Theory and applications of Benford's law to fraud detection, or: Why the IRS should care about number theory!

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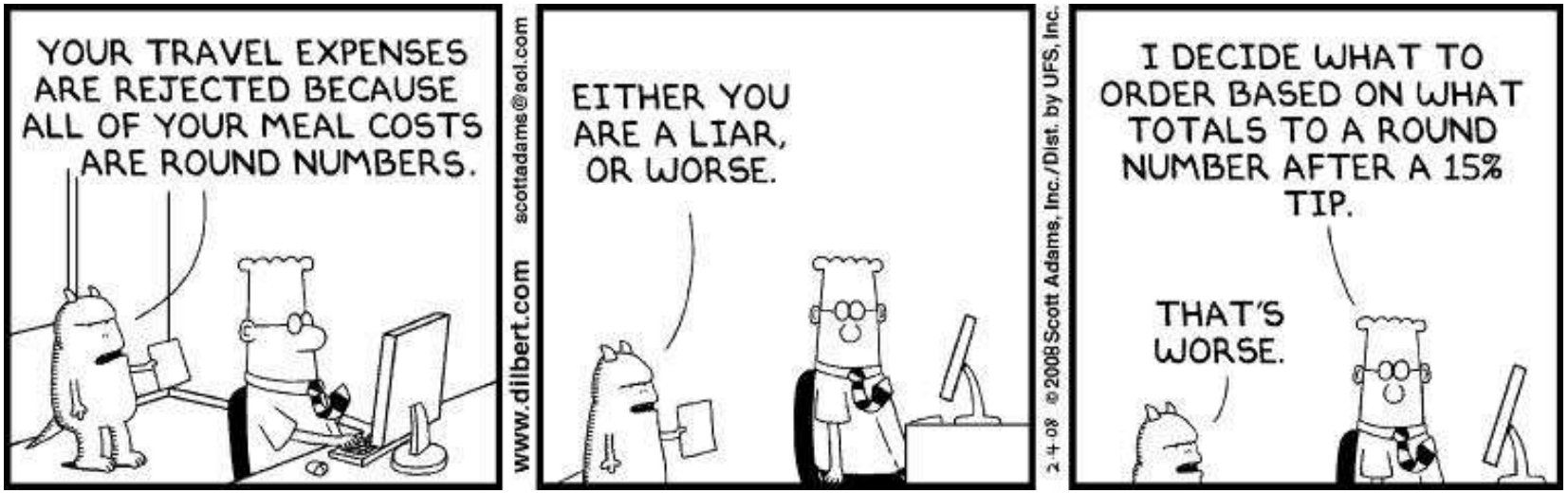
Summary

- Review Benford's Law.
- Discuss examples and applications.
- Sketch proofs.
- Describe open problems.

Caveats!

- Not all fraud can be detected by Benford's Law.
- A math test indicating fraud is *not* proof of fraud: unlikely events, alternate reasons.



Notation

- Logarithms: $\log_B x = y$ means $x = B^y$.
  - Example: $\log_{10} 100 = 2$ as $100 = 10^2$.
  - $\log_B(uv) = \log_B u + \log_B v$.
  - $\log_{10}(100 \cdot 1000) = \log_{10}(100) + \log_{10}(1000)$.
- Set Theory:
  - $\mathbb{Q}$ = rational numbers $= \{p/q : p, q \text{ integers}\}$.
  - $x \in S$ means x belongs to S.
  - $[a, b] = \{x : a \le x \le b\}$.
- Modulo 1:
  - Any x can be written as integer + fraction.
  - x mod 1 means just the fractional part.
  - Example: π mod 1 is about .14159.

Benford's Law: Newcomb (1881), Benford (1938)

Statement

For many data sets, probability of observing a first digit of d base B is $\log_B\left(\frac{d+1}{d}\right)$; base 10 about 30% are 1s.

- Not all data sets satisfy Benford's Law.
  - ◇ Long street $[1, L]$: $L = 199$ versus $L = 999$.
  - ◇ Oscillates between $1/9$ and $5/9$ with first digit 1.
  - ◇ Many streets of different sizes: close to Benford.

Examples

- recurrence relations
- special functions (such as $n!$)
- iterates of power, exponential, rational maps
- products of random variables
- L-functions, characteristic polynomials
- iterates of the $3x+1$ map
- differences of order statistics
- hydrology and financial data
- many hierarchical Bayesian models

Applications

- analyzing round-off errors
- determining the optimal way to store numbers
- detecting tax and image fraud, and data integrity

General Theory

Mantissas

Mantissa: $x = M_{10}(x) \cdot 10^k$, k integer.

$M_{10}(x) = M_{10}(\widetilde{x})$ if and only if x and $\widetilde{x}$ have the same leading digits.

Key observation: $\log_{10}(x) = \log_{10}(\widetilde{x}) \bmod 1$ if and only if x and $\widetilde{x}$ have the same leading digits. Thus often study $y = \log_{10} x$.

Equidistribution and Benford's Law

Equidistribution

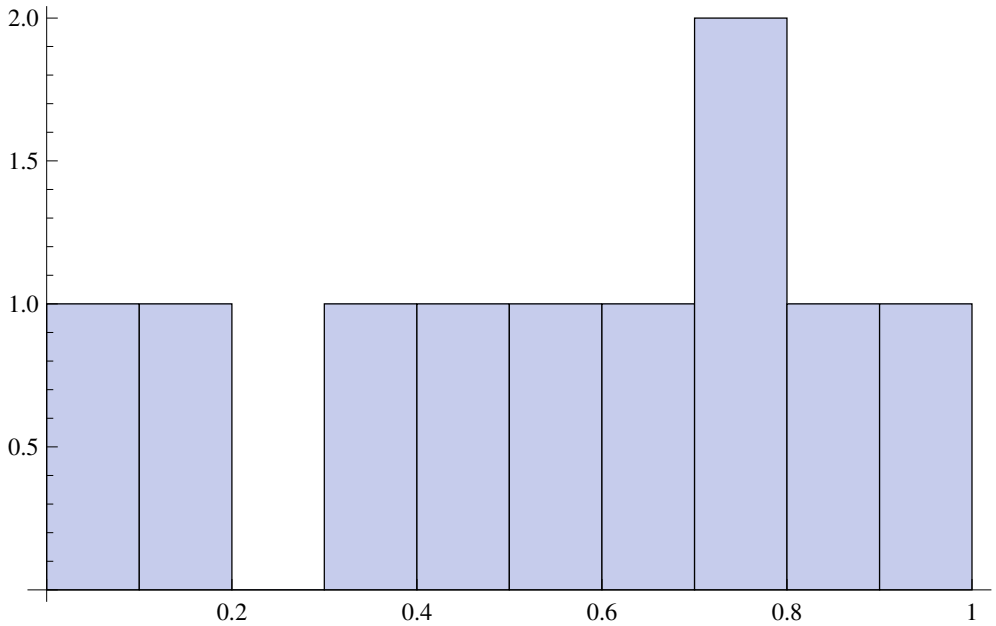
$\{y_n\}_{n=1}^{\infty}$ is equidistributed modulo 1 if probability $y_n \bmod 1 \in [a, b]$ tends to $b - a$:

$$\frac{\#\{n \leq N : y_n \bmod 1 \in [a, b]\}}{N} \rightarrow b - a.$$

- Thm: $\beta \notin \mathbb{Q}$, $n\beta$ is equidistributed mod 1.
- Examples: $\log_{10} 2, \log_{10}\left(\frac{1+\sqrt{5}}{2}\right) \notin \mathbb{Q}$.

