Benford's law, or: Why the IRS cares about number theory!

Steven J. Miller
Smith, Mount Holyoke and Williams Colleges

sjm1@williams.edu
http://www.williams.edu/Mathematics/sjmiller/

Mount Holyoke, February 8, 2012

Interesting Question

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Plausible answers:10%, 11%

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Plausible answers:10%, 11%, about 30%.

Summary

- State Benford's Law.
- Discuss examples and applications.
- Sketch proofs.
- Describe open problems.

Caveats!

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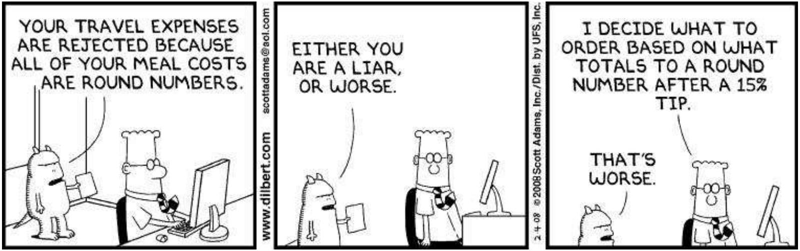

Benford's Law: Newcomb (1881), Benford (1938)

Statement

For many data sets, probability of observing a first digit of d base B is $\log_B\left(\frac{d+1}{d}\right)$; base 10 about 30% are 1s.

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  - ◇ Oscillates between $1/9$ and $5/9$ with first digit 1.
  - ◇ Many streets of different sizes: close to Benford.

Examples

- recurrence relations
- special functions (such as $n!$)
- iterates of power, exponential, rational maps
- products of random variables
- L-functions, characteristic polynomials
- iterates of the $3x + 1$ map
- differences of order statistics
- hydrology and financial data
- many hierarchical Bayesian models

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \quad (\text{if } \mathrm{Re}(s) > 1).$$

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$$\left|\zeta\left(\tfrac{1}{2} + \mathrm{i}\tfrac{k}{4}\right)\right|, k \in \{0, 1, \ldots, 65535\}.$$

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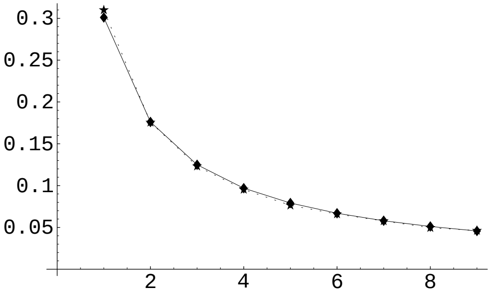

Applications

- analyzing round-off errors
- determining the optimal way to store numbers
- detecting tax and image fraud, and data integrity

General Theory

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Key observation: $\log_{10}(x) = \log_{10}(\widetilde{x}) \bmod 1$ if and only if x and $\widetilde{x}$ have the same leading digits. Thus often study $y = \log_{10} x$.

Equidistribution and Benford's Law

Equidistribution

$\{y_n\}_{n=1}^{\infty}$ is equidistributed modulo 1 if probability $y_n \bmod 1 \in [a,b]$ tends to $b-a$:

$$\frac{\#\{n \leq N : y_n \bmod 1 \in [a,b]\}}{N} \to b-a.$$

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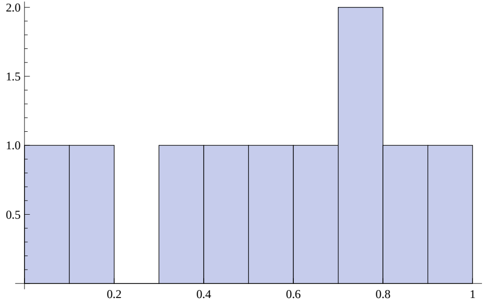
  Proof: if rational: $2 = 10^{p/q}$.
  Thus $2^q = 10^p$ or $2^{q-p} = 5^p$, impossible.

