Why the IRS cares about the Riemann Zeta Function and Number Theory (and why you should too!)

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Faculty Lecture Series, Williams College, 2/12/15

Interesting Question

Motivating Question: For a nice data set, such as the Fibonacci numbers, stock prices, street addresses of Williams employees and students, ..., what percent of the leading digits are 1?

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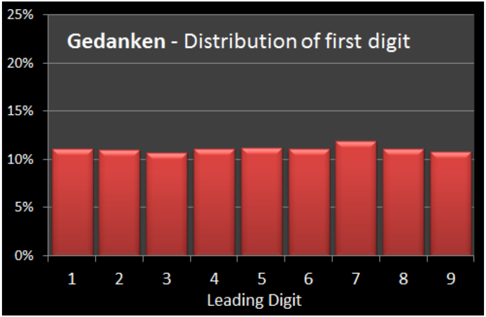

Natural guess: 10% (but immediately correct to 11%!).

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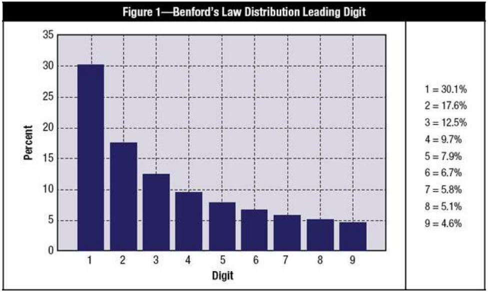
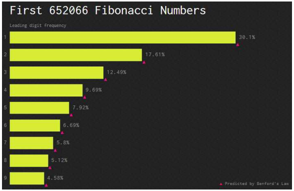


Answer: Benford's law!

Examples with First Digit Bias

Fibonacci numbers



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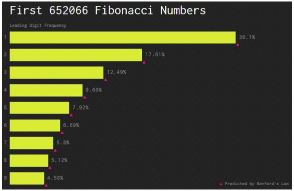
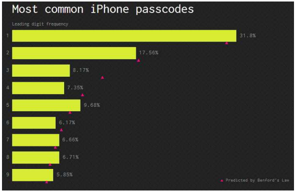


Most common iPhone passcodes



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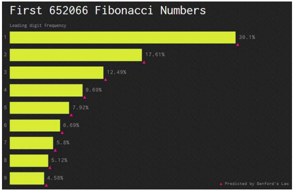

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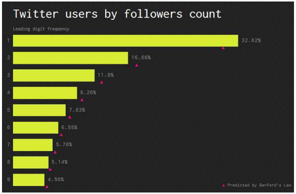

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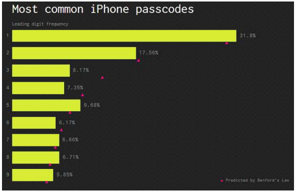

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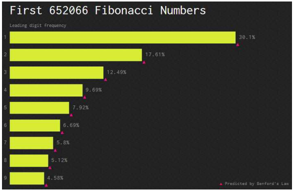

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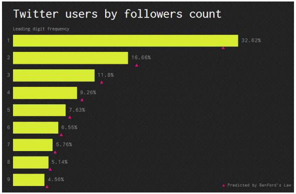

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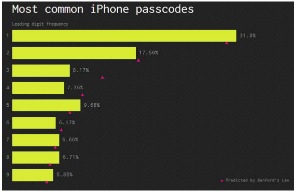
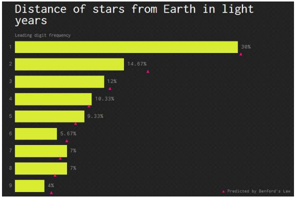


Distance of stars from Earth



Summary

- Explain Benford's Law.
- Discuss examples and applications.
- Sketch proofs.
- Describe open problems.

Caveats!

