Why the IRS cares about Algebra and Number Theory (and why you should too!)

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Interesting Question

Motivating Question: For a nice data set, such as the Fibonacci numbers, stock prices, street addresses of college employees and students, ..., what percent of the leading digits are 1?

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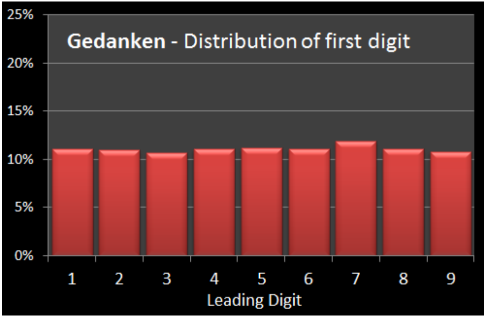

Natural guess: 10% (but immediately correct to 11%!).

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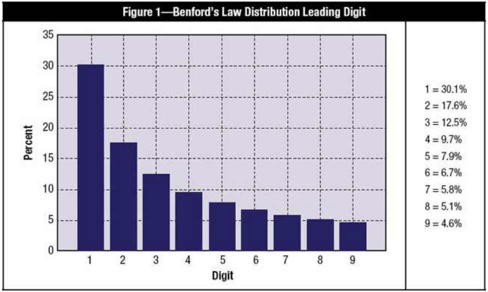
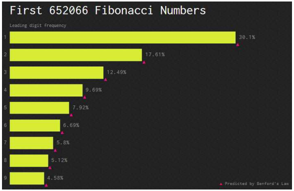


Answer: Benford's law!

Examples with First Digit Bias

Fibonacci numbers



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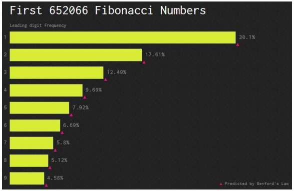
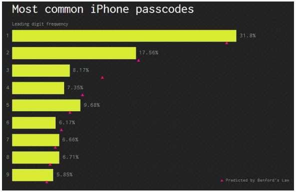


Most common iPhone passcodes



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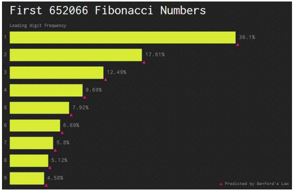
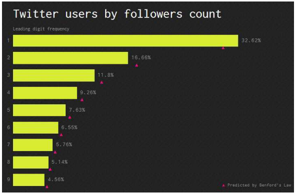


Twitter users by # followers



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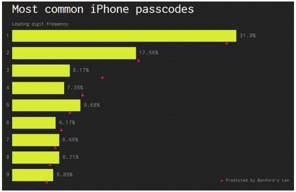

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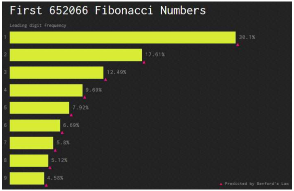

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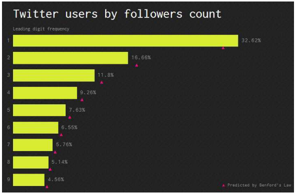

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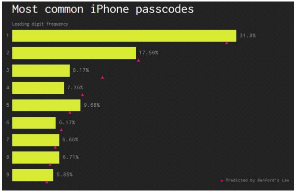
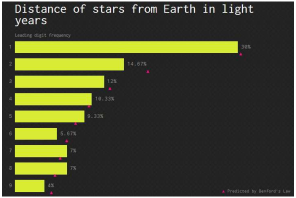


Distance of stars from Earth



Summary

- Explain Benford's Law.
- Discuss examples and applications.
- Sketch proofs.
- Describe open problems.

Caveats!

- A math test indicating fraud is *not* proof of fraud: unlikely events, alternate reasons.

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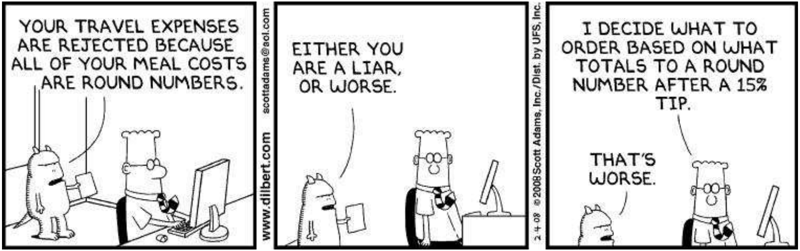


Examples

- recurrence relations
- special functions (such as $n!$)
- iterates of power, exponential, rational maps
- products of random variables
- L-functions, characteristic polynomials
- iterates of the $3x + 1$ map
- differences of order statistics
- hydrology and financial data
- many hierarchical Bayesian models

Applications

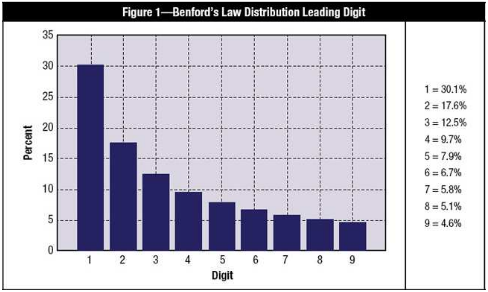
- Analyzing round-off errors.
- Determining the optimal way to store numbers.
- Detecting tax and image fraud, and data integrity.

General Theory

Benford's Law: Newcomb (1881), Benford (1938)

Statement

For many data sets, probability of observing a first digit of d base B is $\log_B\left(\frac{d+1}{d}\right)$; base 10 about 30% are 1s.



Benford's Law (probabilities)

Background Material

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- Key observation: $\log_{10}(x) = \log_{10}(\widetilde{x}) \mod 1$ if and only if x and $\widetilde{x}$ have the same leading digits.

Thus often study $y = \log_{10} x \mod 1$.
Advanced: $e^{2\pi iu} = e^{2\pi i(u \bmod 1)}$.