Example of Equidistribution: $n\sqrt{\pi} \bmod 1$



$n\sqrt{\pi} \bmod 1$ for $n \leq 10$

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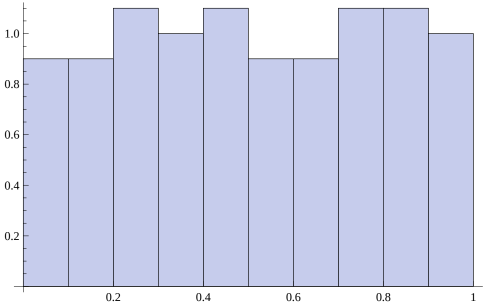

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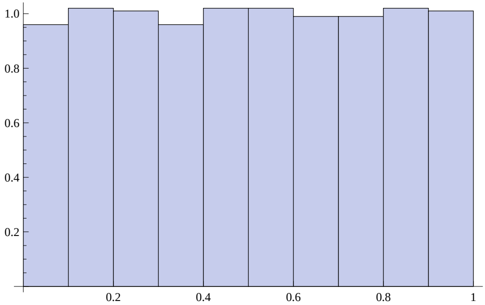

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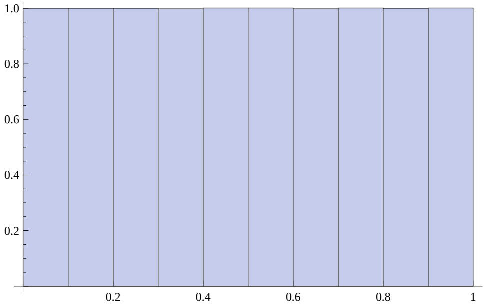

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Logarithms and Benford's Law

Fundamental Equivalence

Data set $\{x_i\}$ is Benford base B if $\{y_i\}$ is equidistributed mod 1, where $y_i = \log_B x_i$.

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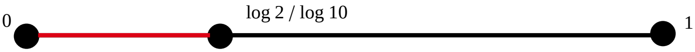

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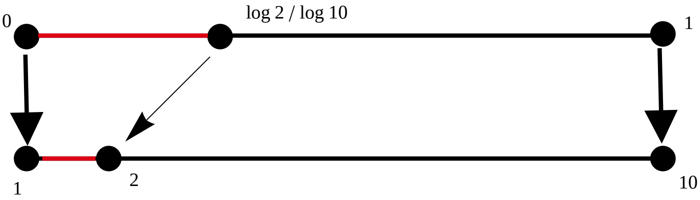

Examples

- 2^n is Benford base 10 as $\log_{10} 2 \notin \mathbb{Q}$.

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- Most linear recurrence relations Benford:
  - $\diamond$ $a_{n+1} = 2a_n - a_{n-1}$
  - $\diamond$ take $a_0 = a_1 = 1$ or $a_0 = 0$, $a_1 = 1$.

Digits of 2^n

First 60 values of 2^n (only displaying 30)

			digit	#	Obs Prob	Benf Prob
1	1024	1048576	1	18	.300	.301
2	2048	2097152	2	12	.200	.176
4	4096	4194304	3	6	.100	.125
8	8192	8388608	4	6	.100	.097
16	16384	16777216	5	6	.100	.079
32	32768	33554432	6	4	.067	.067
64	65536	67108864	7	2	.033	.058
128	131072	134217728	8	5	.083	.051
256	262144	268435456	9	1	.017	.046
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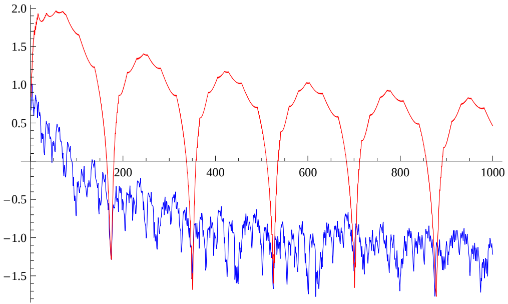
Logarithms and Benford's Law

χ^2 values for α^n, $1 \leq n \leq N$ (5% 15.5).