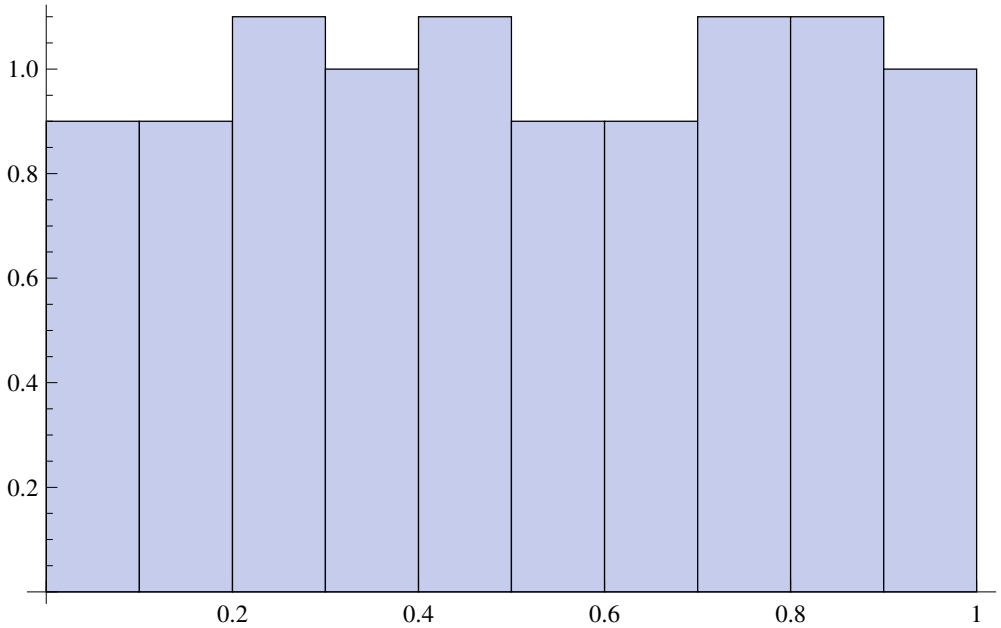
  Proof: if rational: $2 = 10^{p/q}$.
  Thus $2^q = 10^p$ or $2^{q-p} = 5^p$, impossible.

Example of Equidistribution: $n\sqrt{\pi}$ mod 1



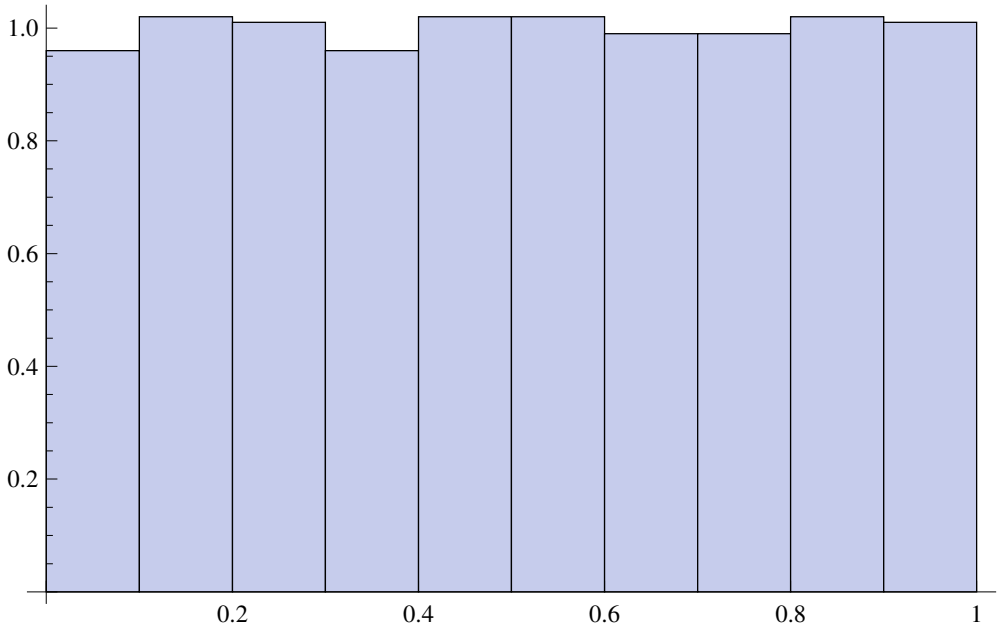
$n\sqrt{\pi}$ mod 1 for $n \leq 10$

Example of Equidistribution: $n\sqrt{\pi}$ mod 1



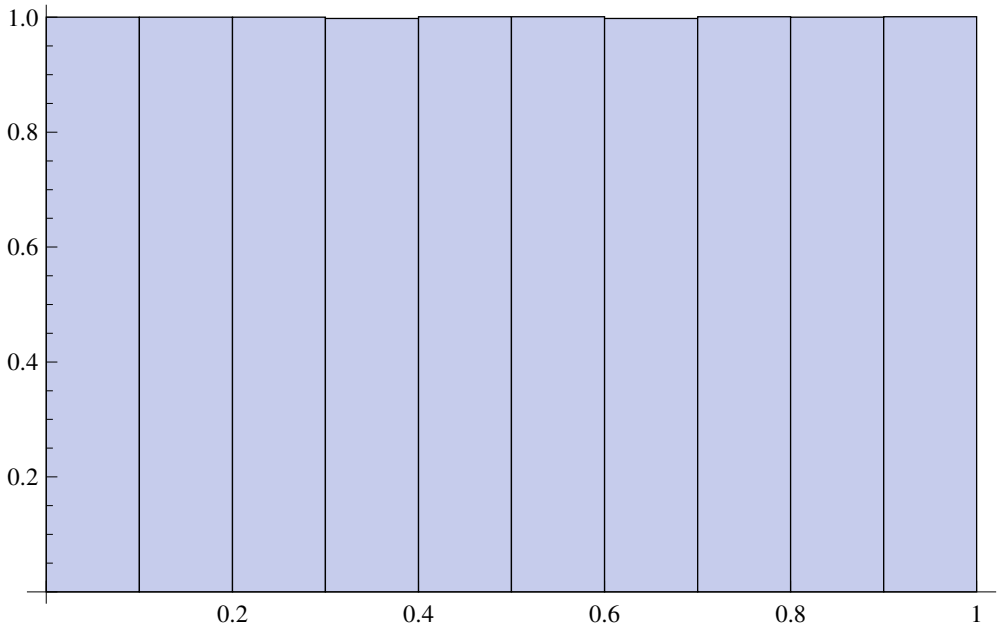
$n\sqrt{\pi}$ mod 1 for $n \leq 100$

Example of Equidistribution: $n\sqrt{\pi}$ mod 1



$n\sqrt{\pi}$ mod 1 for $n \leq 1000$

Example of Equidistribution: $n\sqrt{\pi}$ mod 1



$n\sqrt{\pi}$ mod 1 for $n \leq 10,000$

Denseness

Dense

A sequence $\{z_n\}_{n=1}^{\infty}$ of numbers in $[0, 1]$ is dense if for any interval $[a, b]$ there are infinitely many z_n in $[a, b]$.

- Dirichlet's Box (or Pigeonhole) Principle:
  If $n + 1$ objects are placed in n boxes, at least one box has two objects.

- Denseness of $n\alpha$:
  Thm: If $\alpha \notin \mathbb{Q}$ then $z_n = n\alpha \bmod 1$ is dense.

Proof $n\alpha$ mod 1 dense if $\alpha \notin \mathbb{Q}$

- Enough to show in $[0, b]$ infinitely often for any b.
- Choose any integer $Q > 1/b$.
- Q bins: $\left[0, \frac{1}{Q}\right], \left[\frac{1}{Q}, \frac{2}{Q}\right], \ldots, \left[\frac{Q-1}{Q}, Q\right]$.
- $Q+1$ objects: $\{\alpha \bmod 1, 2\alpha \bmod 1, \ldots, (Q+1)\alpha \bmod 1\}$.
- Two in same bin, say $q_1\alpha$ mod 1 and $q_2\alpha$ mod 1.
- Exists integer p with $0 < q_2\alpha - q_1\alpha - p < \frac{1}{Q}$.
- Get $(q_2 - q_1)\alpha \bmod 1 \in [0, b]$.

Logarithms and Benford's Law

Fundamental Equivalence

Data set $\{x_i\}$ is Benford base B if $\{y_i\}$ is equidistributed mod 1, where $y_i = \log_B x_i$.