Equidistribution and Benford's Law

Equidistribution

$\{y_n\}_{n=1}^{\infty}$ is equidistributed modulo 1 if probability $y_n \bmod 1 \in [a,b]$ tends to $b-a$:

$$\frac{\#\{n \le N : y_n \bmod 1 \in [a,b]\}}{N} \ \to \ b-a.$$

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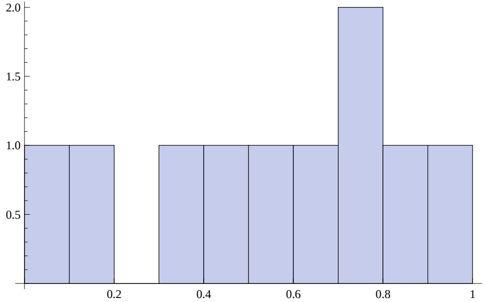
  Proof: if rational: $2 = 10^{p/q}$.
  Thus $2^q = 10^p$ or $2^{q-p} = 5^p$, impossible.

Example of Equidistribution: $n\sqrt{\pi}$ mod 1



$n\sqrt{\pi}$ mod 1 for $n \leq 10$

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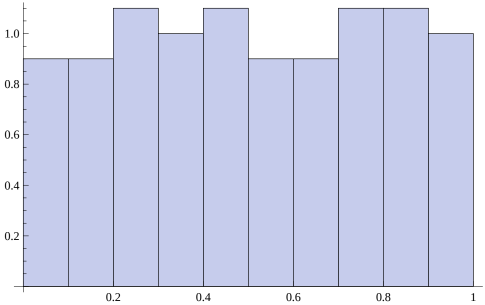

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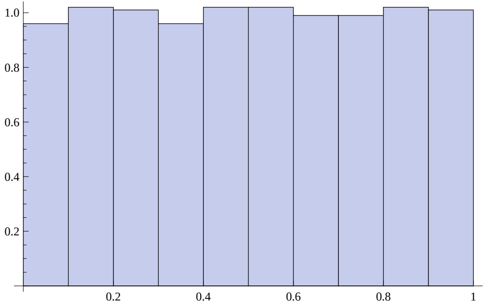

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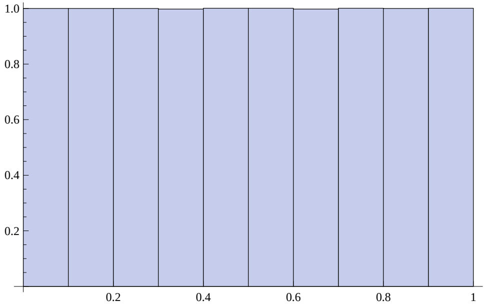

$n\sqrt{\pi}$ mod 1 for $n \le 10,000$

Logarithms and Benford's Law

Fundamental Equivalence

Data set $\{x_i\}$ is Benford base B if $\{y_i\}$ is equidistributed mod 1, where $y_i = \log_B x_i$.

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$$\log_{10} x = \log_{10} S_{10}(x) + k = \log_{10} S_{10} x \text{ mod } 1.$$

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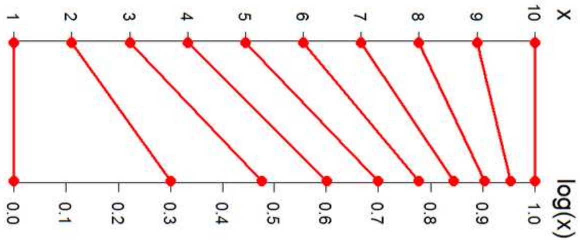
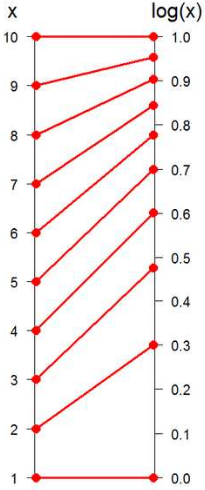


Logarithms and Benford's Law

$$\begin{aligned}&\text{Prob}(\text{leading digit } d)\\ &= \log_{10}(d+1) - \log_{10}(d)\\ &= \log_{10}\left(\frac{d+1}{d}\right)\\ &= \log_{10}\left(1+\frac{1}{d}\right).\end{aligned}$$

Have Benford's law $\leftrightarrow$ mantissa of logarithms of data are uniformly distributed



Examples

- 2^n is Benford base 10 as $\log_{10} 2 \notin \mathbb{Q}$.

- Fibonacci numbers are Benford base 10.
  $a_{n+1} = a_n + a_{n-1}$. Guess $a_n = r^n$:
  $r^{n+1} = r^n + r^{n-1}$ or $r^2 = r + 1$. Roots
  $r = (1 \pm \sqrt{5})/2$. General solution:
  $a_n = c_1 r_1^n + c_2 r_2^n$. Binet:
  $a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$.

- Most linear recurrence relations Benford:

Digits of 2^n

First 60 values of 2^n (only displaying 30)

			digit	#	Obs Prob	Benf Prob
1	1024	1048576	1	18	.300	.301
2	2048	2097152	2	12	.200	.176
4	4096	4194304	3	6	.100	.125
8	8192	8388608	4	6	.100	.097
16	16384	16777216	5	6	.100	.079
32	32768	33554432	6	4	.067	.067
64	65536	67108864	7	2	.033	.058
128	131072	134217728	8	5	.083	.051
256	262144	268435456	9	1	.017	.046
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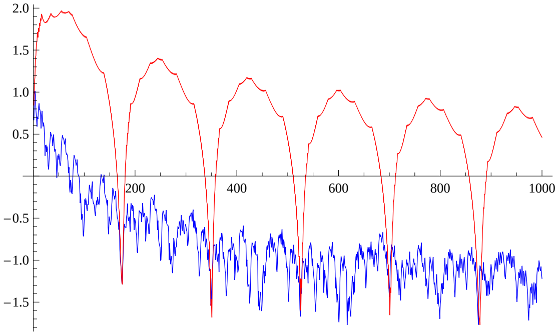
Logarithms and Benford's Law

χ^2 values for α^n, $1 \le n \le N$ (5% 15.5).

