Benford's Law: Why the IRS might care about the $3x + 1$ problem and $\zeta(s)$.

Steven J Miller
Williams College

Steven.J.Miller@williams.edu
http://www.williams.edu/go/math/sjmiller/

Smith College, October 7th, 2008

Summary

- Review Benford's Law.
- Discuss examples and applications.
- Sketch proofs.
- Describe open problems.

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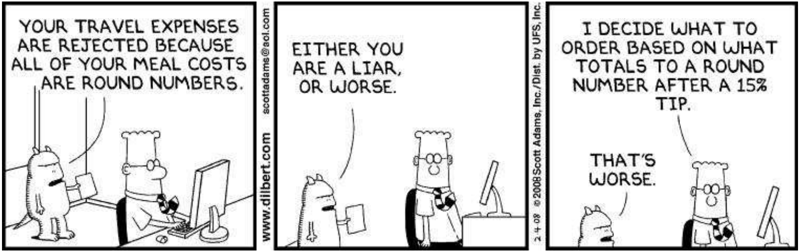

Benford's Law: Newcomb (1881), Benford (1938)

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  - ◇ Long street $[1, L]$: $L = 199$ versus $L = 999$.
  - ◇ Oscillates between $1/9$ and $5/9$ with first digit 1.
  - ◇ Many streets of different sizes: close to Benford.

Examples

- recurrence relations
- special functions (such as $n!$)
- iterates of power, exponential, rational maps
- products of random variables
- L-functions, characteristic polynomials
- iterates of the $3x + 1$ map
- differences of order statistics
- hydrology and financial data
- many hierarchical Bayesian models

Applications

- analyzing round-off errors
- determining the optimal way to store numbers
- detecting tax and image fraud, and data integrity

General Theory

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Key observation: $\log_{10}(x) = \log_{10}(\widetilde{x}) \bmod 1$ if and only if x and $\widetilde{x}$ have the same leading digits. Thus often study $y = \log_{10} x$.

Equidistribution and Benford's Law

Equidistribution

$\{y_n\}_{n=1}^{\infty}$ is equidistributed modulo 1 if probability $y_n \bmod 1 \in [a, b]$ tends to $b - a$:

$$\frac{\#\{n \le N : y_n \bmod 1 \in [a, b]\}}{N} \rightarrow b - a.$$

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  Proof: if rational: $2 = 10^{p/q}$.

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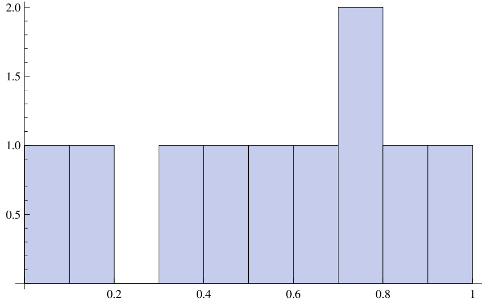
  Thus $2^q = 10^p$ or $2^{q-p} = 5^p$, impossible.

Example of Equidistribution: $n\sqrt{\pi}$ mod 1



$n\sqrt{\pi}$ mod 1 for $n \leq 10$

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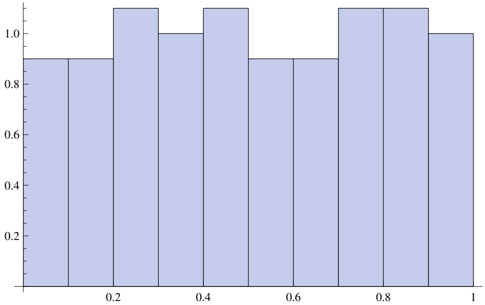

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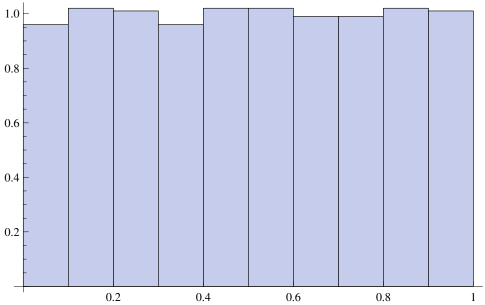

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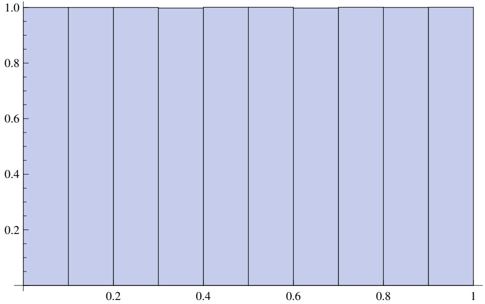

$n\sqrt{\pi} \bmod 1$ for $n \leq 10{,}000$

Denseness

Dense

A sequence $\{z_n\}_{n=1}^{\infty}$ of numbers in $[0,1]$ is dense if for any interval $[a,b]$ there are infinitely many z_n in $[a,b]$.