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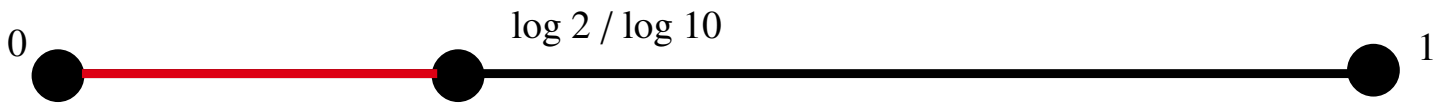

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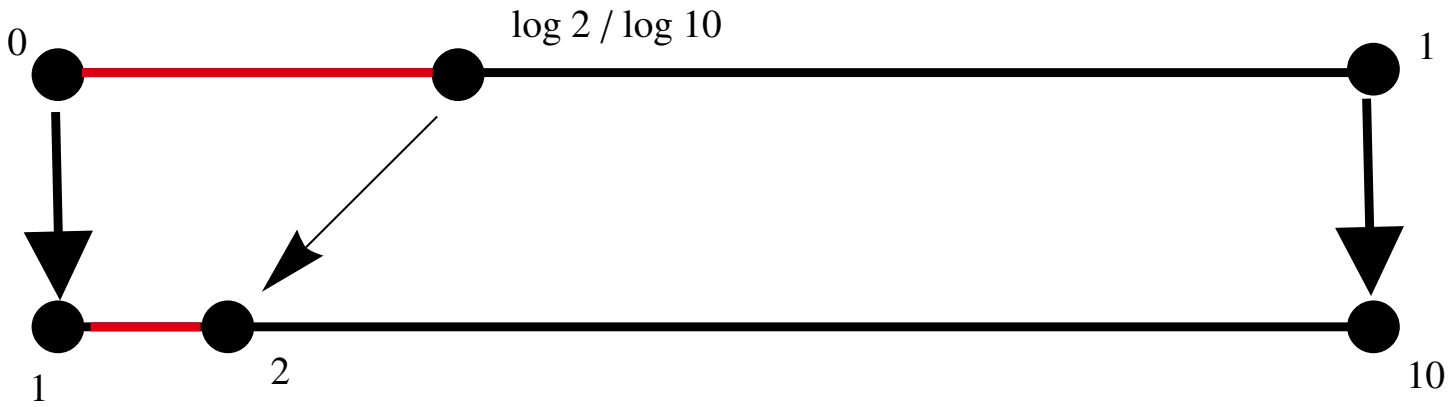

Logarithms and Benford's Law

Fundamental Equivalence

Data set $\{x_i\}$ is Benford base B if $\{y_i\}$ is equidistributed mod 1, where $y_i = \log_B x_i$.

Proof:

- $x = M_B(x) \cdot B^k$ for some $k \in \mathbb{Z}$.
- $\mathrm{FD}_B(x) = d$ iff $d \leq M_B(x) < d+1$.
- $\log_B d \leq y < \log_B(d+1)$, $y = \log_B x \bmod 1$.
- If $Y \sim \mathrm{Unif}(0,1)$ then above probability is $\log_B\left(\frac{d+1}{d}\right)$.

Examples

- 2^n is Benford base 10 as $\log_{10} 2 \notin \mathbb{Q}$.
- Fibonacci numbers are Benford base 10.

  $a_{n+1} = a_n + a_{n-1}$.
  Guess $a_n = n^r$: $r^{n+1} = r^n + r^{n-1}$ or $r^2 = r + 1$.
  Roots $r = (1 \pm \sqrt{5})/2$.
  General solution: $a_n = c_1 r_1^n + c_2 r_2^n$.
  Binet: $a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$.

- Most linear recurrence relations Benford:

Examples

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  $a_{n+1} = a_n + a_{n-1}$.
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- Most linear recurrence relations Benford:

  $\diamond$ $a_{n+1} = 2a_n$ $\diamond$ $a_{n+1} = 2a_n - a_{n-1}$ $\diamond$
  $a_{n+1} = 2a_n - a_{n-1}$
  $\diamond$ take $a_0 = a_1 = 1$ or $a_0 = 0$, $a_1 = 1$.

Digits of 2^n

First 60 values of 2^n (only displaying 30)

1	1024	1048576	digit	#	Obs Prob	Benf Prob
2	2048	2097152	1	18	.300	.301
4	4096	4194304	2	12	.200	.176
8	8192	8388608	3	6	.100	.125
16	16384	16777216	4	6	.100	.097
32	32768	33554432	5	6	.100	.079
64	65536	67108864	6	4	.067	.067
128	131072	134217728	7	2	.033	.058
256	262144	268435456	8	5	.083	.051
512	524288	536870912	9	1	.017	.046

Data Analysis

- χ^2-Tests: Test if theory describes data
  - ◇ Expected probability: $p_d = \log_{10}\left(\frac{d+1}{d}\right)$.
  - ◇ Expect about Np_d will have first digit d.
  - ◇ Observe $\mathrm{Obs}(d)$ with first digit d.
  - ◇ $\chi^2 = \sum_{d=1}^{9} \frac{(\mathrm{Obs}(d) - Np_d)^2}{Np_d}$.
  - ◇ Smaller χ^2, more likely correct model.

- Will study γ^n, e^n, π^n.

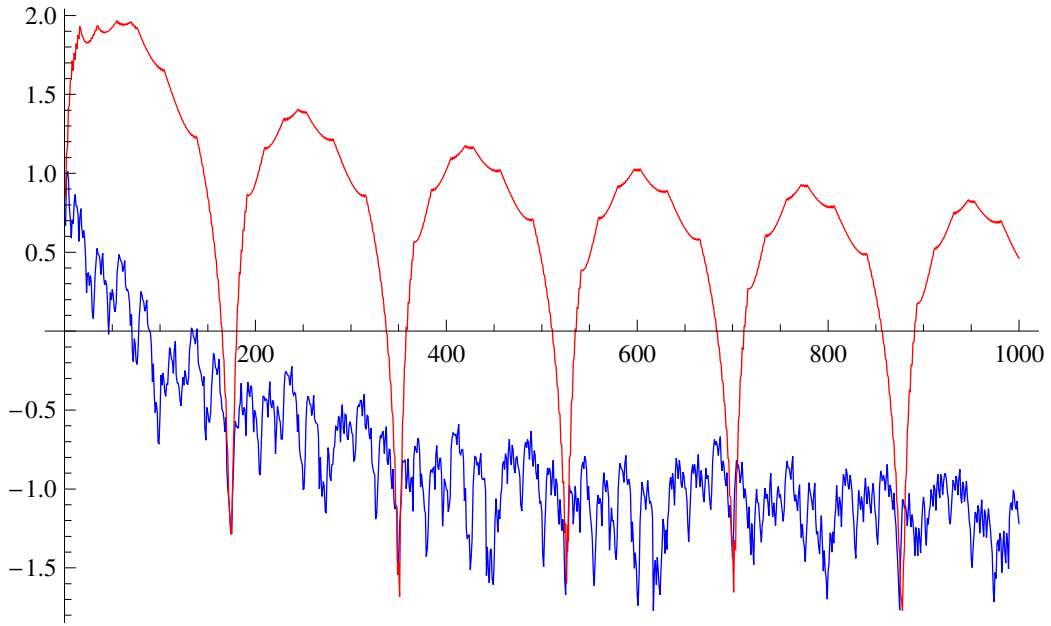
Logarithms and Benford's Law

χ^2 values for α^n, $1 \leq n \leq N$ (5% 15.5).

N	$\chi^2(\gamma)$	$\chi^2(e)$	$\chi^2(\pi)$
100	0.72	0.30	46.65
200	0.24	0.30	8.58
400	0.14	0.10	10.55
500	0.08	0.07	2.69
700	0.19	0.04	0.05
800	0.04	0.03	6.19
900	0.09	0.09	1.71
1000	0.02	0.06	2.90

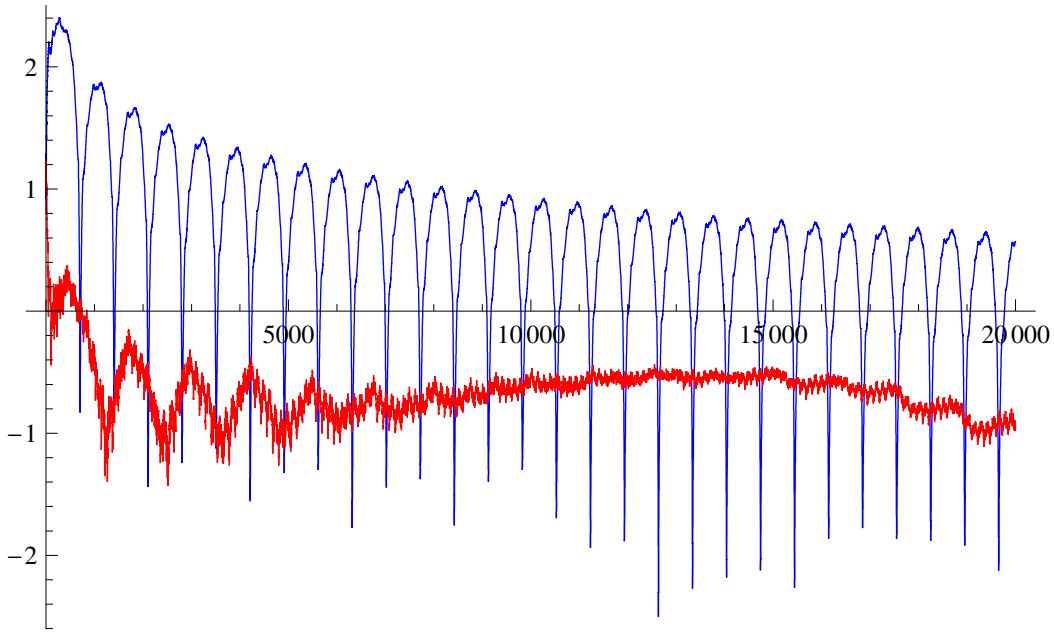
Logarithms and Benford's Law: Base 10

$\log(\chi^2)$ vs N for π^n (red) and e^n (blue), $n \in \{1, \ldots, N\}$. Note $\pi^{175} \approx 1.0028 \cdot 10^{87}$, (5%, $\log(\chi^2) \approx 2.74$).



