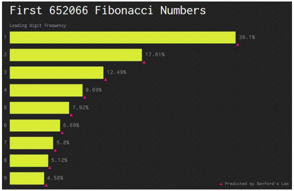
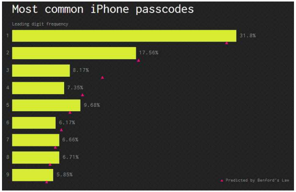


Most common iPhone passcodes



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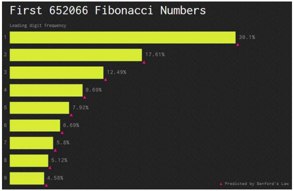

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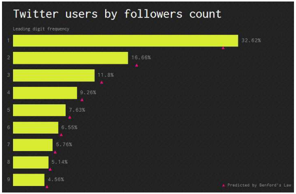

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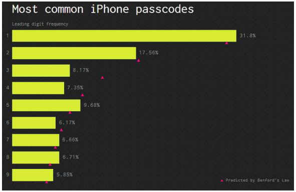

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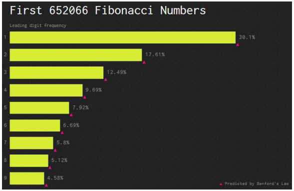

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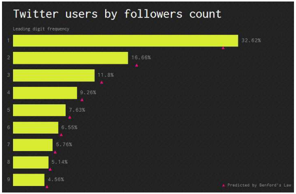

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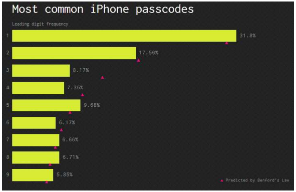
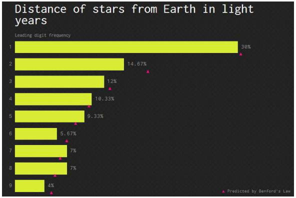


Distance of stars from Earth



Summary

- Explain Benford's Law.
- Discuss examples and applications.
- Sketch proofs.
- Describe open problems.

Caveats!

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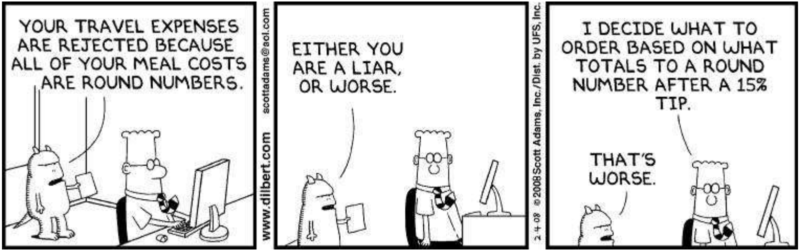


Examples

- recurrence relations
- special functions (such as $n!$)
- iterates of power, exponential, rational maps
- products of random variables
- L-functions, characteristic polynomials
- iterates of the $3x + 1$ map
- differences of order statistics
- hydrology and financial data
- many hierarchical Bayesian models

Applications

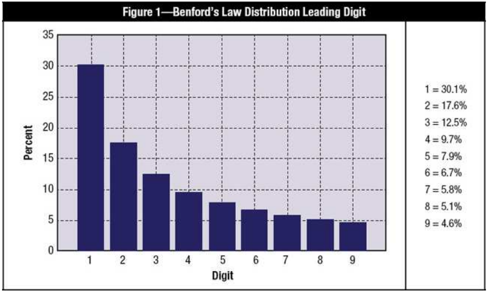
- Analyzing round-off errors.
- Determining the optimal way to store numbers.
- Detecting tax and image fraud, and data integrity.

General Theory

Benford's Law: Newcomb (1881), Benford (1938)

Statement

For many data sets, probability of observing a first digit of d base B is $\log_B\left(\frac{d+1}{d}\right)$; base 10 about 30% are 1s.



Benford's Law (probabilities)

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- Key observation: $\log_{10}(x) = \log_{10}(\widetilde{x}) \mod 1$ if and only if x and $\widetilde{x}$ have the same leading digits.

Thus often study $y = \log_{10} x \mod 1$.
Advanced: $e^{2\pi i u} = e^{2\pi i (u \bmod 1)}$.

Equidistribution and Benford's Law

Equidistribution

$\{y_n\}_{n=1}^{\infty}$ is equidistributed modulo 1 if probability $y_n \bmod 1 \in [a,b]$ tends to $b-a$:

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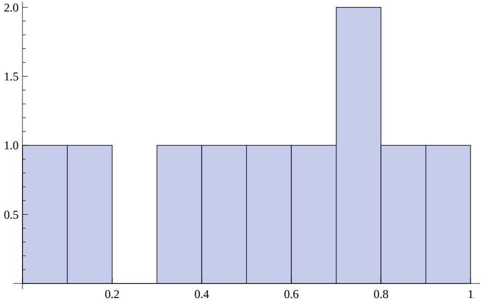
  Thus $2^q = 10^p$ or $2^{q-p} = 5^p$, impossible.

Example of Equidistribution: $n\sqrt{\pi} \bmod 1$



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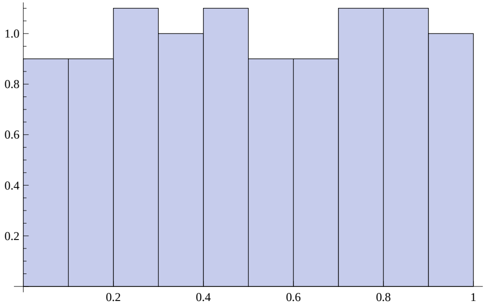

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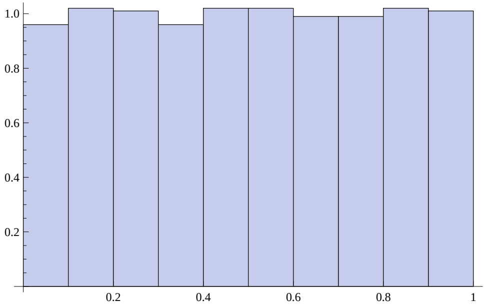

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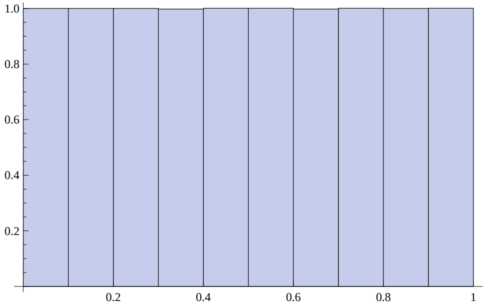

$n\sqrt{\pi}$ mod 1 for $n \le 10,000$

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Fundamental Equivalence

Data set $\{x_i\}$ is Benford base B if $\{y_i\}$ is equidistributed mod 1, where $y_i = \log_B x_i$.

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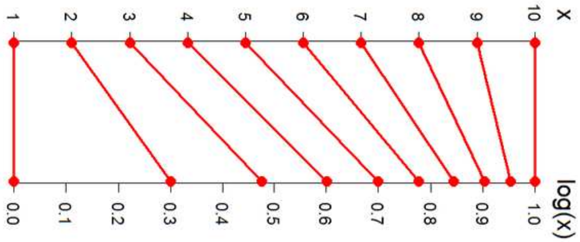
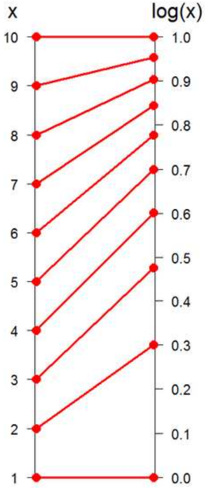


Logarithms and Benford's Law

$$\begin{aligned} &\mathrm{Prob}(\text{leading digit } d) \\ &= \log_{10}(d+1) - \log_{10}(d) \\ &= \log_{10}\left(\frac{d+1}{d}\right) \\ &= \log_{10}\left(1+\frac{1}{d}\right). \end{aligned}$$

Have Benford's law $\leftrightarrow$ mantissa of logarithms of data are uniformly distributed



Examples

- 2^n is Benford base 10 as $\log_{10} 2 \notin \mathbb{Q}$.