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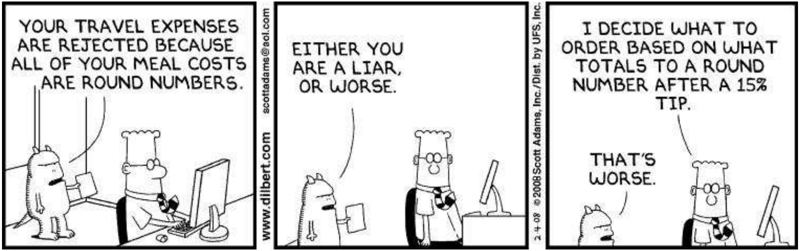


Examples

- recurrence relations
- special functions (such as $n!$)
- iterates of power, exponential, rational maps
- products of random variables
- L-functions, characteristic polynomials
- iterates of the $3x+1$ map
- differences of order statistics
- hydrology and financial data
- many hierarchical Bayesian models

Applications

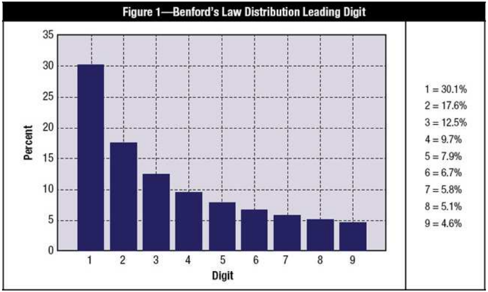
- Analyzing round-off errors.
- Determining the optimal way to store numbers.
- Detecting tax and image fraud, and data integrity.

General Theory

Benford's Law: Newcomb (1881), Benford (1938)

Statement

For many data sets, probability of observing a first digit of d base B is $\log_B\left(\frac{d+1}{d}\right)$; base 10 about 30% are 1s.



Benford's Law (probabilities)

Background Material

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- Key observation: $\log_{10}(x) = \log_{10}(\widetilde{x}) \mod 1$ if and only if x and $\widetilde{x}$ have the same leading digits.

Thus often study $y = \log_{10} x \mod 1$.
Advanced: $e^{2\pi i u} = e^{2\pi i (u \bmod 1)}$.

Equidistribution and Benford's Law

Equidistribution

$\{y_n\}_{n=1}^{\infty}$ is equidistributed modulo 1 if probability $y_n \bmod 1 \in [a,b]$ tends to $b-a$:

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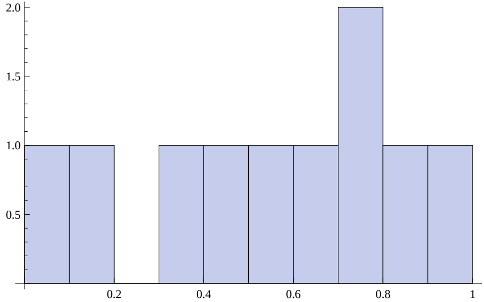
  Thus $2^q = 10^p$ or $2^{q-p} = 5^p$, impossible.

Example of Equidistribution: $n\sqrt{\pi} \bmod 1$



$n\sqrt{\pi} \bmod 1$ for $n \leq 10$

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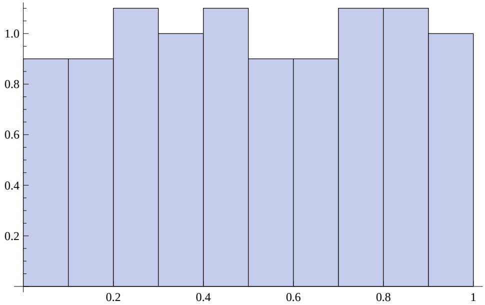

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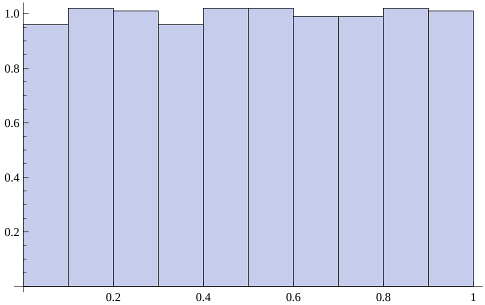

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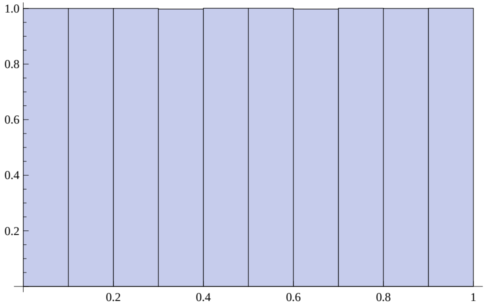

$n\sqrt{\pi}$ mod 1 for $n \le 10,000$

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Fundamental Equivalence

Data set $\{x_i\}$ is Benford base B if $\{y_i\}$ is equidistributed mod 1, where $y_i = \log_B x_i$.