Logarithms and Benford's Law: Base 20

$\log(\chi^2)$ vs N for π^n (red) and e^n (blue), $n \in \{1, \ldots, N\}$. Note $e^3 \approx 20.0855$, (5%, $\log(\chi^2) \approx 2.74$).

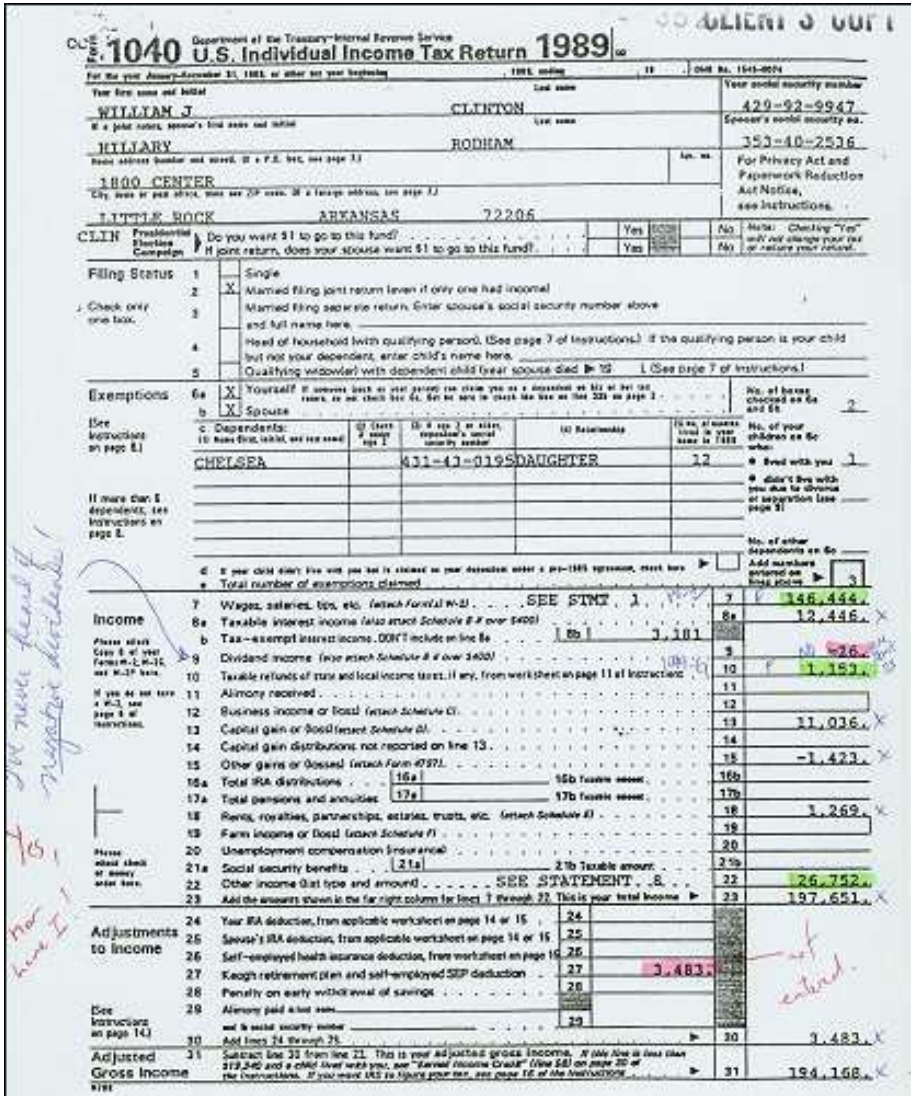


Applications

Stock Market

Milestone	Date	Effective Rate from last milestone
108.35	Jan 12, 1906	
500.24	Mar 12, 1956	3.0%
1003.16	Nov 14, 1972	4.2%
2002.25	Jan 8, 1987	4.9%
3004.46	Apr 17, 1991	9.5%
4003.33	Feb 23, 1995	7.4%
5023.55	Nov 21, 1995	30.6%
6010.00	Oct 14, 1996	20.0%
7022.44	Feb 13, 1997	46.6%
8038.88	Jul 16, 1997	32.3%
9033.23	Apr 6, 1998	16.1%
10006.78	Mar 29, 1999	10.5%
11209.84	Jul 16, 1999	38.0%
12011.73	Oct 19, 2006	1.0%
13089.89	Apr 25, 2007	16.7%
14000.41	Jul 19, 2007	28.9%

Applications for the IRS: Detecting Fraud



Form 1040 U.S. Individual Income Tax Return 1989

WILLIAM J CLINTON 429-92-9947
HILARY RODHAM 353-40-2536
1800 CENTER
LITTLE ROCK ARKANSAS 72206

Filing Status: X Married filing joint return (even if only one had income)

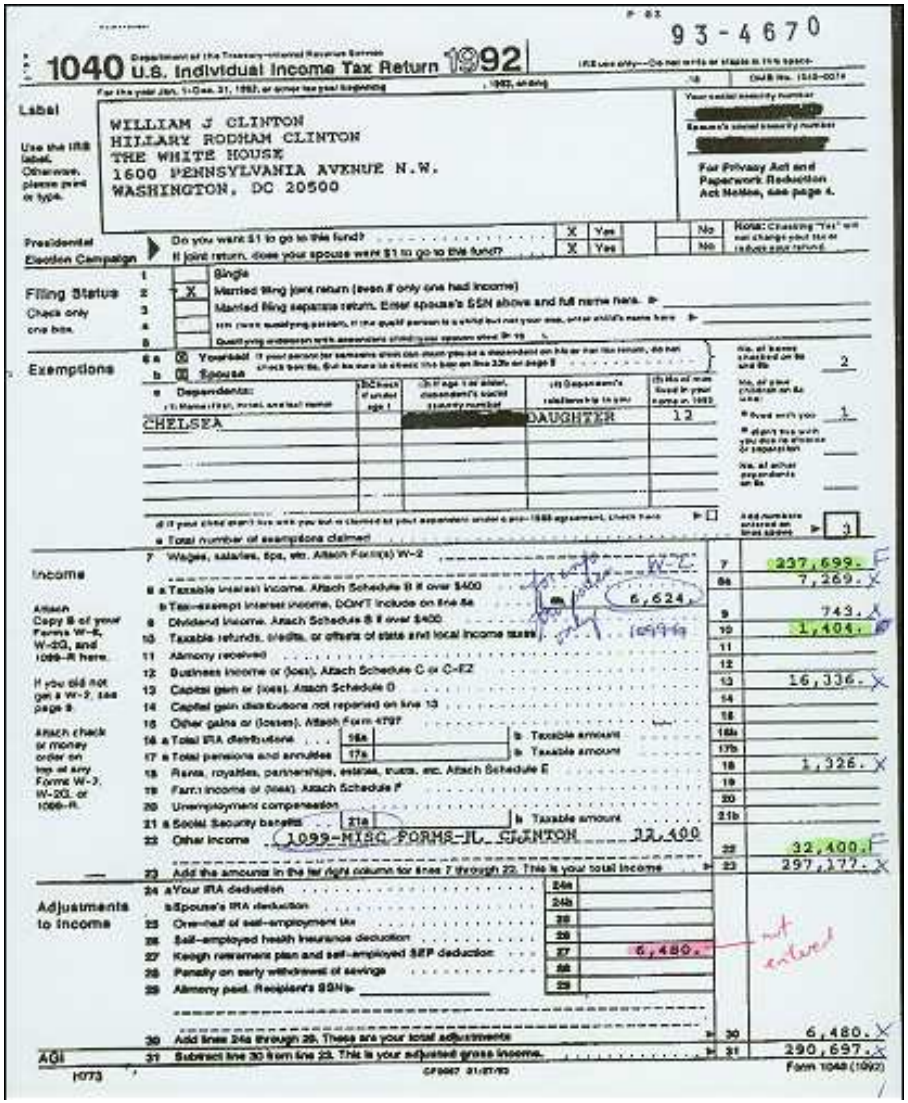
Exemptions: X Yourself; X Spouse; Dependents: CHELSEA 431-47-0195 DAUGHTER 12