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- Most linear recurrence relations Benford:
  - ◇ $a_{n+1} = 2a_n - a_{n-1}$
  - ◇ take $a_0 = a_1 = 1$ or $a_0 = 0$, $a_1 = 1$.

Digits of 2^n

First 60 values of 2^n (only displaying 30)

			digit	#	Obs Prob	Benf Prob
1	1024	1048576	1	18	.300	.301
2	2048	2097152	2	12	.200	.176
4	4096	4194304	3	6	.100	.125
8	8192	8388608	4	6	.100	.097
16	16384	16777216	5	6	.100	.079
32	32768	33554432	6	4	.067	.067
64	65536	67108864	7	2	.033	.058
128	131072	134217728	8	5	.083	.051
256	262144	268435456	9	1	.017	.046
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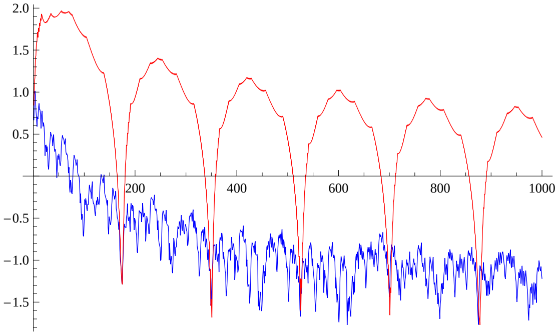
χ^2 values for α^n, $1 \le n \le N$ (5% 15.5).

N	$\chi^2(\gamma)$	$\chi^2(e)$	$\chi^2(\pi)$
100	0.72	0.30	46.65
200	0.24	0.30	8.58
400	0.14	0.10	10.55
500	0.08	0.07	2.69
700	0.19	0.04	0.05
800	0.04	0.03	6.19
900	0.09	0.09	1.71
1000	0.02	0.06	2.90

Logarithms and Benford's Law: Base 10 (5%: $\log(\chi^2) \approx 2.74$)

$\log(\chi^2)$ vs N for π^n (red) and e^n (blue), $n \in \{1, \dots, N\}$.



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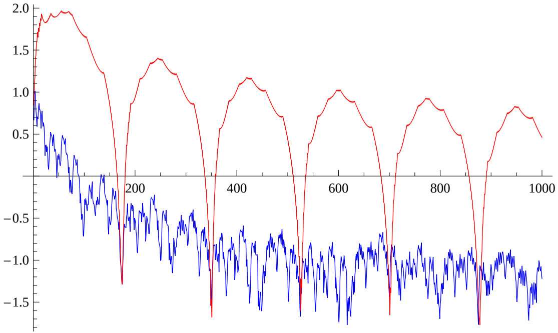
$\log(\chi^2)$ vs N for π^n (red) and e^n (blue), $n \in \{1, \dots, N\}$. Note $\pi^{175} \approx 1.0028 \cdot 10^{87}$.



Why Benford's Law?

Streets

Not all data sets satisfy Benford's Law.

- Long street $[1, L]$: $L = 199$ versus $L = 999$.
- Oscillates b/w $1/9$ and $5/9$ with first digit 1.

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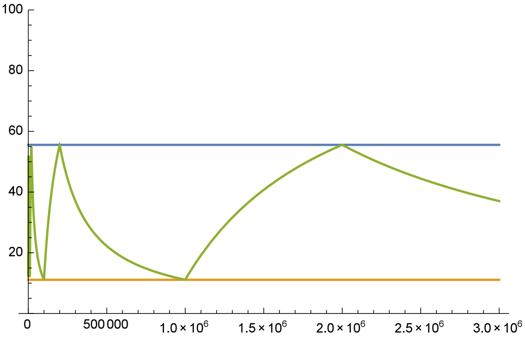

Probability first digit 1 versus street length L.

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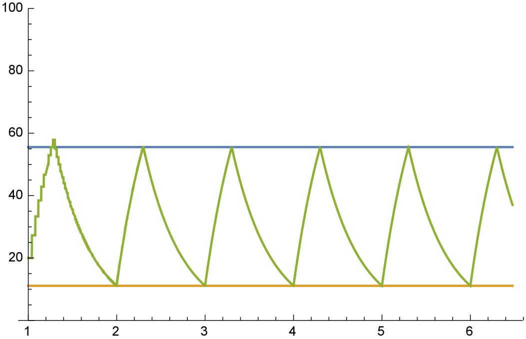

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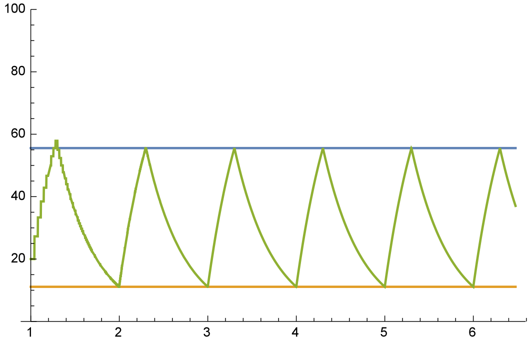


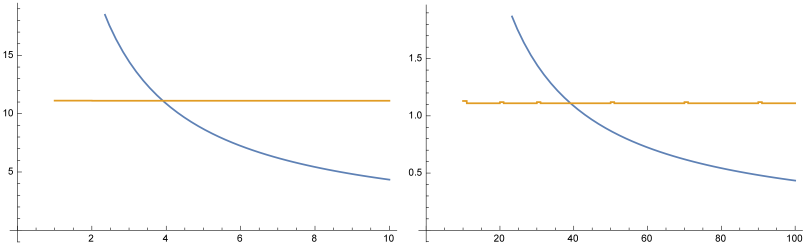
Probability first digit 1 versus log(street length L).

What if we have many streets of different lengths?

Amalgamating Streets

All houses: 1000 Streets,

each from 1 to 10000.



First digit and first two digits vs Benford.

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All houses: 1000 Streets,
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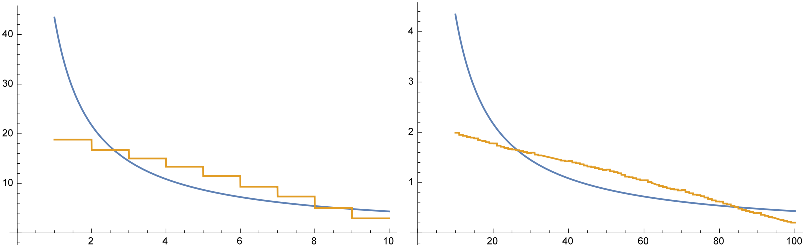
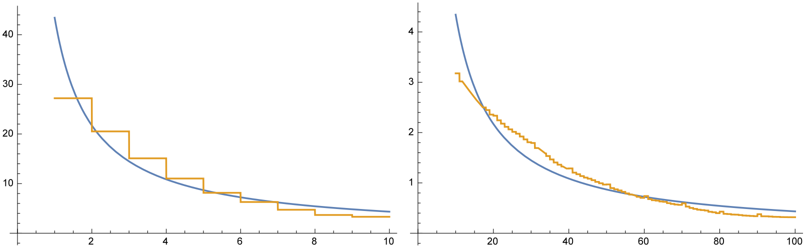


First digit and first two digits vs Benford.