N	$\chi^2(\gamma)$	$\chi^2(e)$	$\chi^2(\pi)$
100	0.72	0.30	46.65
200	0.24	0.30	8.58
400	0.14	0.10	10.55
500	0.08	0.07	2.69
700	0.19	0.04	0.05
800	0.04	0.03	6.19
900	0.09	0.09	1.71
1000	0.02	0.06	2.90

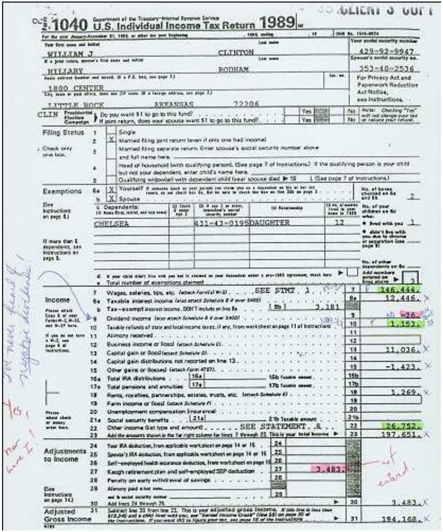
Logarithms and Benford's Law: Base 10

$\log(\chi^2)$ vs N for π^n (red) and e^n (blue), $n \in \{1, \dots, N\}$. Note $\pi^{175} \approx 1.0028 \cdot 10^{87}$, (5%, $\log(\chi^2) \approx 2.74$).



Applications

Applications for the IRS: Detecting Fraud


CLIENT'S COPY

1040 Department of the Treasury—Internal Revenue Service
U.S. Individual Income Tax Return 1989

WILLIAM J CLINTON 429-92-9947
HILLARY RODHAM 353-40-2536
1800 CENTER
LITTLE ROCK ARKANSAS 72206

Presidential Election Campaign: Do you want $1 to go to this fund? If joint return, does your spouse want $1 to go to this fund?

Filing Status
1 Single
2 X Married filing joint return (even if only one had income)
3 Married filing separate return. Enter spouse's social security number above and full name here.
4 Head of household (with qualifying person). (See page 7 of Instructions.) If the qualifying person is your child but not your dependent, enter child's name here.
5 Qualifying widow(er) with dependent child (year spouse died ▶ 19). (See page 7 of Instructions.)

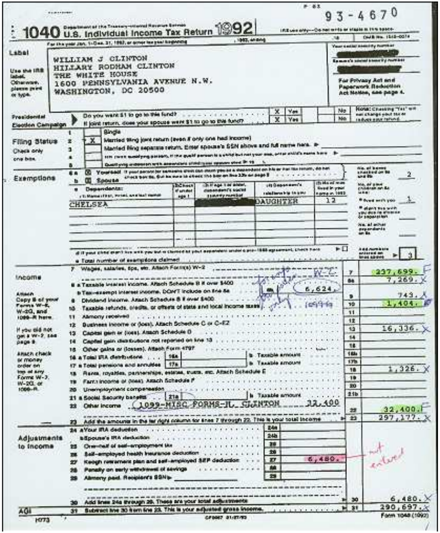
Exemptions
6a X Yourself
b X Spouse
c Dependents: CHELSEA 431-43-0195 DAUGHTER 12
e Total number of exemptions claimed 3

Income
7 Wages, salaries, tips, etc. SEE STMT 1 — 146,444.
8a Taxable interest income — 12,446.
b Tax-exempt interest income. DON'T include on line 8a — 3,181
9 Dividend income — 26.
10 Taxable refunds of state and local income taxes, if any, from worksheet on page 11 of Instructions — 1,153.
11 Alimony received
12 Business income or (loss)
13 Capital gain or (loss) — 11,036.
14 Capital gain distributions not reported on line 13
15 Other gains or (losses) — -1,423.
16a Total IRA distributions / 16b Taxable amount
17a Total pensions and annuities / 17b Taxable amount
18 Rents, royalties, partnerships, estates, trusts, etc. — 1,269.
19 Farm income or (loss)
20 Unemployment compensation (insurance)
21a Social security benefits / 21b Taxable amount
22 Other income SEE STATEMENT 8 — 26,752.
23 Add the amounts shown in the far right column for lines 7 through 22. This is your total income — 197,651.