- Dirichlet's Box (or Pigeonhole) Principle:
  If $n+1$ objects are placed in n boxes, at least one box has two objects.

- Denseness of $n\alpha$:
  Thm: If $\alpha \notin \mathbb{Q}$ then $z_n = n\alpha \bmod 1$ is dense.

Proof $n\alpha$ mod 1 dense if $\alpha \notin \mathbb{Q}$

- Enough to show in $[0, b]$ infinitely often for any b.
- Choose any integer $Q > 1/b$.
- Q bins: $\left[0, \frac{1}{Q}\right], \left[\frac{1}{Q}, \frac{2}{Q}\right], \dots, \left[\frac{Q-1}{Q}, Q\right]$.
- $Q+1$ objects: $\{\alpha \bmod 1, 2\alpha \bmod 1, \dots, (Q+1)\alpha \bmod 1\}$.
- Two in same bin, say $q_1\alpha$ mod 1 and $q_2\alpha$ mod 1.
- Exists integer p with $0 < q_2\alpha - q_1\alpha - p < \frac{1}{Q}$.
- Get $(q_2 - q_1)\alpha$ mod $1 \in [0, b]$.

Logarithms and Benford's Law

Fundamental Equivalence

Data set $\{x_i\}$ is Benford base B if $\{y_i\}$ is equidistributed mod 1, where $y_i = \log_B x_i$.

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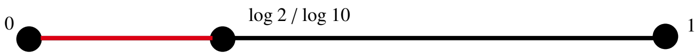

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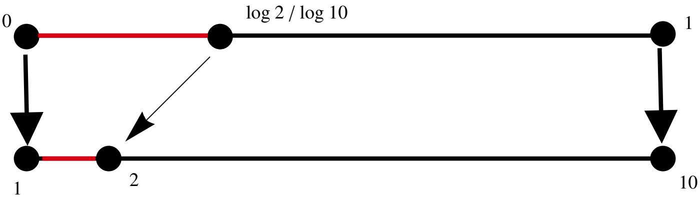

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Proof:

- $x = M_B(x) \cdot B^k$ for some $k \in \mathbb{Z}$.
- $\mathrm{FD}_B(x) = d$ iff $d \le M_B(x) < d + 1$.
- $\log_B d \le y < \log_B(d+1)$, $y = \log_B x$ mod 1.
- If $Y \sim \mathrm{Unif}(0,1)$ then above probability is $\log_B\left(\frac{d+1}{d}\right)$.

Examples

- 2^n is Benford base 10 as $\log_{10} 2 \notin \mathbb{Q}$.

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- Most linear recurrence relations Benford:
  - ◇ $a_{n+1} = 2a_n - a_{n-1}$
  - ◇ take $a_0 = a_1 = 1$ or $a_0 = 0$, $a_1 = 1$.

Digits of 2^n

First 60 values of 2^n (only displaying 30)

			digit	#	Obs Prob	Benf Prob
1	1024	1048576	1	18	.300	.301
2	2048	2097152	2	12	.200	.176
4	4096	4194304	3	6	.100	.125
8	8192	8388608	4	6	.100	.097
16	16384	16777216	5	6	.100	.079
32	32768	33554432	6	4	.067	.067
64	65536	67108864	7	2	.033	.058
128	131072	134217728	8	5	.083	.051
256	262144	268435456	9	1	.017	.046
512	524288	536870912				

Data Analysis

- χ^2-Tests: Test if theory describes data
  - ◇ Expected probability: $p_d = \log_{10}\left(\frac{d+1}{d}\right)$.
  - ◇ Expect about Np_d will have first digit d.
  - ◇ Observe $\mathrm{Obs}(d)$ with first digit d.
  - ◇ $\chi^2 = \sum_{d=1}^{9} \frac{(\mathrm{Obs}(d) - Np_d)^2}{Np_d}$.
  - ◇ Smaller χ^2, more likely correct model.
- Will study γ^n, e^n, π^n.

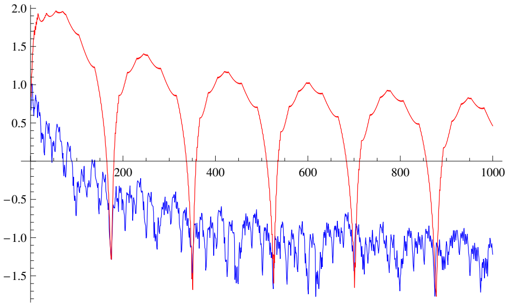
Logarithms and Benford's Law

χ^2 values for α^n, $1 \le n \le N$ (5% 15.5).