Amalgamating Streets

All houses: 1000 Streets,

each 1 to rand(rand(10000)).



First digit and first two digits vs Benford.

Conclusion: More processes, closer to Benford.

Amalgamating Streets

All houses: 1000 Streets,

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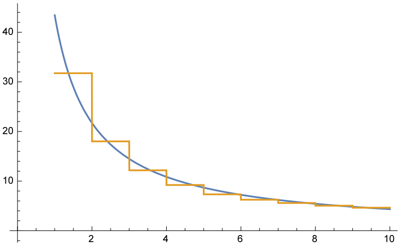

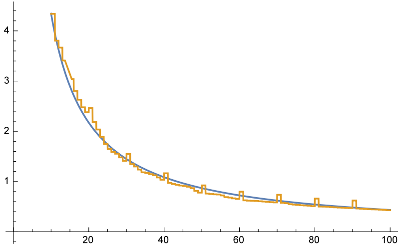

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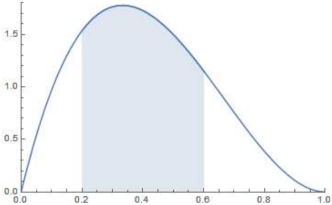
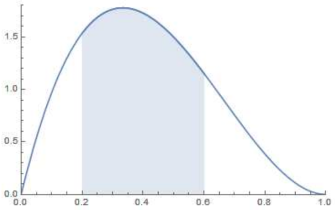
Probability Review



- Let X be random variable with density $p(x)$:
  - $\diamond$ $p(x) \geq 0$; $\int_{-\infty}^{\infty} p(x)\mathrm{d}x = 1$;
  - $\diamond$ $\mathrm{Prob}\,(a \leq X \leq b) = \int_a^b p(x)\mathrm{d}x$.

Probability Review



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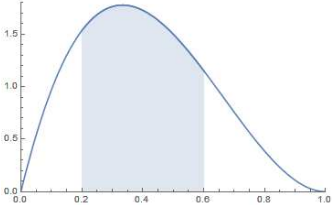
- Mean $\mu = \int_{-\infty}^{\infty} x p(x)\mathrm{d}x$.

Probability Review



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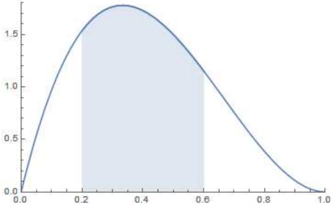
- Mean $\mu = \int_{-\infty}^{\infty} xp(x)\mathrm{d}x$.
- Variance $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 p(x)\mathrm{d}x$.

Probability Review



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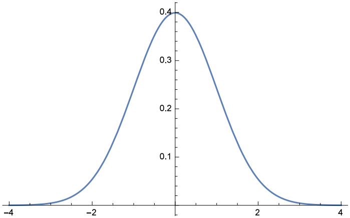
  $\diamond$ $\mathrm{Prob}\,(a \leq X \leq b) = \int_a^b p(x)\mathrm{d}x$.
- Mean $\mu = \int_{-\infty}^{\infty} xp(x)\mathrm{d}x$.
- Variance $\sigma^2 = \int_{-\infty}^{\infty}(x-\mu)^2 p(x)\mathrm{d}x$.
- Independence: knowledge of one random variable gives no knowledge of the other.

Central Limit Theorem

$$\text{Normal } N(\mu, \sigma^2): \; p(x) \; = \; e^{-(x-\mu)^2/2\sigma^2} / \sqrt{2\pi\sigma^2}.$$



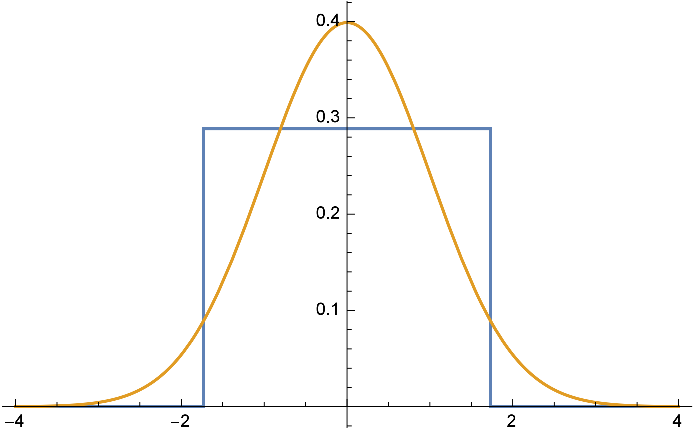
Theorem

If $X_1, X_2, \dots$ independent, identically distributed random variables (mean μ, variance σ^2, finite moments) then

$$S_N \; := \; \frac{X_1 + \cdots + X_N - N\mu}{\sigma\sqrt{N}} \; \text{converges to} \; N(0,1).$$

Central Limit Theorem: Sums of Uniform Random Variables $X_i \sim \mathrm{Unif}(-1/2, 1/2)$ (adjusted to mean 0, variance 1)

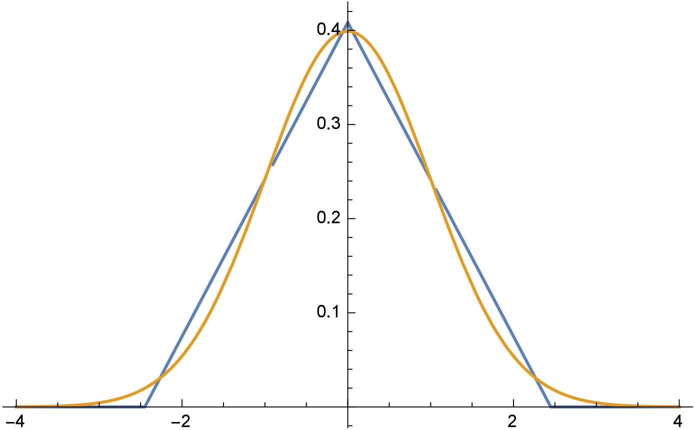
$$Y_1 = X_1/\sigma_{X_1} \text{ vs } N(0,1).$$



Central Limit Theorem: Sums of Uniform Random Variables

$X_i \sim \mathrm{Unif}(-1/2, 1/2)$ **(adjusted to mean 0, variance 1)**

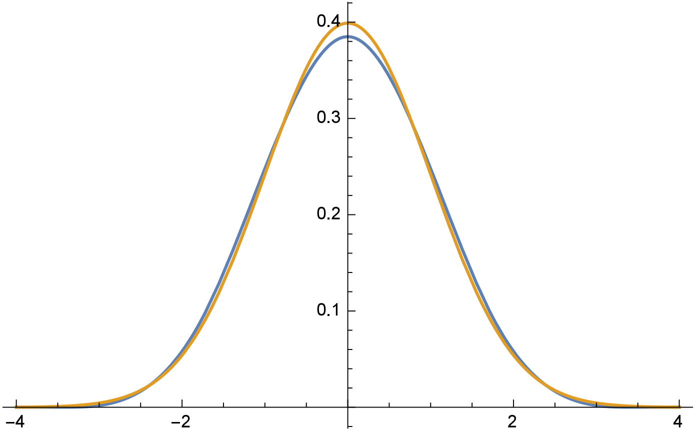
$$Y_2 = (X_1 + X_2)/\sigma_{X_1+X_2} \text{ vs } N(0,1).$$