7 Wages, salaries, tips, etc. SEE STMT 1 146,444.
8a Taxable interest income 12,446.
b Tax-exempt interest income. DON'T include on line 8a 3,101
9 Dividend income -26.
10 Taxable refunds of state and local income taxes, if any, from worksheet on page 11 of instructions 1,153.
11 Alimony received
12 Business income or (loss)
13 Capital gain or (loss) 11,036.
14 Capital gain distributions not reported on line 13
15 Other gains or (losses) -1,423.
16a Total IRA distributions 16a 16b Taxable amount
17a Total pensions and annuities 17a 17b Taxable amount
18 Rents, royalties, partnerships, estates, trusts, etc. 1,269.
19 Farm income or (loss)
20 Unemployment compensation (insurance)
21a Social security benefits 21a 21b Taxable amount
22 Other income (list type and amount) SEE STATEMENT 8 26,752.
23 Add the amounts shown in the far right column for lines 7 through 22. This is your total income 197,651.

Adjustments to Income
24 Your IRA deduction, from applicable worksheet on page 14 or 15
25 Spouse's IRA deduction, from applicable worksheet on page 14 or 15
26 Self-employed health insurance deduction, from worksheet on page 16
27 Keogh retirement plan and self-employed SEP deduction 3,483.
28 Penalty on early withdrawal of savings
29 Alimony paid
30 Add lines 24 through 29 3,483.

Adjusted Gross Income
31 Subtract line 30 from line 23. This is your adjusted gross income. 194,168.

Applications for the IRS: Detecting Fraud



1040 U.S. Individual Income Tax Return 1992

93-4670

WILLIAM J CLINTON
HILLARY RODHAM CLINTON
THE WHITE HOUSE
1600 PENNSYLVANIA AVENUE N.W.
WASHINGTON, DC 20500

Filing Status: X Married filing joint return (even if only one had income)

Exemptions: 6a X Yourself; 6b X Spouse

Dependents:	(2) Check if under age 1	(3) If age 1 or older, dependent's social security number	(4) Dependent's relationship to you	(5) No. of months lived in your home in 1992
CHELSEA			DAUGHTER	12

6a Total number of exemptions claimed: 2

Income

7	Wages, salaries, tips, etc. Attach Form(s) W-2		237,699.
8a	Taxable interest income. Attach Schedule B if over $400	6,624.	7,269.
b	Tax-exempt interest income. DON'T include on line 8a		
9	Dividend income. Attach Schedule B if over $400		743.
10	Taxable refunds, credits, or offsets of state and local income taxes		1,404.
11	Alimony received		
12	Business income or (loss). Attach Schedule C or C-EZ		
13	Capital gain or (loss). Attach Schedule D		16,336.
14	Capital gain distributions not reported on line 13		
15	Other gains or (losses). Attach Form 4797		
16a	Total IRA distributions	16a	b Taxable amount 16b
17a	Total pensions and annuities	17a	b Taxable amount 17b
18	Rents, royalties, partnerships, estates, trusts, etc. Attach Schedule E		1,326.
19	Farm income or (loss). Attach Schedule F		
20	Unemployment compensation		
21a	Social Security benefits	21a	b Taxable amount 21b
22	Other income 1099-MISC FORMS-H. CLINTON 32,400		32,400.
23	Add the amounts in the far right column for lines 7 through 22. This is your total income		297,177.

Adjustments to Income

24a	Your IRA deduction	24a	
b	Spouse's IRA deduction	24b	
25	One-half of self-employment tax	25	
26	Self-employed health insurance deduction	26	
27	Keogh retirement plan and self-employed SEP deduction	27	6,480.
28	Penalty on early withdrawal of savings	28	
29	Alimony paid. Recipient's SSN ▶	29	
30	Add lines 24a through 29. These are your total adjustments		6,480.
31	Subtract line 30 from line 23. This is your adjusted gross income.		290,697.

AGI

Form 1040 (1992)

Applications for the IRS: Detecting Fraud

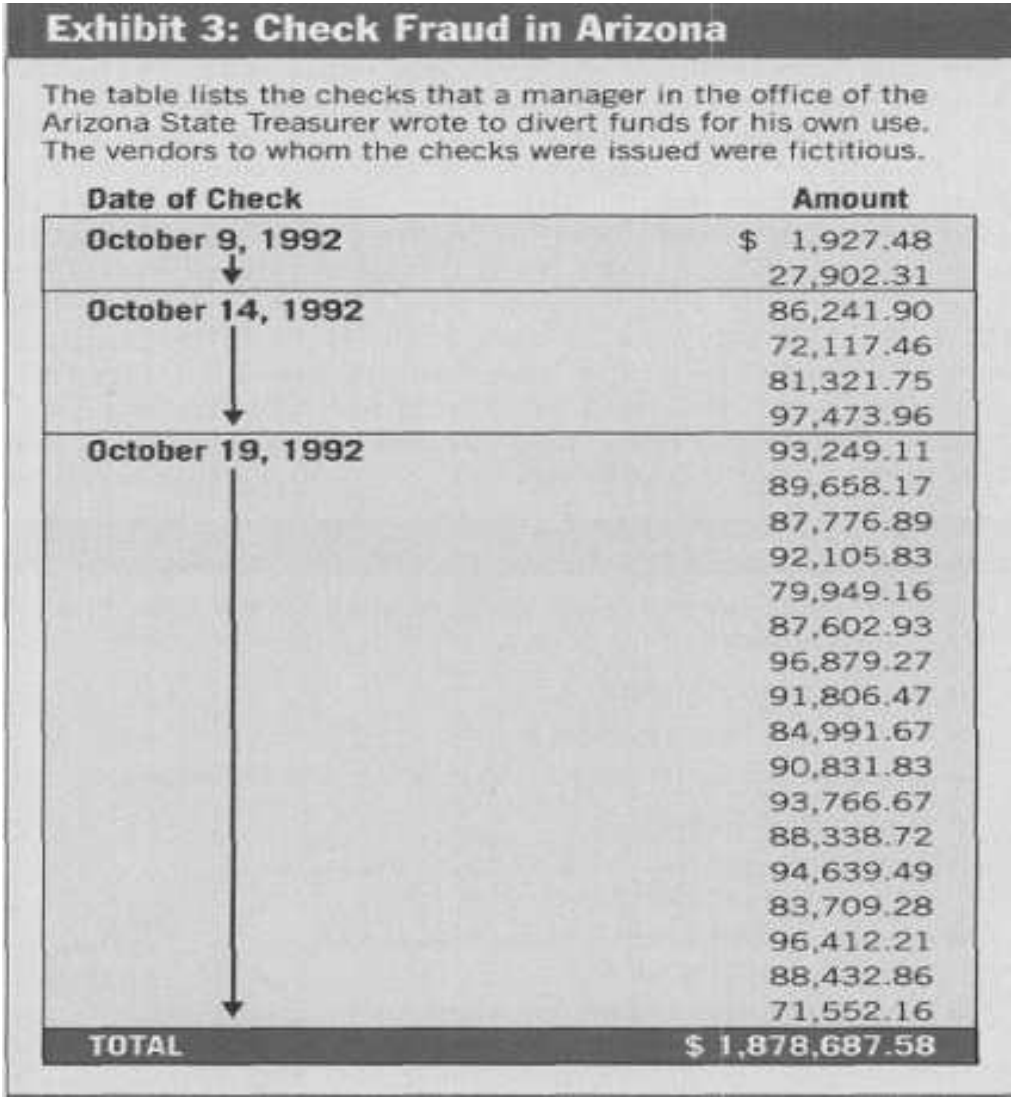


Exhibit 3: Check Fraud in Arizona

The table lists the checks that a manager in the office of the Arizona State Treasurer wrote to divert funds for his own use. The vendors to whom the checks were issued were fictitious.