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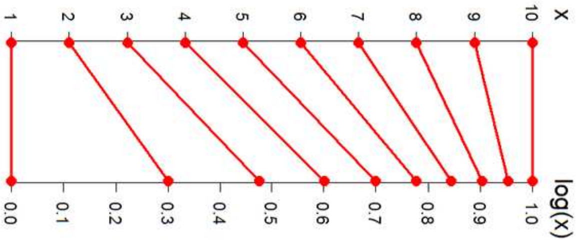
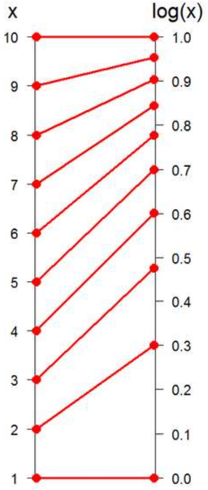


Logarithms and Benford's Law

$$\begin{aligned}&\mathrm{Prob}(\text{leading digit } d)\\ &= \log_{10}(d+1) - \log_{10}(d)\\ &= \log_{10}\left(\frac{d+1}{d}\right)\\ &= \log_{10}\left(1+\frac{1}{d}\right).\end{aligned}$$

Have Benford's law $\leftrightarrow$ mantissa of logarithms of data are uniformly distributed



Examples

- 2^n is Benford base 10 as $\log_{10} 2 \notin \mathbb{Q}$.

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- Most linear recurrence relations Benford:
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  - ◇ take $a_0 = a_1 = 1$ or $a_0 = 0$, $a_1 = 1$.

Digits of 2^n

First 60 values of 2^n (only displaying 30)

			digit	#	Obs Prob	Benf Prob
1	1024	1048576	1	18	.300	.301
2	2048	2097152	2	12	.200	.176
4	4096	4194304	3	6	.100	.125
8	8192	8388608	4	6	.100	.097
16	16384	16777216	5	6	.100	.079
32	32768	33554432	6	4	.067	.067
64	65536	67108864	7	2	.033	.058
128	131072	134217728	8	5	.083	.051
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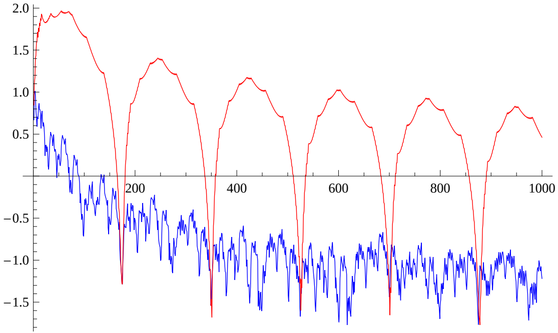
Logarithms and Benford's Law

χ^2 values for α^n, $1 \le n \le N$ (5% 15.5).

N	$\chi^2(\gamma)$	$\chi^2(e)$	$\chi^2(\pi)$
100	0.72	0.30	46.65
200	0.24	0.30	8.58
400	0.14	0.10	10.55
500	0.08	0.07	2.69
700	0.19	0.04	0.05
800	0.04	0.03	6.19
900	0.09	0.09	1.71
1000	0.02	0.06	2.90

Logarithms and Benford's Law: Base 10 (5%: $\log(\chi^2) \approx 2.74$)

$\log(\chi^2)$ vs N for π^n (red) and e^n (blue), $n \in \{1, \dots, N\}$.



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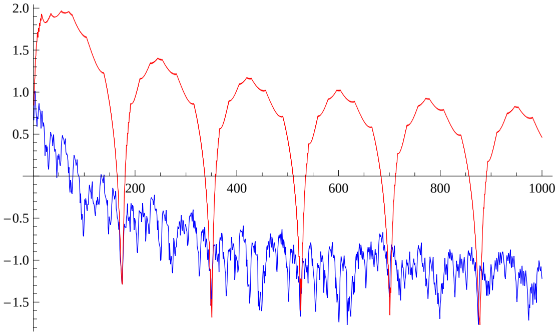
$\log(\chi^2)$ vs N for π^n (red) and e^n (blue), $n \in \{1, \dots, N\}$. Note $\pi^{175} \approx 1.0028 \cdot 10^{87}$.