Adjustments to Income
24 Your IRA deduction, from applicable worksheet on page 14 or 15
25 Spouse's IRA deduction, from applicable worksheet on page 14 or 15
26 Self-employed health insurance deduction, from worksheet on page 15
27 Keogh retirement plan and self-employed SEP deduction — 3,483.
28 Penalty on early withdrawal of savings
29 Alimony paid
30 Add lines 24 through 29 — 3,483.

Adjusted Gross Income
31 Subtract line 30 from line 21. This is your adjusted gross income. — 194,168.

Applications for the IRS: Detecting Fraud


93-4670

1040 U.S. Individual Income Tax Return 1992

Label

WILLIAM J CLINTON
HILLARY RODHAM CLINTON
THE WHITE HOUSE
1600 PENNSYLVANIA AVENUE N.W.
WASHINGTON, DC 20500

Your social security number [redacted]
Spouse's social security number [redacted]

For Privacy Act and Paperwork Reduction Act Notice, see page 4.

Presidential Election Campaign
Do you want $1 to go to this fund? X Yes
If joint return, does your spouse want $1 to go to this fund? X Yes

Filing Status
1 Single
2 X Married filing joint return (even if only one had income)
3 Married filing separate return. Enter spouse's SSN above and full name here.

Exemptions
6a X Yourself
b X Spouse
c Dependents:
CHELSEA [redacted] DAUGHTER 12

e Total number of exemptions claimed 3

Income
7 Wages, salaries, tips, etc. Attach Form(s) W-2 — 237,699.
8a Taxable interest income. Attach Schedule B if over $400 — 7,269.
b Tax-exempt interest income. DON'T include on line 8a — 6,624
9 Dividend income. Attach Schedule B if over $400 — 743.
10 Taxable refunds, credits, or offsets of state and local income taxes — 1,404.
11 Alimony received
12 Business income or (loss). Attach Schedule C or C-EZ
13 Capital gain or (loss). Attach Schedule D — 16,336.
14 Capital gain distributions not reported on line 13
15 Other gains or (losses). Attach Form 4797
16a Total IRA distributions
17a Total pensions and annuities
18 Rents, royalties, partnerships, estates, trusts, etc. Attach Schedule E — 1,326.
19 Farm income or (loss). Attach Schedule F
20 Unemployment compensation
21a Social Security benefits
22 Other income (1099-MISC FORMS-H. CLINTON 32,400) — 32,400.
23 Add the amounts in the far right column for lines 7 through 22. This is your total income — 297,177.

Adjustments to Income
24a Your IRA deduction
b Spouse's IRA deduction
25 One-half of self-employment tax
26 Self-employed health insurance deduction
27 Keogh retirement plan and self-employed SEP deduction — 6,480.
28 Penalty on early withdrawal of savings
29 Alimony paid.
30 Add lines 24a through 29. These are your total adjustments — 6,480.

AGI
31 Subtract line 30 from line 23. This is your adjusted gross income — 290,697.