Central Limit Theorem: Sums of Uniform Random Variables

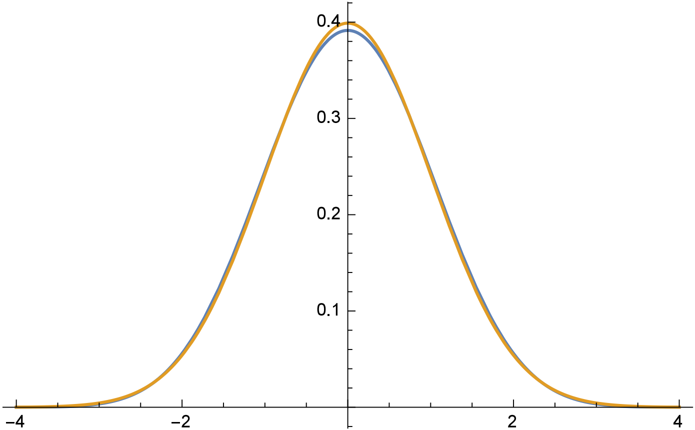
$X_i \sim \mathrm{Unif}(-1/2, 1/2)$ (adjusted to mean 0, variance 1)

$$Y_4 = (X_1 + X_2 + X_3 + X_4)/\sigma_{X_1+X_2+X_3+X_4} \text{ vs } N(0,1).$$



Central Limit Theorem: Sums of Uniform Random Variables
$X_i \sim \mathrm{Unif}(-1/2, 1/2)$ (adjusted to mean 0, variance 1)

$$Y_8 = (X_1 + \cdots + X_8)/\sigma_{X_1+\cdots+X_8} \text{ vs } N(0,1).$$



Central Limit Theorem: Sums of Uniform Random Variables $X_i \sim \mathrm{Unif}(-1/2, 1/2)$ (adjusted to mean 0, variance 1)

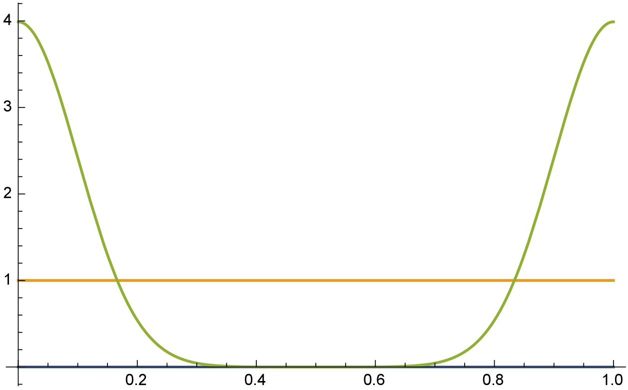
Density of $Y_4 = (X_1 + \cdots + X_4)/\sigma_{X_1+\cdots+X_4}$.

$$\frac{\begin{cases} \frac{1}{27}\left(18 + 9\sqrt{3}\,y - \sqrt{3}\,y^3\right) & y == 0 \\ \frac{1}{18}\left(12 - 6y^2 - \sqrt{3}\,y^3\right) & -\sqrt{3} < y < 0 \\ \frac{1}{54}\left(72 - 36\sqrt{3}\,y + 18y^2 - \sqrt{3}\,y^3\right) & \sqrt{3} < y < 2\sqrt{3} \\ \frac{1}{54}\left(18\sqrt{3}\,y - 18y^2 + \sqrt{3}\,y^3\right) & y == \sqrt{3} \\ \frac{1}{18}\left(12 - 6y^2 + \sqrt{3}\,y^3\right) & 0 < y < \sqrt{3} \\ \frac{1}{54}\left(72 + 36\sqrt{3}\,y + 18y^2 + \sqrt{3}\,y^3\right) & -2\sqrt{3} < y \leq -\sqrt{3} \\ 0 & \text{True} \end{cases}}{\sqrt{3}}$$

(Don't even think of asking to see Y_8's!)

Normal Distributions Mod 1

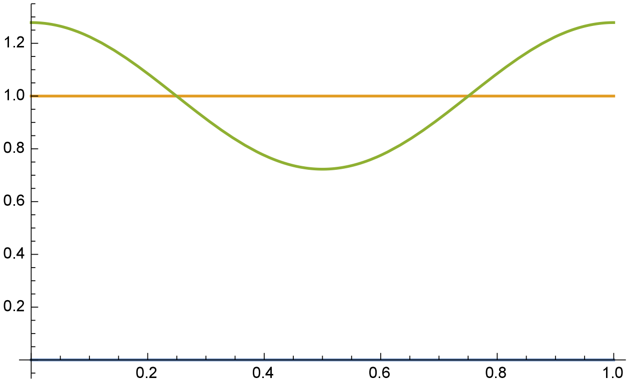
As $\sigma \to \infty$, $N(0,\sigma^2)$ mod 1 $\to \mathrm{Unif}(0,1)$.



Variance is .01.

Normal Distributions Mod 1

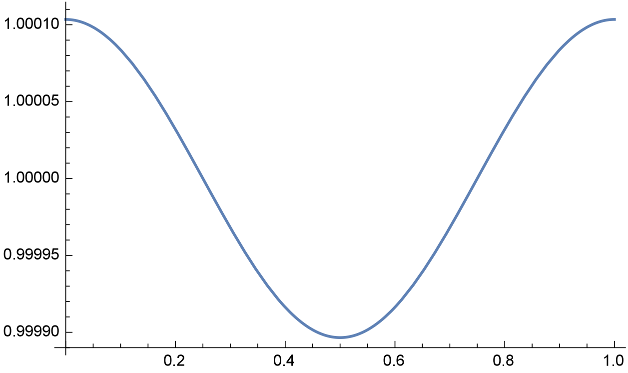
As $\sigma \to \infty$, $N(0, \sigma^2)$ mod 1 $\to \mathrm{Unif}(0, 1)$.



Variance is .1.

Normal Distributions Mod 1

As $\sigma \to \infty$, $N(0, \sigma^2)$ mod 1 $\to \text{Unif}(0, 1)$.



Variance is .5.

Products and Benford's Law

Pavlovian Response: See a product, take a logarithm.

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$X_1, X_2, \dots$ nice, $W_N = X_1 \cdot X_2 \cdots X_N$.

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$Y_i = \log_{10} X_i$, $V_N := \log_{10} W_N$.

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$$V_N = \log_{10}(X_1 \cdot X_2 \cdots X_N)$$

Products and Benford's Law

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$Y_i = \log_{10} X_i$, $V_N := \log_{10} W_N$.

$$\begin{aligned} V_N &= \log_{10}(X_1 \cdot X_2 \cdots X_N) \\ &= \log_{10} X_1 + \log_{10} X_2 + \cdots + \log_{10} X_N \end{aligned}$$

Products and Benford's Law

Pavlovian Response: See a product, take a logarithm.

$X_1, X_2, \dots$ nice, $W_N = X_1 \cdot X_2 \cdots X_N$.