N	$\chi^2(\gamma)$	$\chi^2(e)$	$\chi^2(\pi)$
100	0.72	0.30	46.65
200	0.24	0.30	8.58
400	0.14	0.10	10.55
500	0.08	0.07	2.69
700	0.19	0.04	0.05
800	0.04	0.03	6.19
900	0.09	0.09	1.71
1000	0.02	0.06	2.90

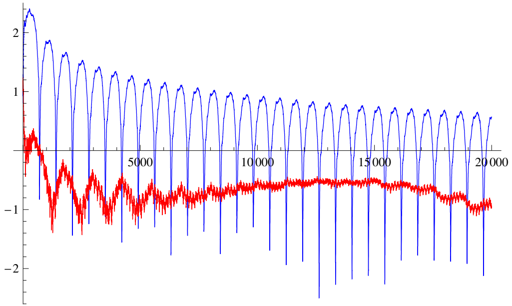
Logarithms and Benford's Law: Base 10

$\log(\chi^2)$ vs N for π^n (red) and e^n (blue), $n \in \{1, \dots, N\}$. Note $\pi^{175} \approx 1.0028 \cdot 10^{87}$, (5%, $\log(\chi^2) \approx 2.74$).



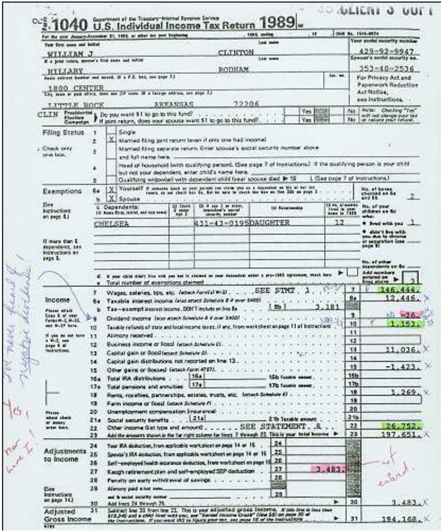
Logarithms and Benford's Law: Base 20

$\log(\chi^2)$ vs N for π^n (red) and e^n (blue), $n \in \{1, \dots, N\}$. Note $e^3 \approx 20.0855$, (5%, $\log(\chi^2) \approx 2.74$).



Applications

Applications for the IRS: Detecting Fraud



CLIENT'S COPY

1040 Department of the Treasury—Internal Revenue Service
U.S. Individual Income Tax Return 1989

WILLIAM J. CLINTON 429-92-9947
HILLARY RODHAM 353-40-2536
1800 CENTER
LITTLE ROCK ARKANSAS 72206

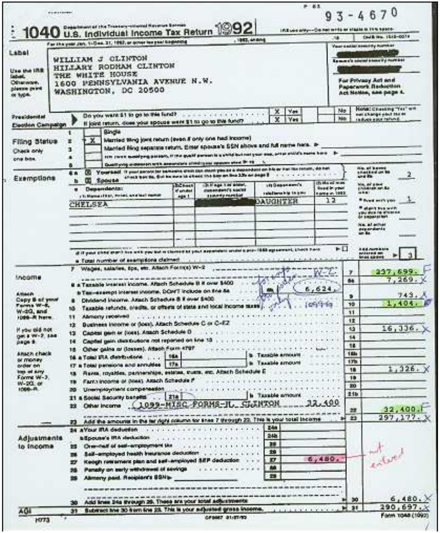
Filing Status
1 Single
2 X Married filing joint return (even if only one had income)
3 Married filing separate return. Enter spouse's social security number above and full name here.
4 Head of household (with qualifying person).
5 Qualifying widow(er) with dependent child

Exemptions
6a X Yourself
b X Spouse
c Dependents: CHELSEA 431-43-0195 DAUGHTER 12
e Total number of exemptions claimed 3

	Income	
7	Wages, salaries, tips, etc. SEE STMT 1	146,444.
8a	Taxable interest income	12,446.
8b	Tax-exempt interest income 3,101	
9	Dividend income	26.
10	Taxable refunds of state and local income taxes	1,153.
11	Alimony received	
12	Business income or (loss)	
13	Capital gain or (loss)	11,036.
14	Capital gain distributions not reported on line 13	
15	Other gains or (losses)	-1,423.
16a	Total IRA distributions	
17a	Total pensions and annuities	
18	Rents, royalties, partnerships, estates, trusts, etc.	1,269.
19	Farm income or (loss)	
20	Unemployment compensation (insurance)	
21a	Social security benefits	
22	Other income SEE STATEMENT 8	26,752.
23	Add the amounts shown in the far right column for lines 7 through 22. This is your total income	197,651.

	Adjustments to Income	
24	Your IRA deduction	
25	Spouse's IRA deduction	
26	Self-employed health insurance deduction	
27	Keogh retirement plan and self-employed SEP deduction	3,483.
28	Penalty on early withdrawal of savings	
29	Alimony paid	
30	Add lines 24 through 29	3,483.
31	Adjusted Gross Income: Subtract line 30 from line 23. This is your adjusted gross income	194,168.

Applications for the IRS: Detecting Fraud



93-4670

1040 U.S. Individual Income Tax Return 1992

Label

WILLIAM J CLINTON
HILLARY RODHAM CLINTON
THE WHITE HOUSE
1600 PENNSYLVANIA AVENUE N.W.
WASHINGTON, DC 20500

Your social security number [redacted]

Spouse's social security number [redacted]

For Privacy Act and Paperwork Reduction Act Notice, see page 4.