Why Benford's Law?

Streets

Not all data sets satisfy Benford's Law.

- Long street $[1, L]$: $L = 199$ versus $L = 999$.
- Oscillates b/w $1/9$ and $5/9$ with first digit 1.

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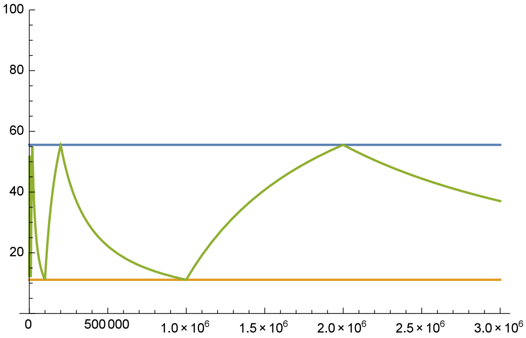

Probability first digit 1 versus street length L.

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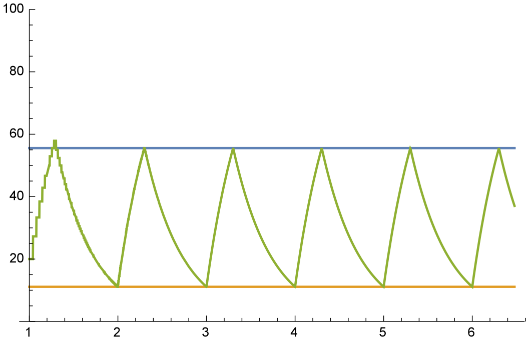

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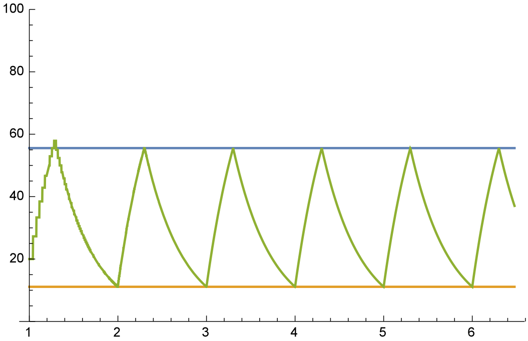
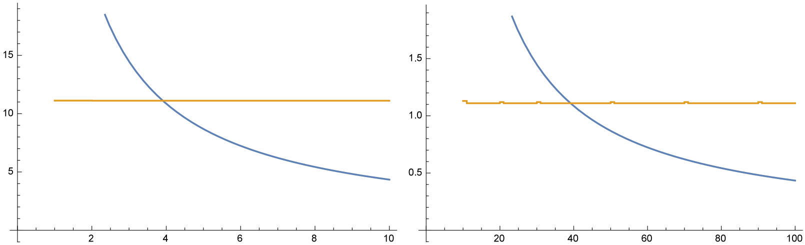


Probability first digit 1 versus log(street length L).

What if we have many streets of different lengths?

Amalgamating Streets

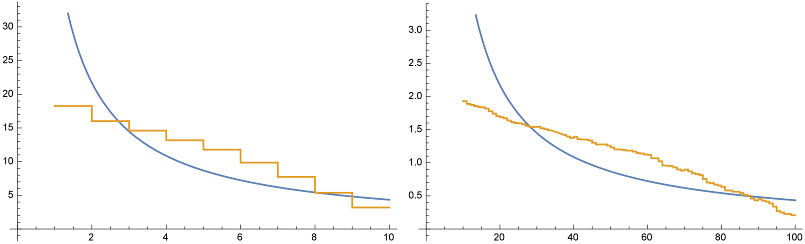
All houses: 100 Streets, each from 1 to 10000.



First digit and first two digits vs Benford.

Amalgamating Streets

All houses: 100 Streets, each from 1 to rand(10000).



First digit and first two digits vs Benford.