$Y_i = \log_{10} X_i$, $V_N := \log_{10} W_N$.

$$\begin{aligned} V_N &= \log_{10}(X_1 \cdot X_2 \cdots X_N) \\ &= \log_{10} X_1 + \log_{10} X_2 + \cdots + \log_{10} X_N \\ &= Y_1 + Y_2 + \cdots + Y_N. \end{aligned}$$

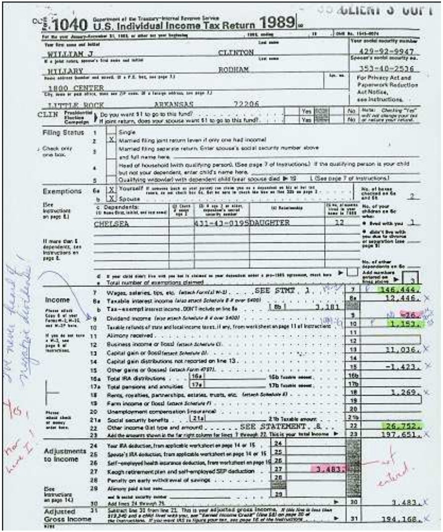
Need distribution of V_N mod 1, which by CLT becomes uniform, implying Benfordness!

Applications

Applications for the IRS: Detecting Fraud

A Tale of Two Steve Millers....

Applications for the IRS: Detecting Fraud



1040 Department of the Treasury—Internal Revenue Service
U.S. Individual Income Tax Return 1989

WILLIAM J	CLINTON
Your social security number	429-92-9947
HILLARY	RODHAM
Spouse's social security no.	353-40-2536
1800 CENTER	
LITTLE ROCK ARKANSAS 72206	

Filing Status

1. Single
2. X Married filing joint return (even if only one had income)
3. Married filing separate return. Enter spouse's social security number above and full name here.
4. Head of household (with qualifying person). (See page 7 of Instructions.) If the qualifying person is your child but not your dependent, enter child's name here.
5. Qualifying widow(er) with dependent child (year spouse died ▶ 19). (See page 7 of Instructions.)

Exemptions

6a X Yourself

b X Spouse

c Dependents: CHELSEA 431-43-0195 DAUGHTER 12

e Total number of exemptions claimed 3

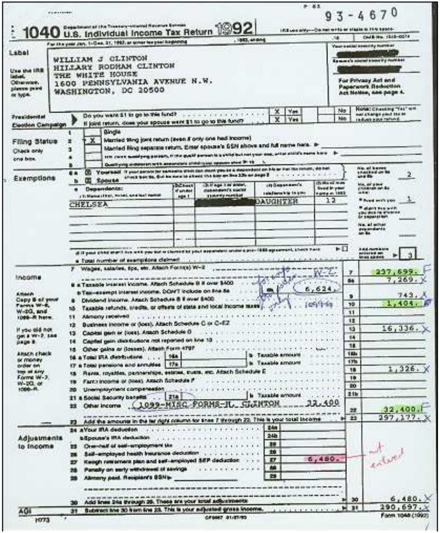
Income

7	Wages, salaries, tips, etc. (attach Form(s) W-2) SEE STMT 1	146,444.
8a	Taxable interest income (also attach Schedule B if over $400)	12,446.
8b	Tax-exempt interest income. DON'T include on line 8a 3,181	
9	Dividend income (also attach Schedule B if over $400)	26.
10	Taxable refunds of state and local income taxes, if any, from worksheet on page 11 of Instructions	1,153.
11	Alimony received	
12	Business income or (loss) (attach Schedule C)	
13	Capital gain or (loss) (attach Schedule D)	11,036.
14	Capital gain distributions not reported on line 13	
15	Other gains or (losses) (attach Form 4797)	-1,423.
16a	Total IRA distributions	
17a	Total pensions and annuities	
18	Rents, royalties, partnerships, estates, trusts, etc. (attach Schedule E)	1,269.
19	Farm income or (loss) (attach Schedule F)	
20	Unemployment compensation (insurance)	
21a	Social security benefits	
22	Other income (list type and amount) SEE STATEMENT 8	26,752.
23	Add the amounts shown in the far right column for lines 7 through 22. This is your total income	197,651.

Adjustments to Income

24	Your IRA deduction, from applicable worksheet on page 14 or 15	
25	Spouse's IRA deduction, from applicable worksheet on page 14 or 15	
26	Self-employed health insurance deduction, from worksheet on page 15	
27	Keogh retirement plan and self-employed SEP deduction	3,483.
28	Penalty on early withdrawal of savings	
29	Alimony paid	
30	Add lines 24 through 29	3,483.
31	Subtract line 30 from line 23. This is your adjusted gross income.	194,168.

Applications for the IRS: Detecting Fraud



93-4670

1040 U.S. Individual Income Tax Return 1992

Label

WILLIAM J CLINTON
HILLARY RODHAM CLINTON
THE WHITE HOUSE
1600 PENNSYLVANIA AVENUE N.W.
WASHINGTON, DC 20500

Your social security number [redacted]

Spouse's social security number [redacted]

For Privacy Act and Paperwork Reduction Act Notice, see page 4.

Presidential Election Campaign — Do you want $1 to go to this fund? X Yes — If joint return, does your spouse want $1 to go to this fund? X Yes

Filing Status — 1 Single; 2 X Married filing joint return (even if only one had income); 3 Married filing separate return. Enter spouse's SSN above and full name here.

Exemptions — 6a Yourself; b Spouse; c Dependents: CHELSEA [redacted] DAUGHTER 12

No. of boxes checked on 6a and 6b: 2; lived with you: 1; Add numbers entered on lines above: 3

Income

7	Wages, salaries, tips, etc. Attach Form(s) W-2	237,699.
8a	Taxable interest income. Attach Schedule B if over $400	7,269.
8b	Tax-exempt interest income. DON'T include on line 8a	6,624.
9	Dividend income. Attach Schedule B if over $400	743.
10	Taxable refunds, credits, or offsets of state and local income taxes	1,404.
11	Alimony received	
12	Business income or (loss). Attach Schedule C or C-EZ	
13	Capital gain or (loss). Attach Schedule D	16,336.
14	Capital gain distributions not reported on line 13	
15	Other gains or (losses). Attach Form 4797	
16a	Total IRA distributions	
17a	Total pensions and annuities	
18	Rents, royalties, partnerships, estates, trusts, etc. Attach Schedule E	1,326.
19	Farm income or (loss). Attach Schedule F	
20	Unemployment compensation	
21a	Social Security benefits	
22	Other income 1099-MISC FORMS-H. CLINTON 32,400	32,400.
23	Add the amounts in the far right column for lines 7 through 22. This is your total income	297,177.

Adjustments to Income

24a	Your IRA deduction	
24b	Spouse's IRA deduction	
25	One-half of self-employment tax	
26	Self-employed health insurance deduction	
27	Keogh retirement plan and self-employed SEP deduction	6,480.
28	Penalty on early withdrawal of savings	
29	Alimony paid. Recipient's SSN	
30	Add lines 24a through 29. These are your total adjustments	6,480.
31	Subtract line 30 from line 23. This is your adjusted gross income.	290,697.

W-2 1099-G not entered

Form 1040 (1992)

Detecting Fraud

Bank Fraud

- Audit of a bank revealed huge spike of numbers starting with

Detecting Fraud

Bank Fraud

- Audit of a bank revealed huge spike of numbers starting with 4

Detecting Fraud

Bank Fraud

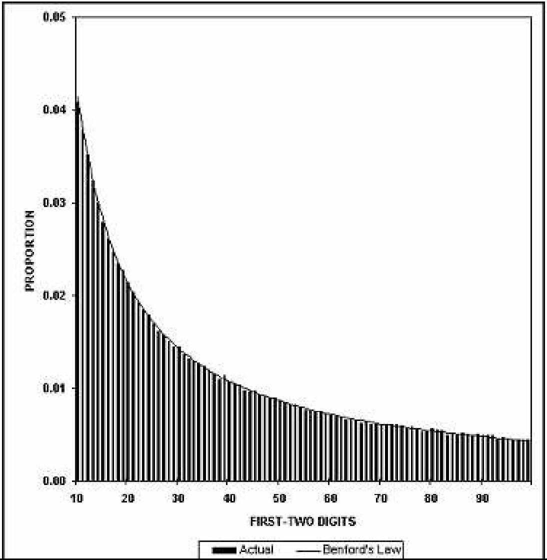
- Audit of a bank revealed huge spike of numbers starting with 48 and 49, most due to one person.