Presidential Election Campaign — Do you want $1 to go to this fund? X Yes — If joint return, does your spouse want $1 to go to this fund? X Yes

Filing Status — 1 Single; 2 X Married filing joint return (even if only one had income); 3 Married filing separate return. Enter spouse's SSN above and full name here.

Exemptions — 6a X Yourself; b X Spouse; c Dependents: CHELSEA [redacted] DAUGHTER 12

Income
7 Wages, salaries, tips, etc. Attach Form(s) W-2 — 237,699.
8a Taxable interest income. Attach Schedule B if over $400 — 7,269.
b Tax-exempt interest income. DON'T include on line 8a — 6,624
9 Dividend income. Attach Schedule B if over $400 — 743.
10 Taxable refunds, credits, or offsets of state and local income taxes — 1,404.
11 Alimony received
12 Business income or (loss). Attach Schedule C or C-EZ
13 Capital gain or (loss). Attach Schedule D — 16,336.
14 Capital gain distributions not reported on line 13
15 Other gains or (losses). Attach Form 4797
16a Total IRA distributions
17a Total pensions and annuities
18 Rents, royalties, partnerships, estates, trusts, etc. Attach Schedule E — 1,326.
19 Farm income or (loss). Attach Schedule F
20 Unemployment compensation
21a Social Security benefits
22 Other income (1099-MISC FORMS-HL CLINTON 32,400) — 32,400.
23 Add the amounts in the far right column for lines 7 through 22. This is your total income — 297,177.

Adjustments to Income
24a Your IRA deduction
b Spouse's IRA deduction
25 One-half of self-employment tax
26 Self-employed health insurance deduction
27 Keogh retirement plan and self-employed SEP deduction — 6,480.
28 Penalty on early withdrawal of savings
29 Alimony paid.
30 Add lines 24a through 29. These are your total adjustments — 6,480.

AGI
31 Subtract line 30 from line 23. This is your adjusted gross income — 290,697.

Form 1040 (1992)

Applications for the IRS: Detecting Fraud

Exhibit 3: Check Fraud in Arizona

The table lists the checks that a manager in the office of the Arizona State Treasurer wrote to divert funds for his own use. The vendors to whom the checks were issued were fictitious.

Date of Check	Amount
October 9, 1992	$ 1,927.48
↓	27,902.31
October 14, 1992	86,241.90
	72,117.46
	81,321.75
↓	97,473.96
October 19, 1992	93,249.11
	89,658.17
	87,776.89
	92,105.83
	79,949.16
	87,602.93
	96,879.27
	91,806.47
	84,991.67
	90,831.83
	93,766.67
	88,338.72
	94,639.49
	83,709.28
	96,412.21
	88,432.86
↓	71,552.16
TOTAL	$ 1,878,687.58

Applications for the IRS: Detecting Fraud (cont)

- Embezzler started small and then increased dollar amounts.
- Most amounts below $100,000 (critical threshold for data requiring additional scrutiny).
- Over 90% had first digit of 7, 8 or 9.

Detecting Fraud

Bank Fraud

- Audit of a bank revealed huge spike of numbers starting with 48 and 49, most due to one person.
- Write-off limit of $5,000. Officer had friends applying for credit cards, ran up balances just under $5,000 then he would write the debts off.

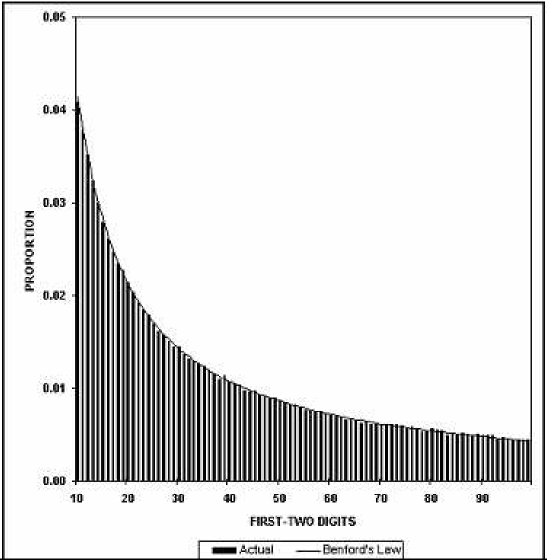
Detecting Fraud

Enron

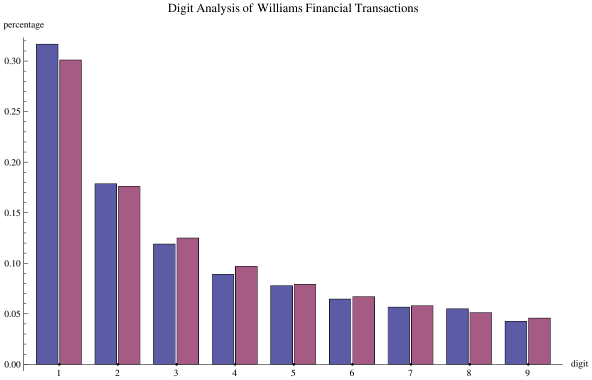
- Benford's Law detected manipulation of revenue numbers.