Date of Check	Amount
October 9, 1992	$ 1,927.48
	27,902.31
October 14, 1992	86,241.90
	72,117.46
	81,321.75
	97,473.96
October 19, 1992	93,249.11
	89,658.17
	87,776.89
	92,105.83
	79,949.16
	87,602.93
	96,879.27
	91,806.47
	84,991.67
	90,831.83
	93,766.67
	88,338.72
	94,639.49
	83,709.28
	96,412.21
	88,432.86
	71,552.16
TOTAL	$ 1,878,687.58

Applications for the IRS: Detecting Fraud (cont)

- Embezzler started small and then increased dollar amounts.
- Most amounts below $100,000 (critical threshold for data requiring additional scrutiny).
- Over 90% had first digit of 7, 8 or 9.

Detecting Fraud

Bank Fraud

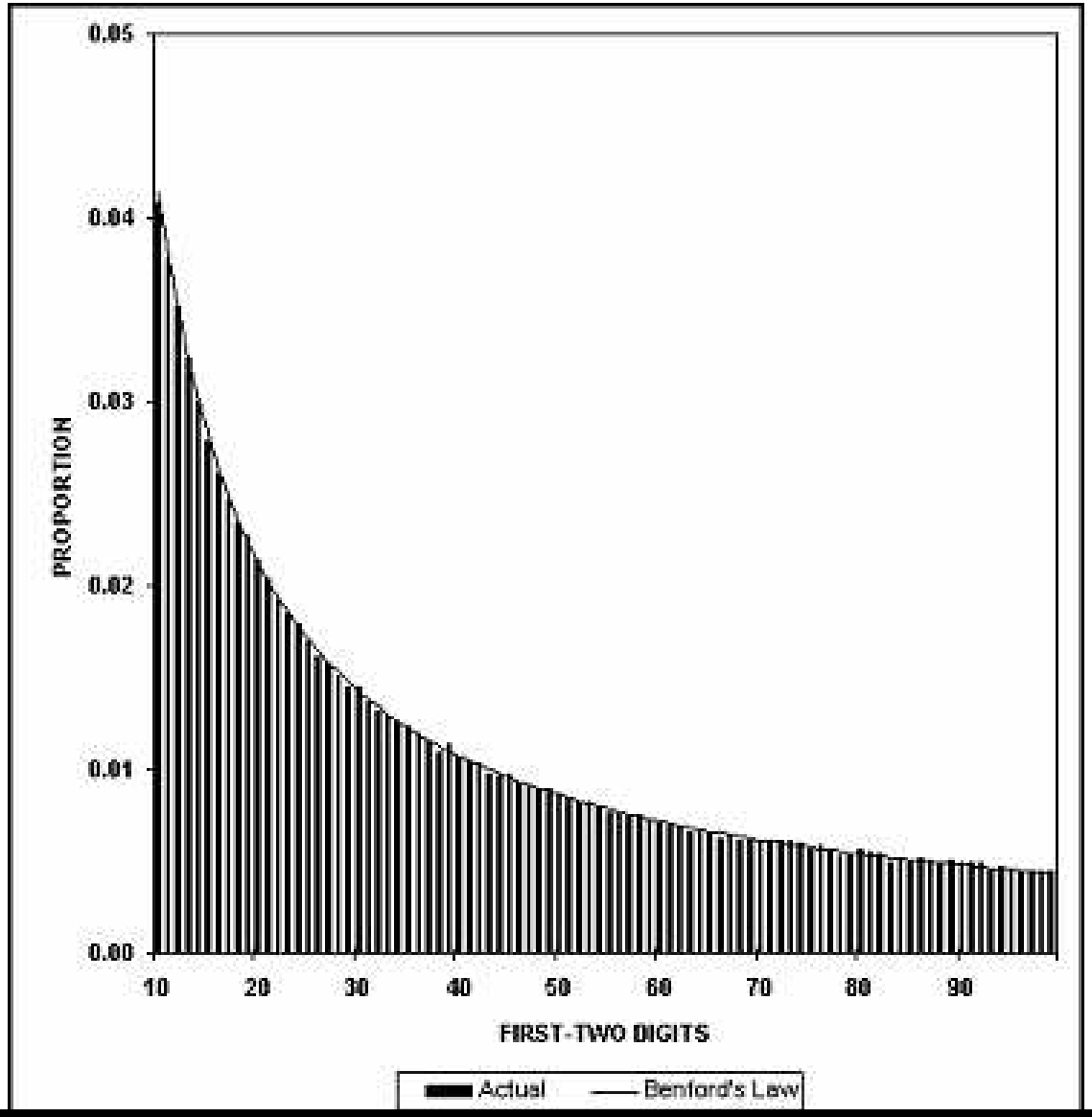
- Audit of a bank revealed huge spike of numbers starting with 48 and 49, most due to one person.
- Write-off limit of \$5,000. Officer had friends applying for credit cards, ran up balances just under \$5,000 then he would write the debts off.

Detecting Fraud

Enron

- Benford's Law detected manipulation of revenue numbers.
- Results showed a tendency towards round Earnings Per Share (0.10, 0.20, etc.).
  Consistent with a small but noticeable increase in earnings management in 2002.

Data Integrity: Stream Flow Statistics: 130 years, 457,440 records



Benford Good Processes

Poisson Summation and Benford's Law: Definitions

- Feller, Pinkham (often exact processes)
- data $Y_{T,B} = \log_B \overrightarrow{X}_T$ (discrete/continuous):

$$\mathbb{P}(A) = \lim_{T\to\infty} \frac{\#\{n \in A : n \le T\}}{T}$$

- Poisson Summation Formula: f nice:

$$\sum_{\ell=-\infty}^{\infty} f(\ell) = \sum_{\ell=-\infty}^{\infty} \widehat{f}(\ell),$$

$$\text{Fourier transform } \widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi} dx.$$

Benford Good Process

X_T is Benford Good if there is a nice f st

$$\mathrm{CDF}_{\vec{Y}_{T,B}}(y) = \int_{-\infty}^{y} \frac{1}{T} f\left(\frac{t}{T}\right) dt + E_T(y) := G_T(y)$$

and monotonically increasing h ($h(|T|) \to \infty$):

- Small tails: $G_T(\infty) - G_T(Th(T)) = o(1)$, $G_T(-Th(T)) - G_T(-\infty) = 0(1)$.
- Decay of the Fourier Transform: $\sum_{\ell \neq 0} \left| \frac{\widehat{f}(T\ell)}{\ell} \right| = o(1)$.
- Small translated error: $\mathcal{E}(a, b, T)) = \sum_{|\ell| \leq Th(T)} [E_T(b+\ell) - E_T(a+\ell)] = o(1)$.

Main Theorem

Theorem (Kontorovich and M–, 2005)

X_T converging to X as $T \to \infty$ (think spreading Gaussian). If X_T is Benford good, then X is Benford.

- Examples
  - L-functions
  - characteristic polynomials (RMT)
  - $3x+1$ problem
  - geometric Brownian motion.

Sketch of the proof

- Structure Theorem:
  - ◇ main term is something nice spreading out
  - ◇ apply Poisson summation
- Control translated errors:
  - ◇ hardest step
  - ◇ techniques problem specific

Sketch of the proof (continued)

$$\begin{aligned}
& \sum_{\ell=-\infty}^{\infty} \mathbb{P}\left(a+\ell \leq \overrightarrow{Y}_{T,B} \leq b+\ell\right) \\
= & \sum_{|\ell| \leq Th(T)} \left[G_T(b+\ell) - G_T(a+\ell)\right] + o(1) \\
= & \int_a^b \sum_{|\ell| \leq Th(T)} \frac{1}{T} f\left(\frac{t}{T}\right) dt + \mathcal{E}(a,b,T) + o(1) \\
= & \ \widehat{f}(0) \cdot (b-a) + \sum_{\ell \neq 0} \widehat{f}(T\ell) \frac{e^{2\pi i b\ell} - e^{2\pi i a\ell}}{2\pi i \ell} + o(1).
\end{aligned}$$

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

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$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_{p \text{ prime}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right)$$

$$= \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \cdots\right)\left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \cdots\right)$$

$$= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{(2\cdot 3)^s} + \cdots$$

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

$\lim_{s \to 1^+} \zeta(s) = \infty$ implies infinitely many primes.