Detecting Fraud

Bank Fraud

- Audit of a bank revealed huge spike of numbers starting with 48 and 49, most due to one person.
- Write-off limit of $5,000. Officer had friends applying for credit cards, ran up balances just under $5,000 then he would write the debts off.

Data Integrity: Stream Flow Statistics: 130 years, 457,440 records



Election Fraud: Iran 2009

Numerous questions over Iran's 2009 elections.

Lot of analysis; data moderately suspicious:

- First and second leading digits;
- Last two digits (should almost be uniform);
- Last two digits differing by at least 2.

Warning: enough tests, even if nothing wrong will find a suspicious result (but when all tests are on the boundary...).

Stick Decomposition

Fixed Proportion Decomposition Process

Decomposition Process

1. Consider a stick of length $\mathcal{L}$.

Fixed Proportion Decomposition Process

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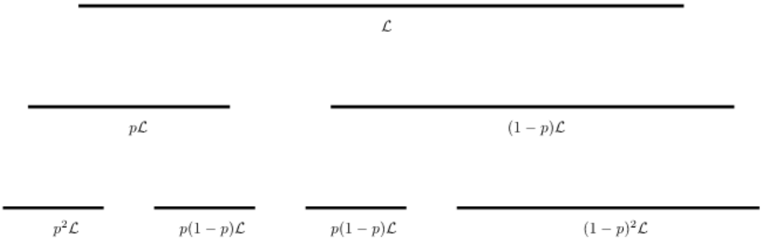
3. Break the stick into two pieces—lengths $p\mathcal{L}$ and $(1-p)\mathcal{L}$.

Fixed Proportion Decomposition Process

Decomposition Process

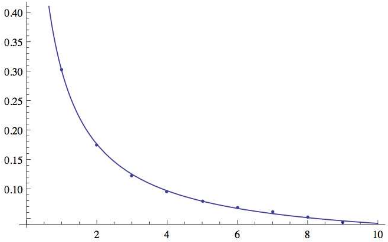
1. Consider a stick of length $\mathcal{L}$.
2. Uniformly choose a proportion $p \in (0, 1)$.
3. Break the stick into two pieces—lengths $p\mathcal{L}$ and $(1-p)\mathcal{L}$.
4. Repeat N times (using the same proportion).

Fixed Proportion Decomposition Process

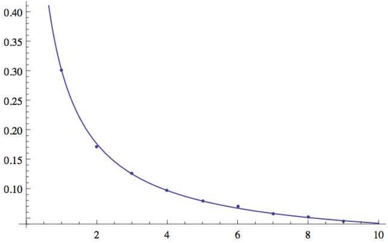


Fixed Proportion Conjecture (Joy Jing '13)

Conjecture: The above decomposition process is Benford as $N \to \infty$ for any $p \in (0,1)$, $p \neq \frac{1}{2}$.



(B) $p = 0.51$ and $N = 10000$.



(B) $p = 0.99$ and $N = 50000$. Benford distribution overlaid.

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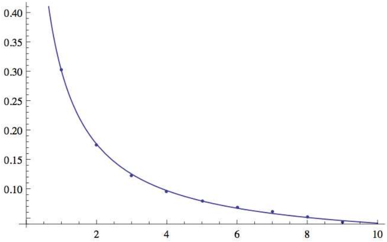
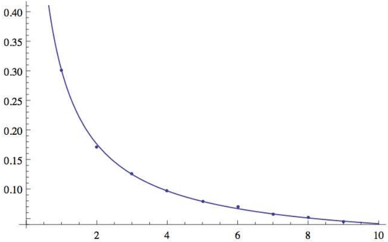


(B) $p = 0.51$ and $N = 10000$.



(B) $p = 0.99$ and $N = 50000$. Benford distribution overlaid.

Counterexample (SMALL '13): $p = \frac{1}{11}$, $1 - p = \frac{10}{11}$.

Benford Analysis

At N^{th} level,

- 2^N sticks
- $N+1$ distinct lengths:

$$p^N\left(\frac{1-p}{p}\right)^j,\ \ j\in\{0,\dots,N\},\ \text{have}\ \binom{N}{j}\ \text{times}.$$

Benford Analysis

At N^{th} level,

- 2^N sticks
- $N+1$ distinct lengths:

$$p^N\left(\frac{1-p}{p}\right)^j,\ j\in\{0,\dots,N\},\ \text{have}\ \binom{N}{j}\ \text{times.}$$

(Weighted) Geometric with ratio $\frac{1-p}{p}=10^y$;
behavior depends on irrationality of y!

Benford Analysis

At N^{th} level,

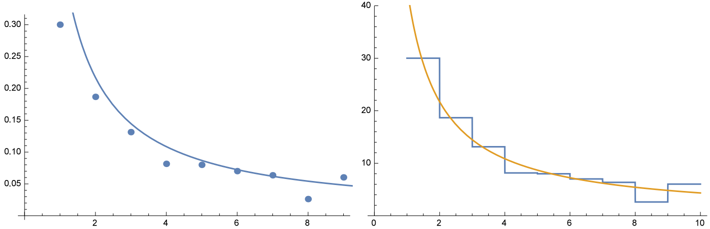
- 2^N sticks
- $N+1$ distinct lengths:

$$p^N\left(\frac{1-p}{p}\right)^j, \ j \in \{0,\dots,N\}, \text{ have } \binom{N}{j} \text{ times.}$$

(Weighted) Geometric with ratio $\frac{1-p}{p} = 10^y$; behavior depends on irrationality of y!

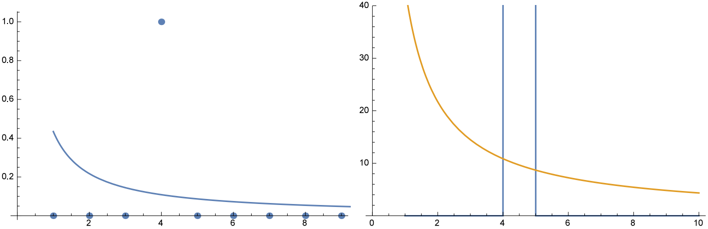
Theorem: Benford if and only if y irrational.

Examples



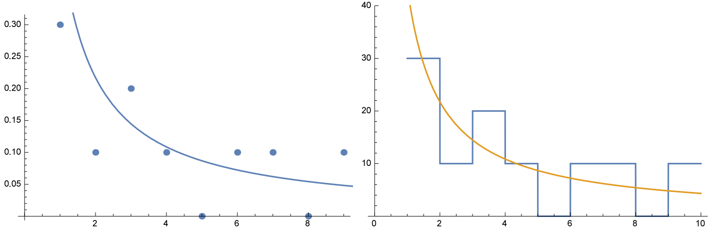
$p = 3/11$, 1000 levels; $y = \log_{10}(8/3) \notin \mathbb{Q}$ (irrational)

Examples



$p = 1/11$, 1000 levels; $y = 1 \in \mathbb{Q}$ (rational)

Examples



$$p = 1/(1+10^{33/10}),\ 1000 \text{ levels};\ y = 33/10 \in \mathbb{Q}$$
(rational)

Conclusions

Current / Future Investigations

- Develop more sophisticated tests for fraud.
- Study digits of other systems.
  ◇ Break rods of variable integer length, each piece breaks until is a prime, or a square, ...
  ◇ Fragmentation models in higher dimensions.