Form 1040 (1992)

Detecting Fraud

Bank Fraud

- Audit of a bank revealed huge spike of numbers starting with

Detecting Fraud

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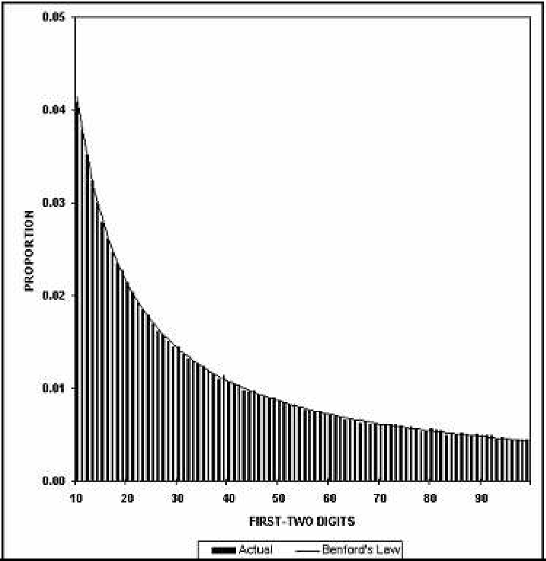
- Write-off limit of \$5,000. Officer had friends applying for credit cards, ran up balances just under \$5,000 then he would write the debts off.

Data Integrity: Stream Flow Statistics: 130 years, 457,440 records



Election Fraud: Iran 2009

Numerous protests/complaints over Iran's 2009 elections.

Lot of analysis; data moderately suspicious:

- First and second leading digits;
- Last two digits (should almost be uniform);
- Last two digits differing by at least 2.

Warning: enough tests, even if nothing wrong will find a suspicious result (but when all tests are on the boundary...).

The $3x + 1$ Problem
and
Benford's Law

$3x+1$ Problem

- Kakutani (conspiracy), Erdös (not ready).
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  2-path $(1, 1)$, 5-path $(1, 1, 2, 3, 4)$.
  m-path: $(k_1, \dots, k_m)$.

Heuristic Proof of $3x + 1$ Conjecture

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&= \log a_n + \log\left(\frac{3}{4}\right).
\end{aligned}$$

Geometric Brownian Motion, drift $\log(3/4) < 1$.

$3x+1$ and Benford

Theorem (Kontorovich and M–, 2005)

As $m \to \infty$, $x_m/(3/4)^m x_0$ is Benford.

Theorem (Lagarias-Soundararajan 2006)

$X \geq 2^N$, for all but at most $c(B)N^{-1/36}X$ initial seeds the distribution of the first N iterates of the $3x+1$ map are within $2N^{-1/36}$ of the Benford probabilities.

$3x+1$ Data: random 10,000 digit number, $2^k || 3x+1$

80,514 iterations ($(4/3)^n = a_0$ predicts 80,319);
$\chi^2 = 13.5$ (5% 15.5).

Digit	Number	Observed	Benford
1	24251	0.301	0.301
2	14156	0.176	0.176
3	10227	0.127	0.125
4	7931	0.099	0.097
5	6359	0.079	0.079
6	5372	0.067	0.067
7	4476	0.056	0.058
8	4092	0.051	0.051
9	3650	0.045	0.046

$3x+1$ Data: random 10,000 digit number, $2|3x+1$

241,344 iterations, $\chi^2 = 11.4$ (5% 15.5).

Digit	Number	Observed	Benford
1	72924	0.302	0.301
2	42357	0.176	0.176
3	30201	0.125	0.125
4	23507	0.097	0.097
5	18928	0.078	0.079
6	16296	0.068	0.067
7	13702	0.057	0.058
8	12356	0.051	0.051
9	11073	0.046	0.046

$5x+1$ Data: random 10,000 digit number, $2^k || 5x+1$

27,004 iterations, $\chi^2 = 1.8$ (5% 15.5).

Digit	Number	Observed	Benford
1	8154	0.302	0.301
2	4770	0.177	0.176
3	3405	0.126	0.125
4	2634	0.098	0.097
5	2105	0.078	0.079
6	1787	0.066	0.067
7	1568	0.058	0.058
8	1357	0.050	0.051
9	1224	0.045	0.046

$5x+1$ Data: random 10,000 digit number, $2|5x+1$

241,344 iterations, $\chi^2 = 3 \cdot 10^{-4}$ (5% 15.5).