N	$\chi^2(\gamma)$	$\chi^2(e)$	$\chi^2(\pi)$
100	0.72	0.30	46.65
200	0.24	0.30	8.58
400	0.14	0.10	10.55
500	0.08	0.07	2.69
700	0.19	0.04	0.05
800	0.04	0.03	6.19
900	0.09	0.09	1.71
1000	0.02	0.06	2.90

Logarithms and Benford's Law: Base 10 (5%: $\log(\chi^2) \approx 2.74$)

$\log(\chi^2)$ vs N for π^n (red) and e^n (blue), $n \in \{1, \dots, N\}$.



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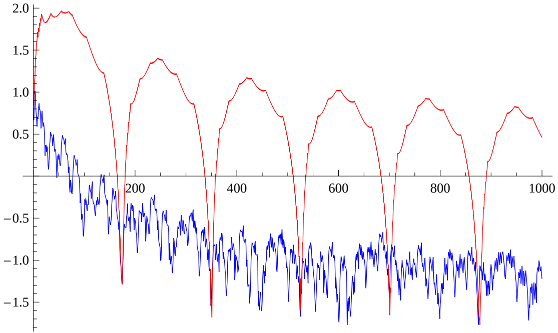
$\log(\chi^2)$ vs N for π^n (red) and e^n (blue), $n \in \{1, \dots, N\}$. Note $\pi^{175} \approx 1.0028 \cdot 10^{87}$.

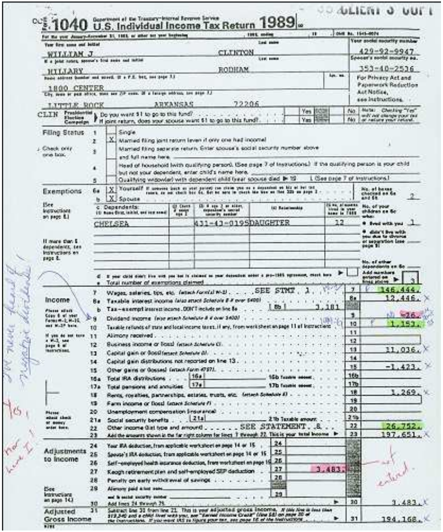


Applications

Applications for the IRS: Detecting Fraud

A Tale of Two Steve Millers....

Applications for the IRS: Detecting Fraud


CLIENT'S COPY

1040 Department of the Treasury—Internal Revenue Service
U.S. Individual Income Tax Return 1989

Your first name and initial: WILLIAM J — Last name: CLINTON — Your social security number: 429-92-9947
If a joint return, spouse's first name and initial: HILLARY — Last name: RODHAM — Spouse's social security no.: 353-40-2536
Present home address: 1800 CENTER
City, town or post office, state and ZIP code: LITTLE ROCK ARKANSAS 72206
For Privacy Act and Paperwork Reduction Act Notice, see Instructions.

Presidential Election Campaign: Do you want $1 to go to this fund? Yes — If joint return, does your spouse want $1 to go to this fund? Yes

Filing Status
1 Single
2 X Married filing joint return (even if only one had income)
3 Married filing separate return. Enter spouse's social security number above and full name here.
4 Head of household (with qualifying person). (See page 7 of Instructions.)
5 Qualifying widow(er) with dependent child (year spouse died ▶ 19). (See page 7 of Instructions.)

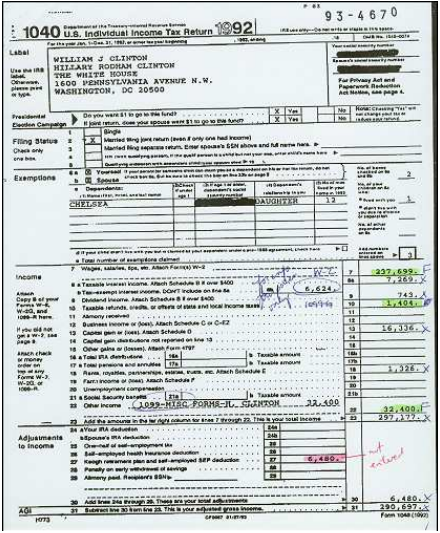
Exemptions
6a X Yourself
b X Spouse
c Dependents: CHELSEA — 431-43-0195 — DAUGHTER — 12
e Total number of exemptions claimed — 3

Income
7 Wages, salaries, tips, etc. — SEE STMT 1 — 146,444.
8a Taxable interest income — 12,446.
b Tax-exempt interest income — 3,181
9 Dividend income — 26.
10 Taxable refunds of state and local income taxes — 1,153.
11 Alimony received
12 Business income or (loss)
13 Capital gain or (loss) — 11,036.
14 Capital gain distributions not reported on line 13
15 Other gains or (losses) — -1,423.
16a Total IRA distributions
17a Total pensions and annuities
18 Rents, royalties, partnerships, estates, trusts, etc. — 1,269.
19 Farm income or (loss)
20 Unemployment compensation (insurance)
21a Social security benefits
22 Other income — SEE STATEMENT 8 — 26,752.
23 Add the amounts shown in the far right column for lines 7 through 22. This is your total income ▶ 197,651.

Adjustments to Income
24 Your IRA deduction
25 Spouse's IRA deduction
26 Self-employed health insurance deduction
27 Keogh retirement plan and self-employed SEP deduction — 3,483.
28 Penalty on early withdrawal of savings
29 Alimony paid
30 Add lines 24 through 29 ▶ 3,483.

Adjusted Gross Income
31 Subtract line 30 from line 23. This is your adjusted gross income. — 194,168.

Applications for the IRS: Detecting Fraud


93-4670

1040 U.S. Individual Income Tax Return 1992

Label

WILLIAM J CLINTON
HILLARY RODHAM CLINTON
THE WHITE HOUSE
1600 PENNSYLVANIA AVENUE N.W.
WASHINGTON, DC 20500

Your social security number [redacted]

Spouse's social security number [redacted]

For Privacy Act and Paperwork Reduction Act Notice, see page 4.