Conclusions and Future Investigations

- See many different systems exhibit Benford behavior.
- Ingredients of proofs (logarithms, equidistribution).
- Applications to fraud detection / data integrity.

References

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- A. K. Adhikari and B. P. Sarkar, *Distribution of most significant digit in certain functions whose arguments are random variables*, Sankhyā: The Indian J. of Statistics, Series B **30** (1968), 47–58.
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The Riemann Zeta Function $\zeta(s)$ and Benford's Law

Riemann Zeta Function (for real part of s greater than 1)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

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$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \operatorname{Re}(s) > 1.$$

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Geometric Series Formula: $(1-x)^{-1} = 1 + x + x^2 + \cdots$.
Unique Factorization: $n = p_1^{r_1} \cdots p_m^{r_m}$.

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$$\prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \left[1 + \frac{1}{2^s} + \left(\frac{1}{2^s}\right)^2 + \cdots\right]\left[1 + \frac{1}{3^s} + \left(\frac{1}{3^s}\right)^2 + \cdots\right]\cdots$$

$$= \sum_n \frac{1}{n^s}.$$

Riemann Zeta Function (cont)

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1$$
$$\pi(x) = \#\{p : p \text{ is prime}, p \le x\}$$

Properties of $\zeta(s)$ and Primes:

Riemann Zeta Function (cont)

$$\begin{aligned}\zeta(s) &= \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1 \\ \pi(x) &= \#\{p : p \text{ is prime}, p \leq x\}\end{aligned}$$

Properties of $\zeta(s)$ and Primes:

- $\lim_{s \to 1^+} \zeta(s) = \infty$, $\pi(x) \to \infty$.

Riemann Zeta Function (cont)

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1$$

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Properties of $\zeta(s)$ and Primes:

- $\lim_{s \to 1^+} \zeta(s) = \infty$, $\pi(x) \to \infty$.
- $\zeta(2) = \frac{\pi^2}{6}$, $\pi(x) \to \infty$.

The Riemann Zeta Function and Benford's Law

$$\left|\zeta\left(\frac{1}{2}+\mathrm{i}\frac{k}{4}\right)\right|,\ k\in\{0,1,\dots,65535\}.$$

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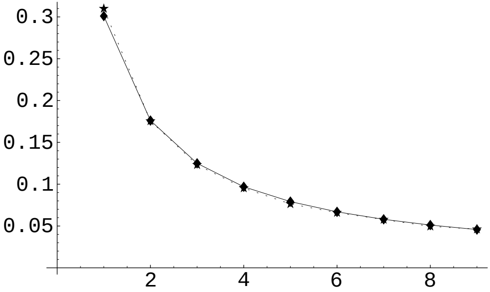
$$\left|\zeta\left(\frac{1}{2}+\mathrm{i}\frac{k}{4}\right)\right|,\ k \in \{0, 1, \dots, 65535\}.$$



First digits of $\left|\zeta\left(\frac{1}{2}+\mathrm{i}\frac{k}{4}\right)\right|$ versus Benford's law.

Proof Sketch: 'Good' L-Functions

We say an L-function is *good* if:

- Euler product:

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_p \prod_{j=1}^{d} \left(1 - \alpha_{f,j}(p)p^{-s}\right)^{-1}.$$

- $L(s,f)$ has a meromorphic continuation to $\mathbb{C}$, is of finite order, and has at most finitely many poles (all on the line $\mathrm{Re}(s) = 1$).
- Functional equation:

$$e^{i\omega}G(s)L(s,f) \;=\; e^{-i\omega}\overline{G(1-\overline{s})L(1-\overline{s})},$$

where $\omega \in \mathbb{R}$ and

$$G(s) \;=\; Q^s \prod_{i=1}^{h} \Gamma(\lambda_i s + \mu_i)$$

with $Q, \lambda_i > 0$ and $\mathrm{Re}(\mu_i) \geq 0$.

Proof Sketch: 'Good' L-Functions (cont)

- For some $\aleph > 0$, $c \in \mathbb{C}$, $x \geq 2$ we have

$$\sum_{p \leq x} \frac{|a_f(p)|^2}{p} \;=\; \aleph \log\log x + c + O\left(\frac{1}{\log x}\right).$$

- The $\alpha_{f,j}(p)$ are (Ramanujan-Petersson) tempered: $|\alpha_{f,j}(p)| \leq 1$.
- If $N(\sigma, T)$ is the number of zeros ρ of $L(s)$ with $\mathrm{Re}(\rho) \geq \sigma$ and $\mathrm{Im}(\rho) \in [0, T]$, then for some $\beta > 0$ we have

$$N(\sigma, T) \;=\; O\left(T^{1-\beta\left(\sigma - \frac{1}{2}\right)} \log T\right).$$

Known in some cases, such as $\zeta(s)$ and Hecke cuspidal forms of full level and even weight $k > 0$.

Log-Normal Law (Hejhal, Laurinčikas, Selberg)

Log-Normal Law

$$\frac{\mu\left(\{t \in [T, 2T] : \log|L(\sigma + it, f)| \in [a, b]\}\right)}{T} =$$

$$\frac{1}{\sqrt{\psi(\sigma, T)}} \int_a^b e^{-\pi u^2/\psi(\sigma,T)} du + \mathrm{Error}$$

$$\psi(\sigma, T) = \aleph \log\left[\min\left(\log T, \frac{1}{\sigma - \frac{1}{2}}\right)\right] + O(1)$$

$$\frac{1}{2} \le \sigma \le \frac{1}{2} + \frac{1}{\log^\delta T}, \quad \delta \in (0, 1).$$

Result: Values of L-functions and Benford's Law

Theorem (Kontorovich and M–, 2005)

$L(s,f)$ a good L-function, as $T \to \infty$, $L(\sigma_T + it, f)$ is Benford.

Ingredients

- Approximate $\log L(\sigma_T + it, f)$ with $\sum_{n \le x} \frac{c(n)\Lambda(n)}{\log n} \frac{1}{n^{\sigma_T + it}}$.
- study moments $\int_T^{2T} |\cdot|$, $k \le \log^{1-\delta} T$.
- Montgomery-Vaughan: $\int_T^{2T} \sum a_n n^{-it} \overline{\sum b_m m^{-it}} dt = H \sum a_n \overline{b}_n + O(1)\sqrt{\sum n|a_n|^2 \sum n|b_n|^2}$.

Results: Explicit L-Function Statement

Theorem (Kontorovich-Miller '05)

Let $L(s,f)$ be a good L-function. Fix a $\delta \in (0,1)$. For each T, let $\sigma_T = \frac{1}{2} + \frac{1}{\log^\delta T}$. Then as $T \to \infty$

$$\frac{\mu\left\{t \in [T,2T]:\ M_B\left(|L(\sigma_T + \mathrm{i}t, f)|\right) \leq \tau\right\}}{T} \to \log_B \tau$$

Thus the values of the L-function satisfy Benford's Law in the limit for any base B.

The $3x + 1$ Problem and Benford's Law

$3x+1$ Problem

- Kakutani (conspiracy), Erdös (not ready).
- x odd, $T(x) = \frac{3x+1}{2^k}$, $2^k || 3x+1$.
- Conjecture: for some $n = n(x)$, $T^n(x) = 1$.