- Results showed a tendency towards round Earnings Per Share (0.10, 0.20, etc.). Consistent with a small but noticeable increase in earnings management in 2002.

Data Integrity: Stream Flow Statistics: 130 years, 457,440 records



Analysis of Williams College Transactions (thanks to Richard McDowell): September 6, 2006 to June 29, 2007: 64,000+ transactions



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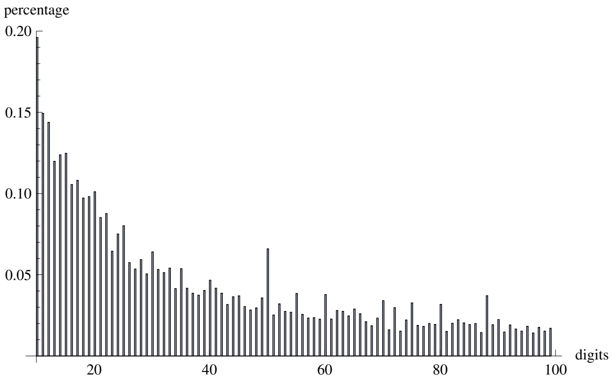

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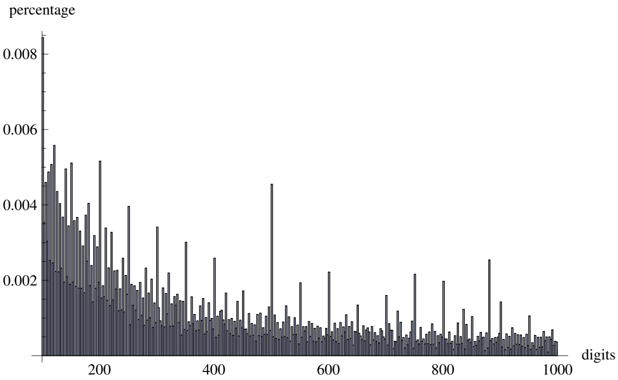

Benford Good Processes

Poisson Summation and Benford's Law: Definitions

- Feller, Pinkham (often exact processes)
- data $Y_{T,B} = \log_B \overrightarrow{X}_T$ (discrete/continuous):

$$\mathbb{P}(A) \ = \ \lim_{T\to\infty} \frac{\#\{n \in A : n \le T\}}{T}$$

- Poisson Summation Formula: f nice:

$$\sum_{\ell=-\infty}^{\infty} f(\ell) \ = \ \sum_{\ell=-\infty}^{\infty} \widehat{f}(\ell),$$

$$\text{Fourier transform } \widehat{f}(\xi) \ = \ \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi} dx.$$

Benford Good Process

X_T is Benford Good if there is a nice f st

$$\mathrm{CDF}_{\overrightarrow{Y}_{T,B}}(y) \ = \ \int_{-\infty}^{y} \frac{1}{T} f\left(\frac{t}{T}\right) dt + E_T(y) \ := \ G_T(y)$$

and monotonically increasing h ($h(|T|) \to \infty$):

- Small tails: $G_T(\infty) - G_T(Th(T)) = o(1)$, $G_T(-Th(T)) - G_T(-\infty) = 0(1)$.
- Decay of the Fourier Transform: $\sum_{\ell \neq 0} \left| \frac{\widehat{f}(T\ell)}{\ell} \right| = o(1)$.
- Small translated error: $\mathcal{E}(a, b, T)) = \sum_{|\ell| \leq Th(T)} [E_T(b+\ell) - E_T(a+\ell)] = o(1)$.

Main Theorem

Theorem (Kontorovich and M–, 2005)

X_T converging to X as $T \to \infty$ (think spreading Gaussian). If X_T is Benford good, then X is Benford.

- Examples
  - ◇ L-functions
  - ◇ characteristic polynomials (RMT)
  - ◇ $3x+1$ problem
  - ◇ geometric Brownian motion.

Sketch of the proof

- Structure Theorem:
  - ◇ main term is something nice spreading out
  - ◇ apply Poisson summation
- Control translated errors:
  - ◇ hardest step
  - ◇ techniques problem specific

Sketch of the proof (continued)

$$\sum_{\ell=-\infty}^{\infty} \mathbb{P}\left(a+\ell \leq \overrightarrow{Y}_{T,B} \leq b+\ell\right)$$

$$= \sum_{|\ell|\leq Th(T)} [G_T(b+\ell) - G_T(a+\ell)] + o(1)$$

$$= \int_a^b \sum_{|\ell|\leq Th(T)} \frac{1}{T} f\left(\frac{t}{T}\right) dt + \mathcal{E}(a,b,T) + o(1)$$

$$= \widehat{f}(0)\cdot(b-a) + \sum_{\ell\neq 0} \widehat{f}(T\ell)\frac{e^{2\pi i b\ell} - e^{2\pi i a\ell}}{2\pi i\ell} + o(1).$$