Presidential Election Campaign — Do you want $1 to go to this fund? X Yes — If joint return, does your spouse want $1 to go to this fund? X Yes

Filing Status — 1 Single; 2 X Married filing joint return (even if only one had income); 3 Married filing separate return; 4 Head of household; 5 Qualifying widow(er) with dependent child

Exemptions — 6a Yourself; b Spouse; c Dependents: CHELSEA [redacted] DAUGHTER 12

2 — 1 — 3

Income

7 Wages, salaries, tips, etc. Attach Form(s) W-2 — 237,699.
8a Taxable interest income. Attach Schedule B if over $400 — 7,269.
b Tax-exempt interest income. DON'T include on line 8a — 6,624
9 Dividend income. Attach Schedule B if over $400 — 743.
10 Taxable refunds, credits, or offsets of state and local income taxes — 1,404.
11 Alimony received
12 Business income or (loss). Attach Schedule C or C-EZ — 16,336.
13 Capital gain or (loss). Attach Schedule D
14 Capital gain distributions not reported on line 13
15 Other gains or (losses). Attach Form 4797
16a Total IRA distributions — b Taxable amount
17a Total pensions and annuities — b Taxable amount
18 Rents, royalties, partnerships, estates, trusts, etc. Attach Schedule E — 1,326.
19 Farm income or (loss). Attach Schedule F
20 Unemployment compensation
21a Social Security benefits — b Taxable amount
22 Other income (1099-MISC FORMS-H. CLINTON 32,400) — 32,400.
23 Add the amounts in the far right column for lines 7 through 22. This is your total income — 297,177.

Adjustments to Income

24a Your IRA deduction
b Spouse's IRA deduction
25 One-half of self-employment tax
26 Self-employed health insurance deduction
27 Keogh retirement plan and self-employed SEP deduction — 6,480.
28 Penalty on early withdrawal of savings
29 Alimony paid. Recipient's SSN
30 Add lines 24a through 29. These are your total adjustments — 6,480.
31 Subtract line 30 from line 23. This is your adjusted gross income — 290,697.

AGI — Form 1040 (1992)

Detecting Fraud

Bank Fraud

- Audit of a bank revealed huge spike of numbers starting with

Detecting Fraud

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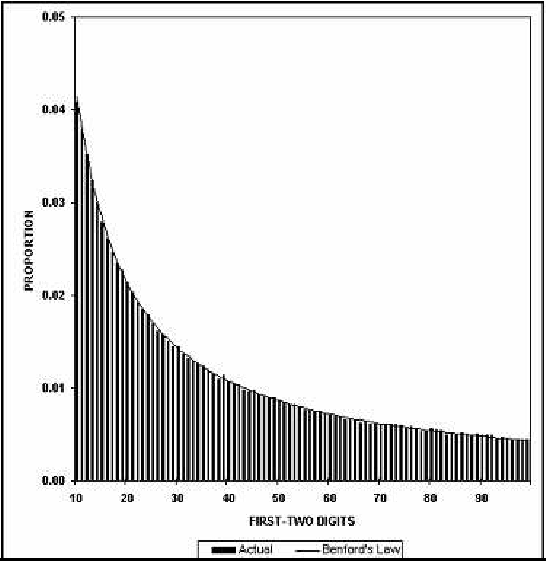
- Write-off limit of $5,000. Officer had friends applying for credit cards, ran up balances just under $5,000 then he would write the debts off.

Data Integrity: Stream Flow Statistics: 130 years, 457,440 records



Election Fraud: Iran 2009

Numerous questions over Iran's 2009 elections.

Lot of analysis; data moderately suspicious:

- First and second leading digits;
- Last two digits (should almost be uniform);
- Last two digits differing by at least 2.

Warning: enough tests, even if nothing wrong will find a suspicious result (but when all tests are on the boundary...).

The $3x + 1$ Problem
and
Benford's Law

$3x+1$ Problem

- Kakutani (conspiracy), Erdös (not ready).
- x odd, $T(x) = \frac{3x+1}{2^k}$, $2^k || 3x+1$.
- Conjecture: for some $n = n(x)$, $T^n(x) = 1$.

$3x + 1$ Problem

- Kakutani (conspiracy), Erdös (not ready).
- x odd, $T(x) = \frac{3x+1}{2^k}$, $2^k || 3x + 1$.
- Conjecture: for some $n = n(x)$, $T^n(x) = 1$.
- 7

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- $7 \to_1 11 \to_1 17 \to_2 13 \to_3 5 \to_4 1 \to_2 1$,
  2-path $(1, 1)$, 5-path $(1, 1, 2, 3, 4)$.
  m-path: $(k_1, \dots, k_m)$.

Heuristic Proof of $3x+1$ Conjecture

$$\begin{aligned}
a_{n+1} &= T(a_n) \\
\mathbb{E}[\log a_{n+1}] &\approx \sum_{k=1}^{\infty} \frac{1}{2^k} \log\left(\frac{3a_n}{2^k}\right) \\
&= \log a_n + \log 3 - \log 2 \sum_{k=1}^{\infty} \frac{k}{2^k} \\
&= \log a_n + \log\left(\frac{3}{4}\right).
\end{aligned}$$

Geometric Brownian Motion, drift $\log(3/4) < 1$.

$3x + 1$ and Benford

Theorem (Kontorovich and M–, 2005)

As $m \to \infty$, $x_m/(3/4)^m x_0$ is Benford.

Theorem (Lagarias-Soundararajan, 2006)

$X \geq 2^N$, for all but at most $c(B)N^{-1/36}X$ initial seeds the distribution of the first N iterates of the $3x + 1$ map are within $2N^{-1/36}$ of the Benford probabilities.

Structure Theorem: Sinai, Kontorovich-Sinai

$\mathbb{P}(A) = \lim_{N\to\infty} \frac{\#\{n\le N: n\equiv 1,5 \bmod 6, n\in A\}}{\#\{n\le N: n\equiv 1,5 \bmod 6\}}$.

$(k_1,\dots,k_m)$: two full arithm progressions:

$6\cdot 2^{k_1+\cdots+k_m}p+q$.