Digit	Number	Observed	Benford
1	72652	0.301	0.301
2	42499	0.176	0.176
3	30153	0.125	0.125
4	23388	0.097	0.097
5	19110	0.079	0.079
6	16159	0.067	0.067
7	13995	0.058	0.058
8	12345	0.051	0.051
9	11043	0.046	0.046

Conclusions

Current / Future Investigations

- Develop more sophisticated tests for fraud.

- Study digits of other systems.
  ◇ Break rod of fixed length a variable number of times.
  ◇ Break rods of variable length a variable number of times.
  ◇ Break rods of variable length, each piece then breaks with given probability.
  ◇ Break rods of variable integer length, each piece breaks until is a prime, or a square, ...

Conclusions and Future Investigations

- See many different systems exhibit Benford behavior.
- Ingredients of proofs (logarithms, equidistribution).
- Applications to fraud detection / data integrity.

References

A. K. Adhikari, *Some results on the distribution of the most significant digit*, Sankhyā: The Indian Journal of Statistics, Series B **31** (1969), 413–420.

A. K. Adhikari and B. P. Sarkar, *Distribution of most significant digit in certain functions whose arguments are random variables*, Sankhyā: The Indian Journal of Statistics, Series B **30** (1968), 47–58.

R. N. Bhattacharya, *Speed of convergence of the n-fold convolution of a probability measure ona compact group*, Z. Wahrscheinlichkeitstheorie verw. Geb. **25** (1972), 1–10.

F. Benford, *The law of anomalous numbers*, Proceedings of the American Philosophical Society **78** (1938), 551–572.

A. Berger, Leonid A. Bunimovich and T. Hill, *One-dimensional dynamical systems and Benford's Law*, Trans. Amer. Math. Soc. **357** (2005), no. 1, 197–219.

A. Berger and T. Hill, *Newton's method obeys Benford's law*, The Amer. Math. Monthly **114** (2007), no. 7, 588-601.

J. Boyle, *An application of Fourier series to the most significant digit problem* Amer. Math. Monthly **101** (1994), 879–886.

J. Brown and R. Duncan, *Modulo one uniform distribution of the sequence of logarithms of certain recursive sequences*, Fibonacci Quarterly **8** (1970) 482–486.

P. Diaconis, *The distribution of leading digits and uniform distribution mod 1*, Ann. Probab. **5** (1979), 72–81.

W. Feller, *An Introduction to Probability Theory and its Applications, Vol. II*, second edition, John Wiley & Sons, Inc., 1971.

- R. W. Hamming, *On the distribution of numbers*, Bell Syst. Tech. J. **49** (1970), 1609-1625.
- T. Hill, *The first-digit phenomenon*, American Scientist **86** (1996), 358–363.
- T. Hill, *A statistical derivation of the significant-digit law*, Statistical Science **10** (1996), 354–363.
- P. J. Holewijn, *On the uniform distribuiton of sequences of random variables*, Z. Wahrscheinlichkeitstheorie verw. Geb. **14** (1969), 89–92.
- W. Hurlimann, *Benford's Law from 1881 to 2006: a bibliography*, `http://arxiv.org/abs/math/0607168`.
- D. Jang, J. U. Kang, A. Kruckman, J. Kudo and S. J. Miller, *Chains of distributions, hierarchical Bayesian models and Benford's Law*, Journal of Algebra, Number Theory: Advances and Applications, volume 1, number 1 (March 2009), 37–60.

- E. Janvresse and T. de la Rue, *From uniform distribution to Benford's law*, Journal of Applied Probability **41** (2004) no. 4, 1203–1210.
- A. Kontorovich and S. J. Miller, *Benford's Law, Values of L-functions and the* $3x+1$ *Problem*, Acta Arith. **120** (2005), 269–297.
- D. Knuth, *The Art of Computer Programming, Volume 2: Seminumerical Algorithms*, Addison-Wesley, third edition, 1997.
- J. Lagarias and K. Soundararajan, *Benford's Law for the* $3x+1$ *Function*, J. London Math. Soc. (2) **74** (2006), no. 2, 289–303.
- S. Lang, *Undergraduate Analysis*, 2nd edition, Springer-Verlag, New York, 1997.