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

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$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_{p \text{ prime}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right)$$
$$= \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \cdots\right)\left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \cdots\right)$$
$$= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{(2 \cdot 3)^s} + \cdots$$

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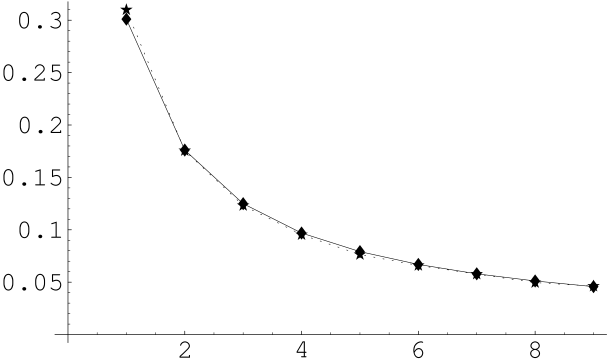
$\zeta(2) = \pi^2/6$ implies infinitely many primes.

Riemann Zeta Function

$$\left|\zeta\left(\frac{1}{2}+i\frac{k}{4}\right)\right|,\ k \in \{0, 1, \dots, 65535\}.$$



The $3x + 1$ Problem and Benford's Law

$3x + 1$ Problem

- Kakutani (conspiracy), Erdös (not ready).
- x odd, $T(x) = \frac{3x+1}{2^k}$, $2^k || 3x + 1$.

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- $7 \to_1 11 \to_1 17 \to_2 13 \to_3 5 \to_4 1 \to_2 1$,
  2-path $(1, 1)$, 5-path $(1, 1, 2, 3, 4)$.
  m-path: $(k_1, \dots, k_m)$.

Heuristic Proof of $3x + 1$ Conjecture

$$a_{n+1} = T(a_n)$$

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$$\begin{aligned} a_{n+1} &= T(a_n) \\ \mathbb{E}[\log a_{n+1}] &\approx \sum_{k=1}^{\infty} \frac{1}{2^k} \log\left(\frac{3a_n}{2^k}\right) \end{aligned}$$

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Geometric Brownian Motion, drift $\log(3/4) < 1$.

Structure Theorem: Sinai, Kontorovich-Sinai

$$\mathbb{P}(A) = \lim_{N\to\infty} \frac{\#\{n\le N : n\equiv 1,5 \bmod 6, n\in A\}}{\#\{n\le N : n\equiv 1,5 \bmod 6\}}.$$

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$(k_1,\dots,k_m)$: two full arithm progressions:

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Theorem (Sinai, Kontorovich-Sinai)

k_i-values are i.i.d.r.v. (geometric, 1/2):

$$\mathbb{P}\left(\frac{\log_2\left[\frac{x_m}{\left(\frac{3}{4}\right)^m x_0}\right]}{\sqrt{2m}} \le a\right) = \mathbb{P}\left(\frac{S_m - 2m}{\sqrt{2m}} \le a\right)$$

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$3x+1$ and Benford

Theorem (Kontorovich and M–, 2005)

As $m \to \infty$, $x_m/(3/4)^m x_0$ is Benford.

Theorem (Lagarias-Soundararajan 2006)

$X \geq 2^N$, for all but at most $c(B)N^{-1/36}X$ initial seeds the distribution of the first N iterates of the $3x+1$ map are within $2N^{-1/36}$ of the Benford probabilities.

Sketch of the proof

- Failed Proof: lattices, bad errors.
- CLT: $(S_m - 2m)/\sqrt{2m} \to N(0,1)$:

$$\mathbb{P}\left(S_m - 2m = k\right) = \frac{\eta(k/\sqrt{m})}{\sqrt{m}} + O\left(\frac{1}{g(m)\sqrt{m}}\right).$$

- Quantified Equidistribution:
  $I_\ell = \{\ell M, \dots, (\ell+1)M - 1\}$, $M = m^c$, $c < 1/2$
  $k_1, k_2 \in I_\ell$: $\left|\eta\left(\frac{k_1}{\sqrt{m}}\right) - \eta\left(\frac{k_2}{\sqrt{m}}\right)\right|$ small
  $C = \log_B 2$ of irrationality type $\kappa < \infty$:

$$\#\{k \in I_\ell : \overline{kC} \in [a,b]\} = M(b-a) + O(M^{1+\epsilon-1/\kappa}).$$

Irrationality Type

Irrationality type

α has irrationality type κ if κ is the supremum of all γ with

$$\underline{\lim}_{q\to\infty} q^{\gamma+1} \min_p \left|\alpha - \frac{p}{q}\right| = 0.$$

- Algebraic irrationals: type 1 (Roth's Thm).
- Theory of Linear Forms: $\log_B 2$ of finite type.