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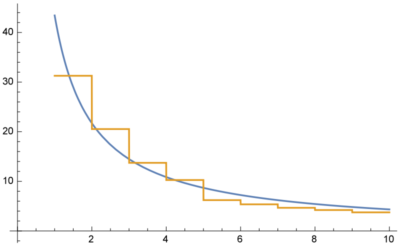
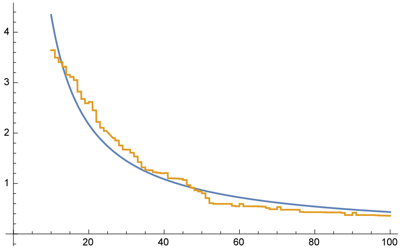
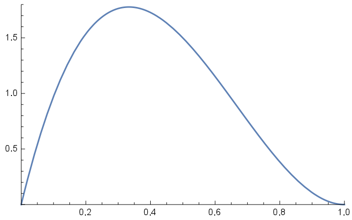




First digit and first two digits vs Benford.

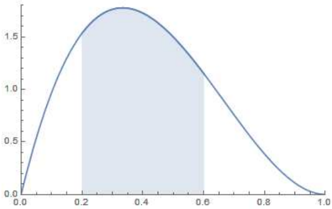
Conclusion: More processes, closer to Benford.

Probability Review



- Let X be random variable with density $p(x)$:
  - $\diamond$ $p(x) \geq 0$; $\int_{-\infty}^{\infty} p(x)\mathrm{d}x = 1$;
  - $\diamond$ $\mathrm{Prob}\left(a \leq X \leq b\right) = \int_a^b p(x)\mathrm{d}x$.

Probability Review



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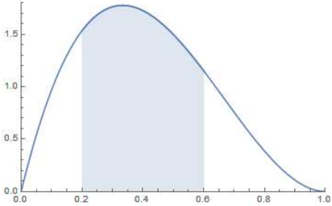
- Mean $\mu = \int_{-\infty}^{\infty} xp(x)\mathrm{d}x$.

Probability Review



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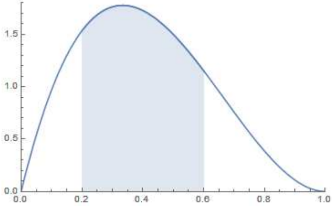
- Mean $\mu = \int_{-\infty}^{\infty} xp(x)\mathrm{d}x$.
- Variance $\sigma^2 = \int_{-\infty}^{\infty}(x-\mu)^2 p(x)\mathrm{d}x$.

Probability Review



- Let X be random variable with density $p(x)$:
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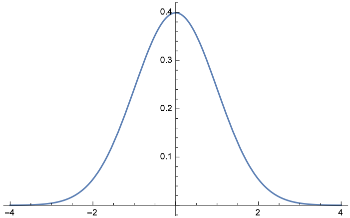
  - $\diamond$ $\text{Prob}\,(a \leq X \leq b) = \int_a^b p(x)\mathrm{d}x$.
- Mean $\mu = \int_{-\infty}^{\infty} x p(x)\mathrm{d}x$.
- Variance $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 p(x)\mathrm{d}x$.
- Independence: knowledge of one random variable gives no knowledge of the other.

Central Limit Theorem

$$\text{Normal } N(\mu, \sigma^2): \ p(x) \ = \ e^{-(x-\mu)^2/2\sigma^2}/\sqrt{2\pi\sigma^2}.$$



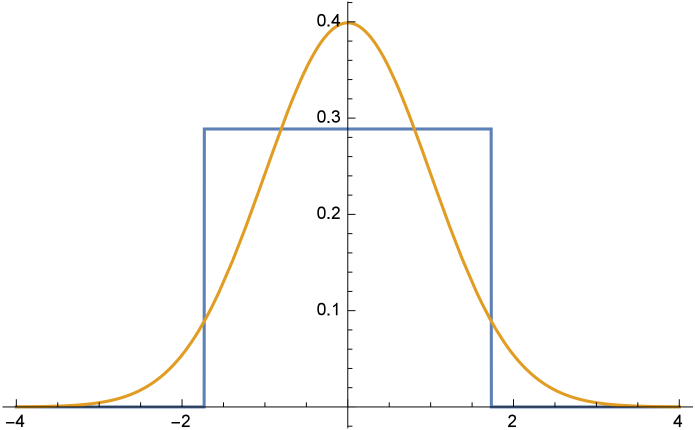
Theorem

If $X_1, X_2, \dots$ independent, identically distributed random variables (mean μ, variance σ^2, finite moments) then

$$S_N \ := \ \frac{X_1 + \cdots + X_N - N\mu}{\sigma\sqrt{N}} \ \text{converges to} \ N(0,1).$$