Riemann Zeta Function

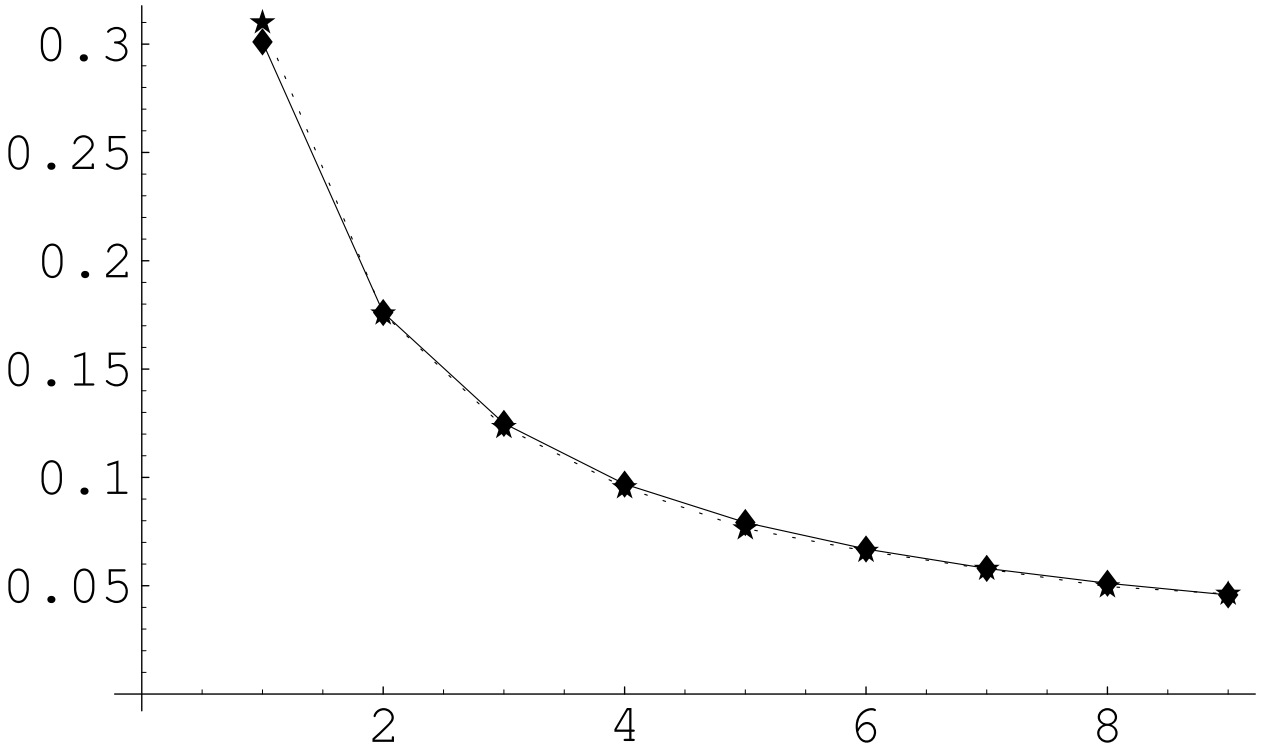
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

$\lim_{s \to 1^+} \zeta(s) = \infty$ implies infinitely many primes.

$\zeta(2) = \pi^2/6$ implies infinitely many primes.

Riemann Zeta Function

$$\left|\zeta\left(\tfrac{1}{2}+\mathrm{i}\tfrac{k}{4}\right)\right|,\ k\in\{0,1,\ldots,65535\}.$$



The $3x + 1$ Problem and Benford's Law

$3x + 1$ Problem

- Kakutani (conspiracy), Erdös (not ready).
- x odd, $T(x) = \frac{3x+1}{2^k}$, $2^k || 3x + 1$.
- Conjecture: for some $n = n(x)$, $T^n(x) = 1$.
- $7 \to_1 11 \to_1 17 \to_2 13 \to_3 5 \to_4 1 \to_2 1$,
  2-path $(1, 1)$, 5-path $(1, 1, 2, 3, 4)$.
  m-path: $(k_1, \ldots, k_m)$.

Heuristic Proof of $3x+1$ Conjecture

$$
\begin{aligned}
a_{n+1} &= T(a_n) \\
\mathbb{E}[\log a_{n+1}] &\approx \sum_{k=1}^{\infty} \frac{1}{2^k} \log\left(\frac{3a_n}{2^k}\right) \\
&= \log a_n + \log 3 - \log 2 \sum_{k=1}^{\infty} \frac{k}{2^k} \\
&= \log a_n + \log\left(\frac{3}{4}\right).
\end{aligned}
$$

Geometric Brownian Motion, drift $\log(3/4) < 1$.

Structure Theorem: Sinai, Kontorovich-Sinai

$\mathbb{P}(A) = \lim_{N\to\infty} \frac{\#\{n\leq N : n\equiv 1,5 \bmod 6, n\in A\}}{\#\{n\leq N : n\equiv 1,5 \bmod 6\}}$.
$(k_1,\ldots,k_m)$: two full arithm progressions:
$6\cdot 2^{k_1+\cdots+k_m}p+q$.

Theorem (Sinai, Kontorovich-Sinai)

k_i-values are i.i.d.r.v. (geometric, $1/2$):

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$3x+1$ and Benford

Theorem (Kontorovich and M–, 2005)

As $m \to \infty$, $x_m/(3/4)^m x_0$ is Benford.

Theorem (Lagarias-Soundararajan 2006)

$X \geq 2^N$, for all but at most $c(B)N^{-1/36}X$ initial seeds the distribution of the first N iterates of the $3x+1$ map are within $2N^{-1/36}$ of the Benford probabilities.

Sketch of the proof

- Failed Proof: lattices, bad errors.
- CLT: $(S_m - 2m)/\sqrt{2m} \to N(0,1)$:

$$\mathbb{P}(S_m - 2m = k) = \frac{\eta(k/\sqrt{m})}{\sqrt{m}} + O\left(\frac{1}{g(m)\sqrt{m}}\right).$$

- Quantified Equidistribution:
  $I_\ell = \{\ell M, \ldots, (\ell+1)M - 1\}$, $M = m^c$, $c < 1/2$
  $k_1, k_2 \in I_\ell$: $\left|\eta\left(\frac{k_1}{\sqrt{m}}\right) - \eta\left(\frac{k_2}{\sqrt{m}}\right)\right|$ small
  $C = \log_B 2$ of irrationality type $\kappa < \infty$:

$$\#\{k \in I_\ell : \overline{kC} \in [a,b]\} = M(b-a) + O(M^{1+\epsilon-1/\kappa}).$$

Irrationality Type

Irrationality type

α has irrationality type κ if κ is the supremum of all γ with

$$\underline{\lim}_{q\to\infty} q^{\gamma+1} \min_p \left|\alpha - \frac{p}{q}\right| = 0.$$

- Algebraic irrationals: type 1 (Roth's Thm).
- Theory of Linear Forms: $\log_B 2$ of finite type.

Linear Forms

Theorem (Baker)

$\alpha_1, \ldots, \alpha_n$ algebraic numbers height $A_j \geq 4$, $\beta_1, \ldots, \beta_n \in \mathbb{Q}$ with height at most $B \geq 4$,

$$\Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n.$$

If $\Lambda \neq 0$ then $|\Lambda| > B^{-C\Omega \log \Omega'}$, with $d = [\mathbb{Q}(\alpha_i, \beta_j) : \mathbb{Q}]$, $C = (16nd)^{200n}$, $\Omega = \prod_j \log A_j$, $\Omega' = \Omega / \log A_n$.

Gives $\log_{10} 2$ of finite type, with $\kappa < 1.2 \cdot 10^{602}$:

$$|\log_{10} 2 - p/q| = |q \log 2 - p \log 10| \, / q \log 10.$$

Quantified Equidistribution

Theorem (Erdös-Turan)

$$D_N = \frac{\sup_{[a,b]} |N(b-a) - \#\{n \leq N : x_n \in [a,b]\}|}{N}$$

There is a C such that for all m:

$$D_N \leq C \cdot \left(\frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \right)$$

Proof of Erdös-Turan

Consider special case $x_n = n\alpha$, $\alpha \notin \mathbb{Q}$.