Theorem (Sinai, Kontorovich-Sinai)

k_i-values are i.i.d.r.v. (geometric, $1/2$):

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$$\mathbb{P}\left(\frac{\log_2\left[\frac{x_m}{\left(\frac{3}{4}\right)^m x_0}\right]}{\sqrt{2m}}\le a\right) = \mathbb{P}\left(\frac{S_m-2m}{\sqrt{2m}}\le a\right)$$

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Sketch of the proof of Benfordness

- Failed Proof: lattices, bad errors.

- CLT: $(S_m - 2m)/\sqrt{2m} \to N(0,1)$:

$$\mathbb{P}\left(S_m - 2m = k\right) = \frac{\eta(k/\sqrt{m})}{\sqrt{m}} + O\left(\frac{1}{g(m)\sqrt{m}}\right).$$

- Quantified Equidistribution:

$I_\ell = \{\ell M, \dots, (\ell+1)M - 1\}$, $M = m^c$, $c < 1/2$

$k_1, k_2 \in I_\ell$: $\left|\eta\left(\frac{k_1}{\sqrt{m}}\right) - \eta\left(\frac{k_2}{\sqrt{m}}\right)\right|$ small

$C = \log_B 2$ of irrationality type $\kappa < \infty$:

$$\#\{k \in I_\ell : \overline{kC} \in [a,b]\} = M(b-a) + O(M^{1+\epsilon-1/\kappa}).$$

Irrationality Type

Irrationality type

α has irrationality type κ if κ is the supremum of all γ with

$$\underline{\lim}_{q\to\infty} q^{\gamma+1} \min_p \left|\alpha - \frac{p}{q}\right| = 0.$$

- Algebraic irrationals: type 1 (Roth's Thm).
- Theory of Linear Forms: $\log_B 2$ of finite type.

Linear Forms

Theorem (Baker)

$\alpha_1, \dots, \alpha_n$ algebraic numbers height $A_j \geq 4$, $\beta_1, \dots, \beta_n \in \mathbb{Q}$ with height at most $B \geq 4$,

$$\Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n.$$

If $\Lambda \neq 0$ then $|\Lambda| > B^{-C\Omega \log \Omega'}$, with $d = [\mathbb{Q}(\alpha_i, \beta_j) : \mathbb{Q}]$, $C = (16nd)^{200n}$, $\Omega = \prod_j \log A_j$, $\Omega' = \Omega / \log A_n$.

Gives $\log_{10} 2$ of finite type, with $\kappa < 1.2 \cdot 10^{602}$:

$$|\log_{10} 2 - p/q| = |q \log 2 - p \log 10| / q \log 10.$$

Quantified Equidistribution

Theorem (Erdös-Turan)

$$D_N = \frac{\sup_{[a,b]} |N(b-a) - \#\{n \le N : x_n \in [a,b]\}|}{N}$$

There is a C such that for all m:

$$D_N \le C \cdot \left(\frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \right)$$

Proof of Erdös-Turan

Consider special case $x_n = n\alpha$, $\alpha \notin \mathbb{Q}$.

- Exponential sum $\leq \frac{1}{|\sin(\pi h\alpha)|} \leq \frac{1}{2||h\alpha||}$.
- Must control $\sum_{h=1}^{m} \frac{1}{h||h\alpha||}$, see irrationality type enter.
- type κ, $\sum_{h=1}^{m} \frac{1}{h||h\alpha||} = O\left(m^{\kappa-1+\epsilon}\right)$, take $m = \lfloor N^{1/\kappa} \rfloor$.

$3x+1$ Data: random 10,000 digit number, $2^k || 3x+1$

80,514 iterations ($(4/3)^n = a_0$ predicts 80,319); $\chi^2 = 13.5$ (5% 15.5).

Digit	Number	Observed	Benford
1	24251	0.301	0.301
2	14156	0.176	0.176
3	10227	0.127	0.125
4	7931	0.099	0.097
5	6359	0.079	0.079
6	5372	0.067	0.067
7	4476	0.056	0.058
8	4092	0.051	0.051
9	3650	0.045	0.046

$3x+1$ Data: random 10,000 digit number, $2|3x+1$

241,344 iterations, $\chi^2 = 11.4$ (5% 15.5).

Digit	Number	Observed	Benford
1	72924	0.302	0.301
2	42357	0.176	0.176
3	30201	0.125	0.125
4	23507	0.097	0.097
5	18928	0.078	0.079
6	16296	0.068	0.067
7	13702	0.057	0.058
8	12356	0.051	0.051
9	11073	0.046	0.046

$5x+1$ **Data: random 10,000 digit number,** $2^k || 5x+1$

27,004 iterations, $\chi^2 = 1.8$ (5% 15.5).

Digit	Number	Observed	Benford
1	8154	0.302	0.301
2	4770	0.177	0.176
3	3405	0.126	0.125
4	2634	0.098	0.097
5	2105	0.078	0.079
6	1787	0.066	0.067
7	1568	0.058	0.058
8	1357	0.050	0.051
9	1224	0.045	0.046

$5x+1$ Data: random 10,000 digit number, $2|5x+1$

241,344 iterations, $\chi^2 = 3 \cdot 10^{-4}$ (5% 15.5).

Digit	Number	Observed	Benford
1	72652	0.301	0.301
2	42499	0.176	0.176
3	30153	0.125	0.125
4	23388	0.097	0.097
5	19110	0.079	0.079
6	16159	0.067	0.067
7	13995	0.058	0.058
8	12345	0.051	0.051
9	11043	0.046	0.046

Conclusions

Current / Future Investigations

- Develop more sophisticated tests for fraud.
- Study digits of other systems.
- Investigate interplay between algebraic structure of number and rate of convergence to Benford behavior.

Conclusions and Future Investigations

- See many different systems exhibit Benford behavior.
- Ingredients of proofs (logarithms, equidistribution).
- Applications to fraud detection / data integrity.

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Appendix:
Why Benford's Law?

Streets

Not all data sets satisfy Benford's Law.

- Long street $[1, L]$: $L = 199$ versus $L = 999$.
- Oscillates b/w $1/9$ and $5/9$ with first digit 1.