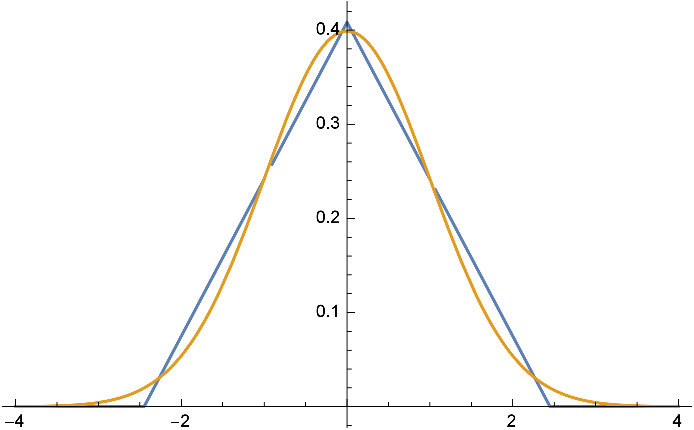
Central Limit Theorem: Sums of Uniform Random Variables $X_i \sim \mathrm{Unif}(-1/2, 1/2)$

$$Y_1 = X_1/\sigma_{X_1} \text{ vs } N(0,1).$$



Central Limit Theorem: Sums of Uniform Random Variables
$X_i \sim \mathrm{Unif}(-1/2, 1/2)$

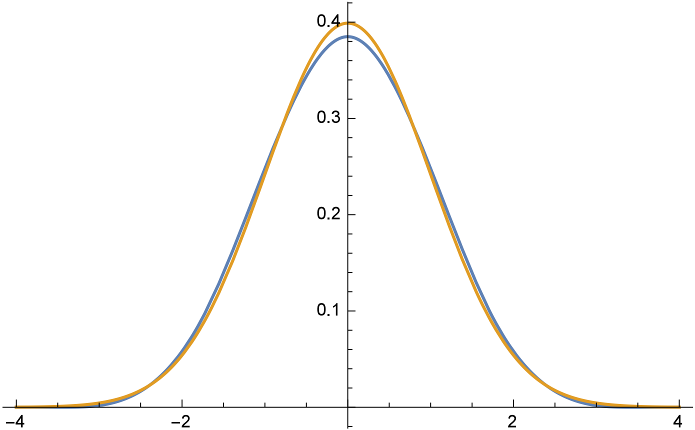
$$Y_2 = (X_1 + X_2)/\sigma_{X_1+X_2} \text{ vs } N(0,1).$$



Central Limit Theorem: Sums of Uniform Random Variables

$X_i \sim \mathrm{Unif}(-1/2, 1/2)$

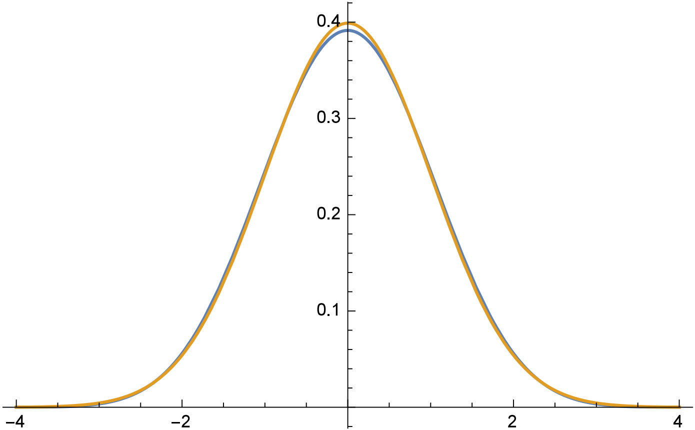
$$Y_4 = (X_1 + X_2 + X_3 + X_4)/\sigma_{X_1+X_2+X_3+X_4} \text{ vs } N(0,1).$$