Linear Forms

Theorem (Baker)

$\alpha_1, \dots, \alpha_n$ algebraic numbers height $A_j \geq 4$, $\beta_1, \dots, \beta_n \in \mathbb{Q}$ with height at most $B \geq 4$,

$$\Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n.$$

If $\Lambda \neq 0$ then $|\Lambda| > B^{-C\Omega \log \Omega'}$, with $d = [\mathbb{Q}(\alpha_i, \beta_j) : \mathbb{Q}]$, $C = (16nd)^{200n}$, $\Omega = \prod_j \log A_j$, $\Omega' = \Omega / \log A_n$.

Gives $\log_{10} 2$ of finite type, with $\kappa < 1.2 \cdot 10^{602}$:

$$|\log_{10} 2 - p/q| = |q \log 2 - p \log 10| / q \log 10.$$

Quantified Equidistribution

Theorem (Erdös-Turan)

$$D_N = \frac{\sup_{[a,b]} \left|N(b-a) - \#\{n \le N : x_n \in [a,b]\}\right|}{N}$$

There is a C such that for all m:

$$D_N \le C \cdot \left(\frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h} \left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h x_n}\right|\right)$$

Proof of Erdös-Turan

Consider special case $x_n = n\alpha$, $\alpha \notin \mathbb{Q}$.

- Exponential sum $\leq \frac{1}{|\sin(\pi h\alpha)|} \leq \frac{1}{2||h\alpha||}$.
- Must control $\sum_{h=1}^{m} \frac{1}{h||h\alpha||}$, see irrationality type enter.
- type κ, $\sum_{h=1}^{m} \frac{1}{h||h\alpha||} = O\left(m^{\kappa-1+\epsilon}\right)$, take $m = \lfloor N^{1/\kappa} \rfloor$.

$3x + 1$ Data: random 10,000 digit number, $2^k || 3x + 1$

80,514 iterations ($(4/3)^n = a_0$ predicts 80,319); $\chi^2 = 13.5$ (5% 15.5).

Digit	Number	Observed	Benford
1	24251	0.301	0.301
2	14156	0.176	0.176
3	10227	0.127	0.125
4	7931	0.099	0.097
5	6359	0.079	0.079
6	5372	0.067	0.067
7	4476	0.056	0.058
8	4092	0.051	0.051
9	3650	0.045	0.046

$3x+1$ Data: random 10,000 digit number, $2|3x+1$

241,344 iterations, $\chi^2 = 11.4$ (5% 15.5).

Digit	Number	Observed	Benford
1	72924	0.302	0.301
2	42357	0.176	0.176
3	30201	0.125	0.125
4	23507	0.097	0.097
5	18928	0.078	0.079
6	16296	0.068	0.067
7	13702	0.057	0.058
8	12356	0.051	0.051
9	11073	0.046	0.046

$5x+1$ Data: random 10,000 digit number, $2^k||5x+1$

27,004 iterations, $\chi^2 = 1.8$ (5% 15.5).

Digit	Number	Observed	Benford
1	8154	0.302	0.301
2	4770	0.177	0.176
3	3405	0.126	0.125
4	2634	0.098	0.097
5	2105	0.078	0.079
6	1787	0.066	0.067
7	1568	0.058	0.058
8	1357	0.050	0.051
9	1224	0.045	0.046

$5x+1$ Data: random 10,000 digit number, $2|5x+1$

241,344 iterations, $\chi^2 = 3 \cdot 10^{-4}$ (5% 15.5).

Digit	Number	Observed	Benford
1	72652	0.301	0.301
2	42499	0.176	0.176
3	30153	0.125	0.125
4	23388	0.097	0.097
5	19110	0.079	0.079
6	16159	0.067	0.067
7	13995	0.058	0.058
8	12345	0.051	0.051
9	11043	0.046	0.046

Conclusions

Conclusions and Future Investigations

- See many different systems exhibit Benford behavior.

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Conclusions and Future Investigations

- See many different systems exhibit Benford behavior.
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- Applications to fraud detection / data integrity.
- Future work:
  - ◇ Study digits of other systems.
  - ◇ Develop more sophisticated tests for fraud

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Products
and
Chains of Random Variables

Key Ingredients

- Mellin transform and Fourier transform related by logarithmic change of variable.
- Poisson summation from collapsing to modulo 1 random variables.

Preliminaries

- $\Xi_1, \dots, \Xi_n$ nice independent r.v.'s on $[0, \infty)$.
- Density $\Xi_1 \cdot \Xi_2$:

$$\int_0^\infty f_2\left(\frac{x}{t}\right) f_1(t) \frac{dt}{t}$$

$\diamond$ Proof: $\mathrm{Prob}(\Xi_1 \cdot \Xi_2 \in [0, x])$:

$$\begin{aligned} & \int_{t=0}^\infty \mathrm{Prob}\left(\Xi_2 \in \left[0, \frac{x}{t}\right]\right) f_1(t) dt \\ = & \int_{t=0}^\infty F_2\left(\frac{x}{t}\right) f_1(t) dt, \end{aligned}$$

differentiate.