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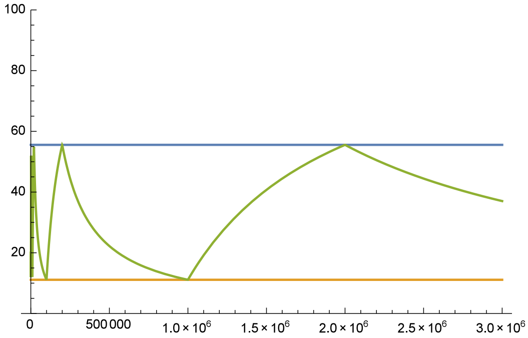

Probability first digit 1 versus street length L.

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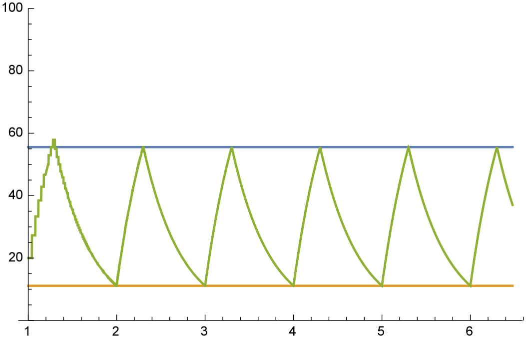

Probability first digit 1 versus log(street length L).

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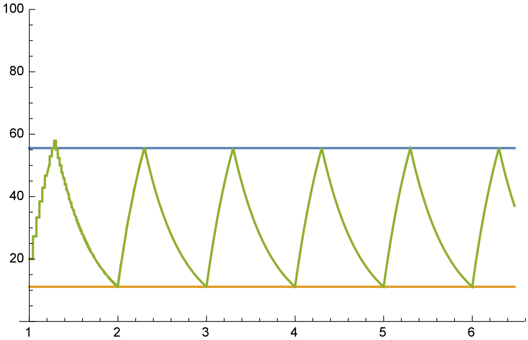


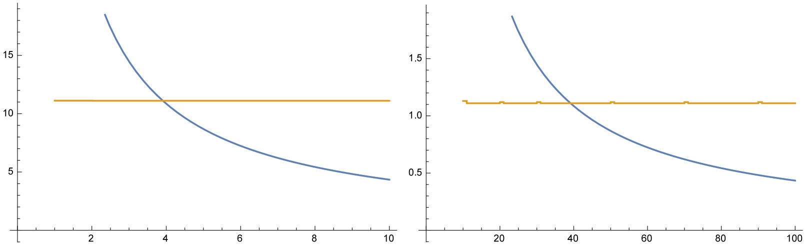
Probability first digit 1 versus log(street length L).

What if we have many streets of different lengths?

Amalgamating Streets

All houses: 1000 Streets,

each from 1 to 10000.



First digit and first two digits vs Benford.

Amalgamating Streets

All houses: 1000 Streets,

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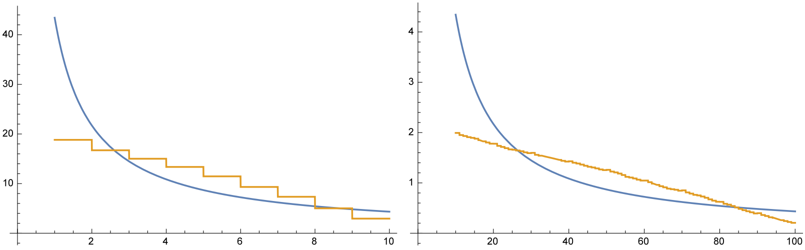

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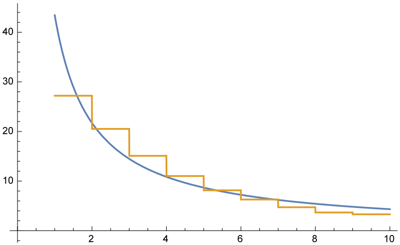

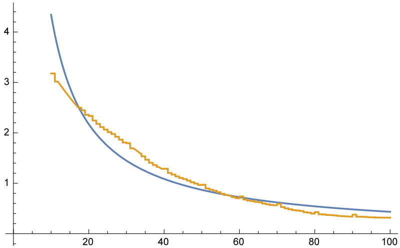

First digit and first two digits vs Benford.

Conclusion: More processes, closer to Benford.

Amalgamating Streets

All houses: 1000 Streets,

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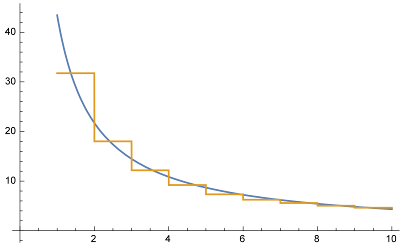
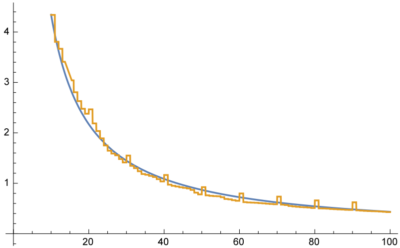
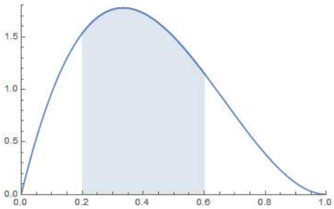




First digit and first two digits vs Benford.

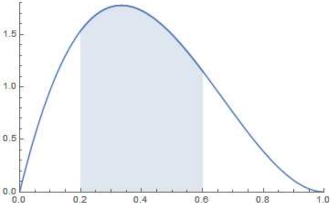
Conclusion: More processes, closer to Benford.

Probability Review



- Let X be random variable with density $p(x)$:
  - $\diamond$ $p(x) \geq 0$; $\int_{-\infty}^{\infty} p(x) \mathrm{d}x = 1$;
  - $\diamond$ $\text{Prob}\,(a \leq X \leq b) = \int_a^b p(x) \mathrm{d}x$.

Probability Review



- Let X be random variable with density $p(x)$:
  - $\diamond$ $p(x) \geq 0$; $\int_{-\infty}^{\infty} p(x)\mathrm{d}x = 1$;
  - $\diamond$ $\mathrm{Prob}\,(a \leq X \leq b) = \int_a^b p(x)\mathrm{d}x$.
- Mean $\mu = \int_{-\infty}^{\infty} x p(x)\mathrm{d}x$.