- Exponential sum $\leq \frac{1}{|\sin(\pi h\alpha)|} \leq \frac{1}{2||h\alpha||}$.
- Must control $\sum_{h=1}^{m} \frac{1}{h||h\alpha||}$, see irrationality type enter.
- type κ, $\sum_{h=1}^{m} \frac{1}{h||h\alpha||} = O\left(m^{\kappa-1+\epsilon}\right)$, take $m = \lfloor N^{1/\kappa} \rfloor$.

$3x+1$ Data: random 10,000 digit number, $2^k || 3x+1$

80,514 iterations ($(4/3)^n = a_0$ predicts 80,319); $\chi^2 = 13.5$ (5% 15.5).

Digit	Number	Observed	Benford
1	24251	0.301	0.301
2	14156	0.176	0.176
3	10227	0.127	0.125
4	7931	0.099	0.097
5	6359	0.079	0.079
6	5372	0.067	0.067
7	4476	0.056	0.058
8	4092	0.051	0.051
9	3650	0.045	0.046

$3x+1$ Data: random 10,000 digit number, $2|3x+1$

241,344 iterations, $\chi^2 = 11.4$ (5% 15.5).

Digit	Number	Observed	Benford
1	72924	0.302	0.301
2	42357	0.176	0.176
3	30201	0.125	0.125
4	23507	0.097	0.097
5	18928	0.078	0.079
6	16296	0.068	0.067
7	13702	0.057	0.058
8	12356	0.051	0.051
9	11073	0.046	0.046

$5x+1$ Data: random 10,000 digit number, $2^k || 5x+1$

27,004 iterations, $\chi^2 = 1.8$ (5% 15.5).

Digit	Number	Observed	Benford
1	8154	0.302	0.301
2	4770	0.177	0.176
3	3405	0.126	0.125
4	2634	0.098	0.097
5	2105	0.078	0.079
6	1787	0.066	0.067
7	1568	0.058	0.058
8	1357	0.050	0.051
9	1224	0.045	0.046

$5x+1$ Data: random 10,000 digit number, $2|5x+1$

241,344 iterations, $\chi^2 = 3 \cdot 10^{-4}$ (5% 15.5).

Digit	Number	Observed	Benford
1	72652	0.301	0.301
2	42499	0.176	0.176
3	30153	0.125	0.125
4	23388	0.097	0.097
5	19110	0.079	0.079
6	16159	0.067	0.067
7	13995	0.058	0.058
8	12345	0.051	0.051
9	11043	0.046	0.046

Products and Chains of Random Variables

Key Ingredients

- Mellin transform and Fourier transform related by logarithmic change of variable.
- Poisson summation from collapsing to modulo 1 random variables.

Preliminaries

- $\Xi_1, \ldots, \Xi_n$ nice independent r.v.'s on $[0, \infty)$.
- Density $\Xi_1 \cdot \Xi_2$:

$$\int_0^\infty f_2\left(\frac{x}{t}\right) f_1(t) \frac{dt}{t}$$

◇ Proof: $\text{Prob}(\Xi_1 \cdot \Xi_2 \in [0, x])$:

$$\begin{aligned} & \int_{t=0}^\infty \text{Prob}\left(\Xi_2 \in \left[0, \frac{x}{t}\right]\right) f_1(t) dt \\ = & \int_{t=0}^\infty F_2\left(\frac{x}{t}\right) f_1(t) dt, \end{aligned}$$

differentiate.

Mellin Transform

$$\begin{aligned}(\mathcal{M}f)(s) &= \int_0^\infty f(x)x^s\frac{dx}{x}\\ (\mathcal{M}^{-1}g)(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g(s)x^{-s}ds\end{aligned}$$

$$g(s) = (\mathcal{M}f)(s), f(x) = (\mathcal{M}^{-1}g)(x).$$

$$\begin{aligned}(f_1 \star f_2)(x) &= \int_0^\infty f_2\left(\frac{x}{t}\right) f_1(t)\frac{dt}{t}\\ (\mathcal{M}(f_1 \star f_2))(s) &= (\mathcal{M}f_1)(s)\cdot(\mathcal{M}f_2)(s).\end{aligned}$$

Mellin Transform Formulation: Products Random Variables

Theorem

X_i's independent, densities f_i. $\Xi_n = X_1 \cdots X_n$,

$$\begin{aligned} h_n(x_n) &= (f_1 \star \cdots \star f_n)(x_n) \\ (\mathcal{M}h_n)(s) &= \prod_{m=1}^{n} (\mathcal{M}f_m)(s). \end{aligned}$$

As $n \to \infty$, Ξ_n becomes Benford: $Y_n = \log_B \Xi_n$,
$|\text{Prob}(Y_n \bmod 1 \in [a,b]) - (b-a)| \leq$

$$(b-a) \cdot \sum_{\ell \neq 0, \ell = -\infty}^{\infty} \prod_{m=1}^{n} (\mathcal{M}f_i)\left(1 - \frac{2\pi i \ell}{\log B}\right).$$

Proof of Kossovsky's Chain Conjecture for certain densities

Conditions

- $\{\mathcal{D}_i(\theta)\}_{i\in I}$: one-parameter distributions, densities $f_{\mathcal{D}_i(\theta)}$ on $[0,\infty)$.
- $p:\mathbb{N}\to I$, $X_1\sim\mathcal{D}_{p(1)}(1)$, $X_m\sim\mathcal{D}_{p(m)}(X_{m-1})$.
- $m\geq 2$,

$$f_m(x_m) \;=\; \int_0^\infty f_{\mathcal{D}_{p(m)}(1)}\left(\frac{x_m}{x_{m-1}}\right) f_{m-1}(x_{m-1})\frac{dx_{m-1}}{x_{m-1}}$$

- $$\lim_{n\to\infty}\sum_{\substack{\ell=-\infty\\ \ell\neq 0}}^{\infty}\prod_{m=1}^{n}(\mathcal{M}f_{\mathcal{D}_{p(m)}(1)})\left(1-\frac{2\pi i\ell}{\log B}\right) \;=\; 0$$

Proof of Kossovsky's Chain Conjecture for certain densities

Theorem (JKKKM)

- *If conditions hold, as $n \to \infty$ the distribution of leading digits of X_n tends to Benford's law.*
- *The error is a nice function of the Mellin transforms: if $Y_n = \log_B X_n$, then*

$$|\text{Prob}(Y_n \bmod 1 \in [a,b]) - (b+a)| \leq$$

$$\left| (b-a) \cdot \sum_{\substack{\ell=-\infty \\ \ell \neq 0}}^{\infty} \prod_{m=1}^{n} (\mathcal{M} f_{\mathcal{D}_{p(m)}(1)}) \left(1 - \frac{2\pi i \ell}{\log B}\right) \right|$$

Example: All $X_i \sim \mathrm{Exp}(1)$

- $X_i \sim \mathrm{Exp}(1)$, $Y_n = \log_B \Xi_n$.
- Needed ingredients:
  - $\int_0^\infty \exp(-x)x^{s-1}dx = \Gamma(s)$.
  - $|\Gamma(1+ix)| = \sqrt{\pi x/\sinh(\pi x)}, x \in \mathbb{R}$.
- $|P_n(s) - \log_{10}(s)| \leq$

$$\log_B s \sum_{\ell=1}^{\infty} \left(\frac{2\pi^2 \ell / \log B}{\sinh(2\pi^2 \ell / \log B)} \right)^{n/2}.$$

Example: All $X_i \sim \mathrm{Exp}(1)$

Bounds on the error

- $|P_n(s) - \log_{10} s| \leq$
  - $3.3 \cdot 10^{-3} \log_B s$ if $n = 2$,
  - $1.9 \cdot 10^{-4} \log_B s$ if $n = 3$,
  - $1.1 \cdot 10^{-5} \log_B s$ if $n = 5$, and
  - $3.6 \cdot 10^{-13} \log_B s$ if $n = 10$.
- Error at most

$$\log_{10} s \sum_{\ell=1}^{\infty} \left(\frac{17.148\ell}{\exp(8.5726\ell)} \right)^{n/2} \leq .057^n \log_{10} s$$

Conclusions

Conclusions and Future Investigations

- See many different systems exhibit Benford behavior.
- Ingredients of proofs (logarithms, equidistribution).
- Applications to fraud detection / data integrity.
- Future work:
  - ◇ Study digits of other systems.
  - ◇ Develop more sophisticated tests for fraud.

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