P. Levy, *L'addition des variables aléatoires définies sur une circonférence*, Bull. de la S. M. F. **67** (1939), 1–41.

E. Ley, *On the peculiar distribution of the U.S. Stock Indices Digits*, The American Statistician **50** (1996), no. 4, 311–313.

R. M. Loynes, *Some results in the probabilistic theory of asympototic uniform distributions modulo* 1, Z. Wahrscheinlichkeitstheorie verw. Geb. **26** (1973), 33–41.

S. J. Miller, *When the Cramér-Rao Inequality provides no information*, Communications in Information and Systems **7** (2007), no. 3, 265–272.

S. J. Miller and M. Nigrini, *The Modulo* 1 *Central Limit Theorem and Benford's Law for Products*, International Journal of Algebra **2** (2008), no. 3, 119–130.

S. J. Miller and M. Nigrini, *Order Statistics and Benford's law*, International Journal of Mathematics and Mathematical Sciences, Volume 2008 (2008), Article ID 382948, 19 pages.

S. J. Miller and R. Takloo-Bighash, *An Invitation to Modern Number Theory*, Princeton University Press, Princeton, NJ, 2006.

S. Newcomb, *Note on the frequency of use of the different digits in natural numbers*, Amer. J. Math. **4** (1881), 39-40.

M. Nigrini, *Digital Analysis and the Reduction of Auditor Litigation Risk*. Pages 69–81 in *Proceedings of the 1996 Deloitte & Touche / University of Kansas Symposium on Auditing Problems*, ed. M. Ettredge, University of Kansas, Lawrence, KS, 1996.

M. Nigrini, *The Use of Benford's Law as an Aid in Analytical Procedures*, Auditing: A Journal of Practice & Theory, **16** (1997), no. 2, 52–67.

M. Nigrini and S. J. Miller, *Benford's Law applied to hydrology data – results and relevance to other geophysical data*, Mathematical Geology **39** (2007), no. 5, 469–490.

M. Nigrini and S. J. Miller, *Data diagnostics using second order tests of Benford's Law*, Auditing: A Journal of Practice and Theory **28** (2009), no. 2, 305–324.

R. Pinkham, *On the Distribution of First Significant Digits*, The Annals of Mathematical Statistics **32**, no. 4 (1961), 1223-1230.

R. A. Raimi, *The first digit problem*, Amer. Math. Monthly **83** (1976), no. 7, 521–538.

H. Robbins, *On the equidistribution of sums of independent random variables*, Proc. Amer. Math. Soc. **4** (1953), 786–799.

H. Sakamoto, *On the distributions of the product and the quotient of the independent and uniformly distributed random variables*, Tôhoku Math. J. **49** (1943), 243–260.

P. Schatte, *On sums modulo 2π of independent random variables*, Math. Nachr. **110** (1983), 243–261.

- P. Schatte, *On the asymptotic uniform distribution of sums reduced mod 1*, Math. Nachr. **115** (1984), 275–281.
- P. Schatte, *On the asymptotic logarithmic distribution of the floating-point mantissas of sums*, Math. Nachr. **127** (1986), 7–20.
- E. Stein and R. Shakarchi, *Fourier Analysis: An Introduction*, Princeton University Press, 2003.
- M. D. Springer and W. E. Thompson, *The distribution of products of independent random variables*, SIAM J. Appl. Math. **14** (1966) 511–526.
- K. Stromberg, *Probabilities on a compact group*, Trans. Amer. Math. Soc. **94** (1960), 295–309.
- P. R. Turner, *The distribution of leading significant digits*, IMA J. Numer. Anal. **2** (1982), no. 4, 407–412.