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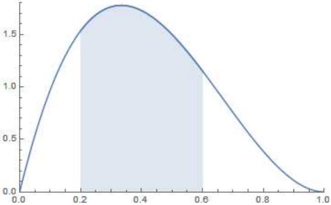

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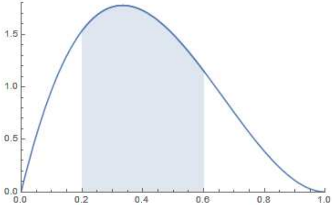
- Variance $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 p(x) \mathrm{d}x$.

Probability Review



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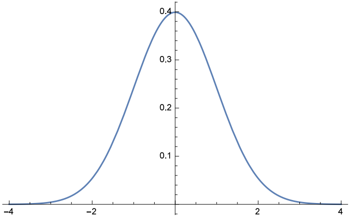
  ◇ $\mathrm{Prob}\,(a \leq X \leq b) = \int_a^b p(x)\mathrm{d}x$.
- Mean $\mu = \int_{-\infty}^{\infty} x p(x)\mathrm{d}x$.
- Variance $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 p(x)\mathrm{d}x$.
- Independence: knowledge of one random variable gives no knowledge of the other.

Central Limit Theorem

$$\text{Normal } N(\mu, \sigma^2): \; p(x) \;=\; e^{-(x-\mu)^2/2\sigma^2}/\sqrt{2\pi\sigma^2}.$$



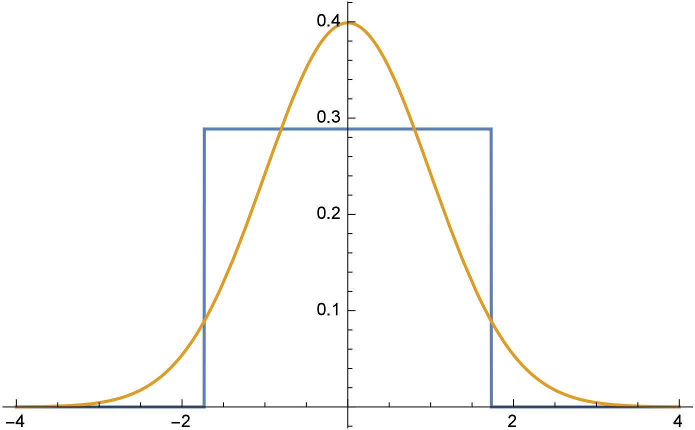
Theorem

If $X_1, X_2, \dots$ independent, identically distributed random variables (mean μ, variance σ^2, finite moments) then

$$S_N \;:=\; \frac{X_1 + \cdots + X_N - N\mu}{\sigma\sqrt{N}} \text{ converges to } N(0,1).$$

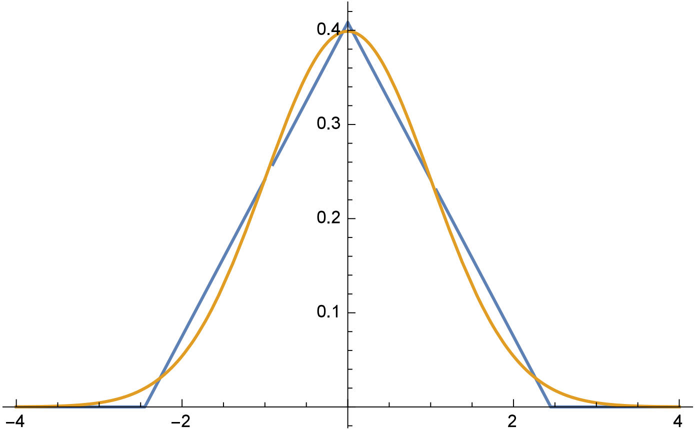
Central Limit Theorem: Sums of Uniform Random Variables
$X_i \sim \mathrm{Unif}(-1/2, 1/2)$ (adjusted to mean 0, variance 1)

$$Y_1 = X_1/\sigma_{X_1} \text{ vs } N(0,1).$$



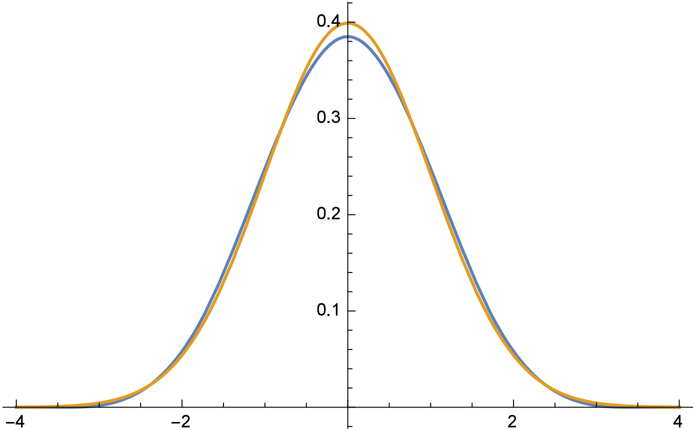
Central Limit Theorem: Sums of Uniform Random Variables $X_i \sim \mathrm{Unif}(-1/2, 1/2)$ (adjusted to mean 0, variance 1)

$$Y_2 = (X_1 + X_2)/\sigma_{X_1+X_2} \text{ vs } N(0,1).$$



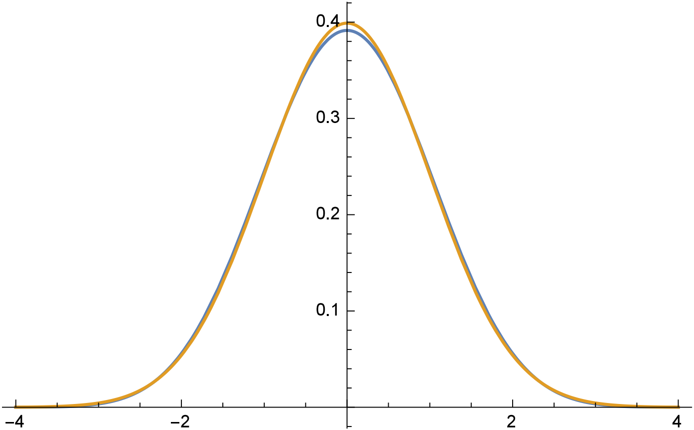
Central Limit Theorem: Sums of Uniform Random Variables $X_i \sim \mathrm{Unif}(-1/2, 1/2)$ (adjusted to mean 0, variance 1)