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- $7 \to_1 11 \to_1 17 \to_2 13 \to_3 5 \to_4 1 \to_2 1$,
  2-path $(1, 1)$, 5-path $(1, 1, 2, 3, 4)$.
  m-path: $(k_1, \dots, k_m)$.

Heuristic Proof of $3x+1$ Conjecture

$$\begin{aligned}
a_{n+1} &= T(a_n) \\
\mathbb{E}[\log a_{n+1}] &\approx \sum_{k=1}^{\infty} \frac{1}{2^k} \log\left(\frac{3a_n}{2^k}\right) \\
&= \log a_n + \log 3 - \log 2 \sum_{k=1}^{\infty} \frac{k}{2^k} \\
&= \log a_n + \log\left(\frac{3}{4}\right).
\end{aligned}$$

Geometric Brownian Motion, drift $\log(3/4) < 1$.

$3x+1$ and Benford

Theorem (Kontorovich and M–, 2005)

As $m \to \infty$, $x_m/(3/4)^m x_0$ is Benford.

Theorem (Lagarias-Soundararajan, 2006)

$X \geq 2^N$, for all but at most $c(B)N^{-1/36}X$ initial seeds the distribution of the first N iterates of the $3x+1$ map are within $2N^{-1/36}$ of the Benford probabilities.

Structure Theorem: Sinai, Kontorovich-Sinai

$\mathbb{P}(A) = \lim_{N\to\infty} \frac{\#\{n \le N : n \equiv 1,5 \bmod 6, n \in A\}}{\#\{n \le N : n \equiv 1,5 \bmod 6\}}$.
$(k_1, \dots, k_m)$: two full arithm progressions:
$6 \cdot 2^{k_1+\cdots+k_m} p + q$.

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$$\mathbb{P}\left(\frac{\log_2\left[\frac{x_m}{\left(\frac{3}{4}\right)^m x_0}\right]}{\sqrt{2m}} \le a\right) = \mathbb{P}\left(\frac{S_m - 2m}{\sqrt{2m}} \le a\right)$$

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$$\mathbb{P}\left(\frac{\log_2\left[\frac{x_m}{\left(\frac{3}{4}\right)^m x_0}\right]}{(\log_2 B)\sqrt{2m}} \le a\right) = \mathbb{P}\left(\frac{S_m - 2m}{(\log_2 B)\sqrt{2m}} \le a\right)$$

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Sketch of the proof of Benfordness

- Failed Proof: lattices, bad errors.

- CLT: $(S_m - 2m)/\sqrt{2m} \to N(0,1)$:
$$\mathbb{P}\left(S_m - 2m = k\right) = \frac{\eta(k/\sqrt{m})}{\sqrt{m}} + O\left(\frac{1}{g(m)\sqrt{m}}\right).$$

- Quantified Equidistribution:
$I_\ell = \{\ell M, \dots, (\ell+1)M - 1\}$, $M = m^c$, $c < 1/2$
$k_1, k_2 \in I_\ell$: $\left|\eta\left(\frac{k_1}{\sqrt{m}}\right) - \eta\left(\frac{k_2}{\sqrt{m}}\right)\right|$ small
$C = \log_B 2$ of irrationality type $\kappa < \infty$:
$$\#\{k \in I_\ell : \overline{kC} \in [a,b]\} = M(b-a) + O(M^{1+\epsilon-1/\kappa}).$$

Irrationality Type

Irrationality type

α has irrationality type κ if κ is the supremum of all γ with

$$\underline{\lim}_{q\to\infty} q^{\gamma+1} \min_p \left|\alpha - \frac{p}{q}\right| = 0.$$

- Algebraic irrationals: type 1 (Roth's Thm).
- Theory of Linear Forms: $\log_B 2$ of finite type.

Linear Forms

Theorem (Baker)

$\alpha_1, \dots, \alpha_n$ *algebraic numbers height* $A_j \geq 4$*,*
$\beta_1, \dots, \beta_n \in \mathbb{Q}$ *with height at most* $B \geq 4$*,*

$$\Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n.$$

If $\Lambda \neq 0$ *then* $|\Lambda| > B^{-C\Omega \log \Omega'}$*, with*
$d = [\mathbb{Q}(\alpha_i, \beta_j) : \mathbb{Q}]$*,* $C = (16nd)^{200n}$*,*
$\Omega = \prod_j \log A_j$*,* $\Omega' = \Omega / \log A_n$*.*

Gives $\log_{10} 2$ of finite type, with $\kappa < 1.2 \cdot 10^{602}$:

$$|\log_{10} 2 - p/q| = |q \log 2 - p \log 10| \,/ q \log 10.$$

Quantified Equidistribution

Theorem (Erdös-Turan)

$$D_N = \frac{\sup_{[a,b]} |N(b-a) - \#\{n \le N : x_n \in [a,b]\}|}{N}$$

There is a C such that for all m:

$$D_N \le C \cdot \left(\frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \right)$$

Proof of Erdös-Turan

Consider special case $x_n = n\alpha$, $\alpha \notin \mathbb{Q}$.

- Exponential sum $\le \frac{1}{|\sin(\pi h\alpha)|} \le \frac{1}{2||h\alpha||}$.
- Must control $\sum_{h=1}^{m} \frac{1}{h||h\alpha||}$, see irrationality type enter.
- type κ, $\sum_{h=1}^{m} \frac{1}{h||h\alpha||} = O\left(m^{\kappa-1+\epsilon}\right)$, take $m = \lfloor N^{1/\kappa} \rfloor$.

$3x+1$ Data: random 10,000 digit number, $2^k || 3x+1$

80,514 iterations ($(4/3)^n = a_0$ predicts 80,319);
$\chi^2 = 13.5$ (5% 15.5).

Digit	Number	Observed	Benford
1	24251	0.301	0.301
2	14156	0.176	0.176
3	10227	0.127	0.125
4	7931	0.099	0.097
5	6359	0.079	0.079
6	5372	0.067	0.067
7	4476	0.056	0.058
8	4092	0.051	0.051
9	3650	0.045	0.046

$3x+1$ Data: random 10,000 digit number, $2|3x+1$

241,344 iterations, $\chi^2 = 11.4$ (5% 15.5).

Digit	Number	Observed	Benford
1	72924	0.302	0.301
2	42357	0.176	0.176
3	30201	0.125	0.125
4	23507	0.097	0.097
5	18928	0.078	0.079
6	16296	0.068	0.067
7	13702	0.057	0.058
8	12356	0.051	0.051
9	11073	0.046	0.046

$5x+1$ Data: random 10,000 digit number, $2^k || 5x+1$

27,004 iterations, $\chi^2 = 1.8$ (5% 15.5).

Digit	Number	Observed	Benford
1	8154	0.302	0.301
2	4770	0.177	0.176
3	3405	0.126	0.125
4	2634	0.098	0.097
5	2105	0.078	0.079
6	1787	0.066	0.067
7	1568	0.058	0.058
8	1357	0.050	0.051
9	1224	0.045	0.046

$5x + 1$ Data: random 10,000 digit number, $2|5x + 1$

241,344 iterations, $\chi^2 = 3 \cdot 10^{-4}$ (5% 15.5).

Digit	Number	Observed	Benford
1	72652	0.301	0.301
2	42499	0.176	0.176
3	30153	0.125	0.125
4	23388	0.097	0.097
5	19110	0.079	0.079
6	16159	0.067	0.067
7	13995	0.058	0.058
8	12345	0.051	0.051
9	11043	0.046	0.046