Mellin Transform

$$(\mathcal{M}f)(s) = \int_0^\infty f(x) x^s \frac{dx}{x}$$
$$(\mathcal{M}^{-1}g)(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g(s) x^{-s} ds$$
$$g(s) = (\mathcal{M}f)(s), f(x) = (\mathcal{M}^{-1}g)(x).$$

$$(f_1 \star f_2)(x) = \int_0^\infty f_2\left(\frac{x}{t}\right) f_1(t) \frac{dt}{t}$$
$$(\mathcal{M}(f_1 \star f_2))(s) = (\mathcal{M}f_1)(s) \cdot (\mathcal{M}f_2)(s).$$

Mellin Transform Formulation: Products Random Variables

Theorem

X_i's independent, densities f_i. $\Xi_n = X_1 \cdots X_n$,

$$\begin{aligned} h_n(x_n) &= (f_1 \star \cdots \star f_n)(x_n) \\ (\mathcal{M}h_n)(s) &= \prod_{m=1}^{n} (\mathcal{M}f_m)(s). \end{aligned}$$

As $n \to \infty$, Ξ_n becomes Benford: $Y_n = \log_B \Xi_n$,
$|\mathrm{Prob}(Y_n \bmod 1 \in [a,b]) - (b-a)| \le$

$$(b-a) \cdot \sum_{\ell \neq 0, \ell=-\infty}^{\infty} \prod_{m=1}^{n} (\mathcal{M}f_i)\left(1 - \frac{2\pi i \ell}{\log B}\right).$$

Proof of Kossovsky's Chain Conjecture for certain densities

Conditions

- $\{\mathcal{D}_i(\theta)\}_{i\in I}$: one-parameter distributions, densities $f_{\mathcal{D}_i(\theta)}$ on $[0,\infty)$.
- $p : \mathbb{N} \to I$, $X_1 \sim \mathcal{D}_{p(1)}(1)$, $X_m \sim \mathcal{D}_{p(m)}(X_{m-1})$.
- $m \geq 2$,

$$f_m(x_m) \ = \ \int_0^\infty f_{\mathcal{D}_{p(m)}(1)}\left(\frac{x_m}{x_{m-1}}\right) f_{m-1}(x_{m-1}) \frac{dx_{m-1}}{x_{m-1}}$$

-

$$\lim_{n\to\infty} \sum_{\substack{\ell=-\infty \\ \ell\neq 0}}^{\infty} \prod_{m=1}^{n} (\mathcal{M} f_{\mathcal{D}_{p(m)}(1)}) \left(1 - \frac{2\pi i \ell}{\log B}\right) \ = \ 0$$

Proof of Kossovsky's Chain Conjecture for certain densities

Theorem (JKKKM)

- *If conditions hold, as $n \to \infty$ the distribution of leading digits of X_n tends to Benford's law.*
- *The error is a nice function of the Mellin transforms: if $Y_n = \log_B X_n$, then*

$$\left|\mathrm{Prob}(Y_n \bmod 1 \in [a,b]) - (b+a)\right| \le$$

$$\left|(b-a)\cdot \sum_{\substack{\ell=-\infty \\ \ell\neq 0}}^{\infty} \prod_{m=1}^{n} \left(\mathcal{M} f_{\mathcal{D}_{p(m)}}(1)\right)\left(1-\frac{2\pi i \ell}{\log B}\right)\right|$$

Example: All $X_i \sim \mathrm{Exp}(1)$

- $X_i \sim \mathrm{Exp}(1)$, $Y_n = \log_B \Xi_n$.
- Needed ingredients:
  - ◇ $\int_0^\infty \exp(-x)x^{s-1}dx = \Gamma(s)$.
  - ◇ $|\Gamma(1+ix)| = \sqrt{\pi x / \sinh(\pi x)}, x \in \mathbb{R}$.
- $|P_n(s) - \log_{10}(s)| \leq$

$$\log_B s \sum_{\ell=1}^{\infty} \left(\frac{2\pi^2 \ell / \log B}{\sinh(2\pi^2 \ell / \log B)} \right)^{n/2}.$$

Example: All $X_i \sim \mathrm{Exp}(1)$

Bounds on the error

- $|P_n(s) - \log_{10} s| \le$
  - $\diamond\ 3.3 \cdot 10^{-3} \log_B s$ if $n = 2$,
  - $\diamond\ 1.9 \cdot 10^{-4} \log_B s$ if $n = 3$,
  - $\diamond\ 1.1 \cdot 10^{-5} \log_B s$ if $n = 5$, and
  - $\diamond\ 3.6 \cdot 10^{-13} \log_B s$ if $n = 10$.
- Error at most

$$\log_{10} s \sum_{\ell=1}^{\infty} \left(\frac{17.148\ell}{\exp(8.5726\ell)} \right)^{n/2} \le .057^n \log_{10} s$$