$$Y_4 = (X_1 + X_2 + X_3 + X_4)/\sigma_{X_1+X_2+X_3+X_4} \text{ vs } N(0,1).$$



Central Limit Theorem: Sums of Uniform Random Variables $X_i \sim \mathrm{Unif}(-1/2, 1/2)$ (adjusted to mean 0, variance 1)

$$Y_8 = (X_1 + \cdots + X_8)/\sigma_{X_1+\cdots+X_8} \text{ vs } N(0,1).$$



Central Limit Theorem: Sums of Uniform Random Variables

$X_i \sim \mathrm{Unif}(-1/2, 1/2)$ **(adjusted to mean 0, variance 1)**

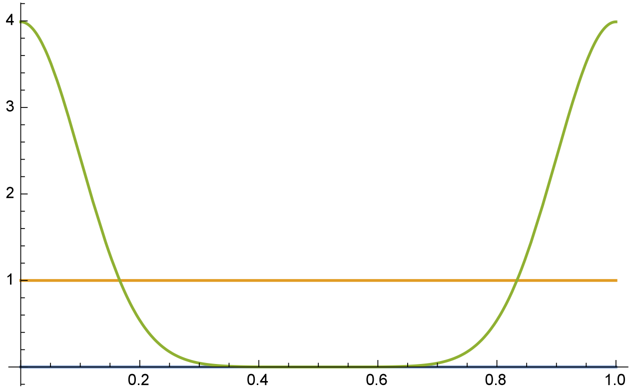
Density of $Y_4 = (X_1 + \cdots + X_4)/\sigma_{X_1+\cdots+X_4}$.

$$\frac{\begin{cases} \frac{1}{27}\left(18 + 9\sqrt{3}\,y - \sqrt{3}\,y^3\right) & y == 0 \\ \frac{1}{18}\left(12 - 6y^2 - \sqrt{3}\,y^3\right) & -\sqrt{3} < y < 0 \\ \frac{1}{54}\left(72 - 36\sqrt{3}\,y + 18y^2 - \sqrt{3}\,y^3\right) & \sqrt{3} < y < 2\sqrt{3} \\ \frac{1}{54}\left(18\sqrt{3}\,y - 18y^2 + \sqrt{3}\,y^3\right) & y == \sqrt{3} \\ \frac{1}{18}\left(12 - 6y^2 + \sqrt{3}\,y^3\right) & 0 < y < \sqrt{3} \\ \frac{1}{54}\left(72 + 36\sqrt{3}\,y + 18y^2 + \sqrt{3}\,y^3\right) & -2\sqrt{3} < y \le -\sqrt{3} \\ 0 & \text{True} \end{cases}}{\sqrt{3}}$$

(Don't even think of asking to see Y_8's!)

Normal Distributions Mod 1

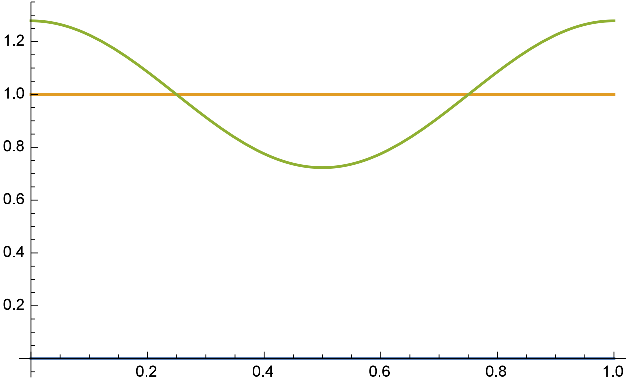
As $\sigma \to \infty$, $N(0, \sigma^2)$ mod 1 $\to \mathrm{Unif}(0, 1)$.



Variance is .01.

Normal Distributions Mod 1

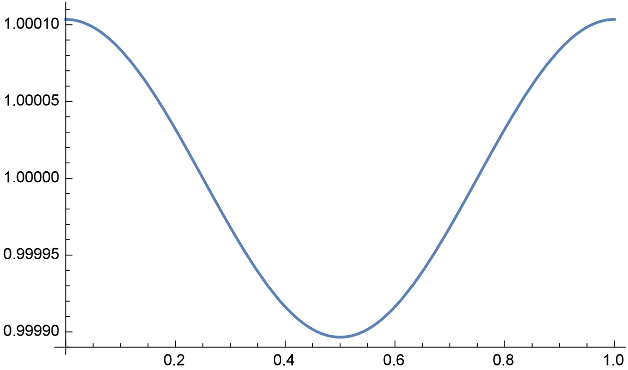
As $\sigma \to \infty$, $N(0, \sigma^2)$ mod 1 $\to \mathrm{Unif}(0, 1)$.



Variance is .1.

Normal Distributions Mod 1

As $\sigma \to \infty$, $N(0, \sigma^2)$ mod 1 $\to \mathrm{Unif}(0, 1)$.



Variance is .5.

Products and Benford's Law

Pavlovian Response: See a product, take a logarithm.

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$$\begin{aligned} V_N &= \log_{10}(X_1 \cdot X_2 \cdots X_N) \\ &= \log_{10} X_1 + \log_{10} X_2 + \cdots + \log_{10} X_N \end{aligned}$$

Products and Benford's Law

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$$\begin{aligned} V_N &= \log_{10}(X_1 \cdot X_2 \cdots X_N) \\ &= \log_{10} X_1 + \log_{10} X_2 + \cdots + \log_{10} X_N \\ &= Y_1 + Y_2 + \cdots + Y_N. \end{aligned}$$

Need distribution of V_N mod 1, which by CLT becomes uniform, implying Benfordness!