Central Limit Theorem: Sums of Uniform Random Variables

$X_i \sim \mathrm{Unif}(-1/2, 1/2)$

$$Y_8 = (X_1 + \cdots + X_8)/\sigma_{X_1+\cdots+X_8} \text{ vs } N(0,1).$$



Central Limit Theorem: Sums of Uniform Random Variables

$X_i \sim \mathrm{Unif}(-1/2, 1/2)$

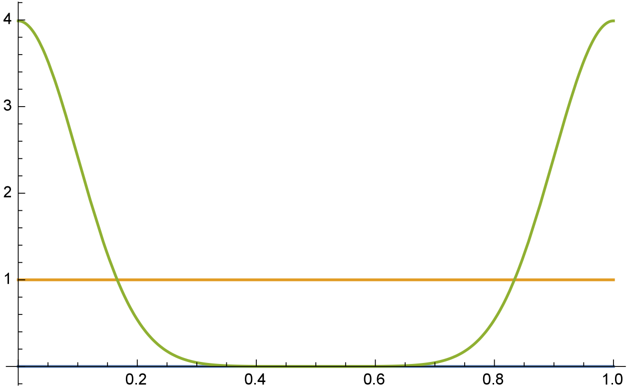
Density of $Y_4 = (X_1 + \cdots + X_4)/\sigma_{X_1+\cdots+X_4}$.

$$\frac{\begin{cases} \frac{1}{27}\left(18 + 9\sqrt{3}\,y - \sqrt{3}\,y^3\right) & y == 0 \\ \frac{1}{18}\left(12 - 6\,y^2 - \sqrt{3}\,y^3\right) & -\sqrt{3} < y < 0 \\ \frac{1}{54}\left(72 - 36\sqrt{3}\,y + 18\,y^2 - \sqrt{3}\,y^3\right) & \sqrt{3} < y < 2\sqrt{3} \\ \frac{1}{54}\left(18\sqrt{3}\,y - 18\,y^2 + \sqrt{3}\,y^3\right) & y == \sqrt{3} \\ \frac{1}{18}\left(12 - 6\,y^2 + \sqrt{3}\,y^3\right) & 0 < y < \sqrt{3} \\ \frac{1}{54}\left(72 + 36\sqrt{3}\,y + 18\,y^2 + \sqrt{3}\,y^3\right) & -2\sqrt{3} < y \le -\sqrt{3} \\ 0 & \mathrm{True} \end{cases}}{\sqrt{3}}$$

(Don't even think of asking to see Y_8's!)

Normal Distributions Mod 1

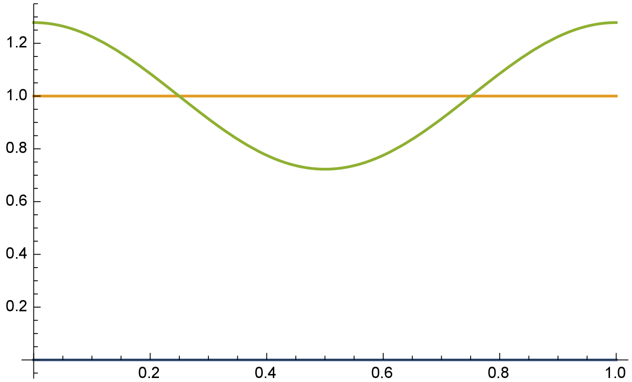
As $\sigma \to \infty$, $N(0, \sigma^2)$ mod 1 $\to \mathrm{Unif}(0, 1)$.



Variance is .01.

Normal Distributions Mod 1

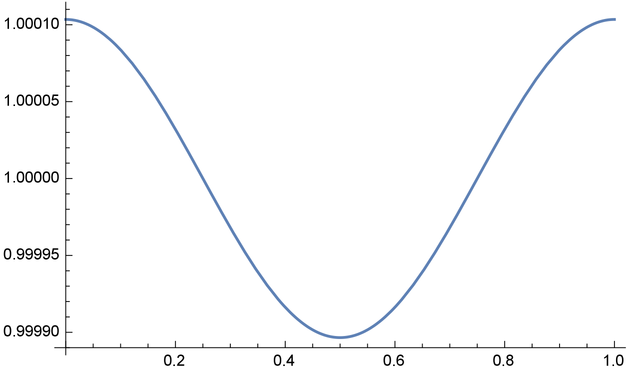
As $\sigma \to \infty$, $N(0,\sigma^2)$ mod 1 $\to \mathrm{Unif}(0,1)$.



Variance is .1.

Normal Distributions Mod 1

As $\sigma \to \infty$, $N(0,\sigma^2)$ mod 1 $\to \text{Unif}(0,1)$.



Variance is .5.

Products and Benford's Law

Pavlovian Response: See a product, take a logarithm.

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$Y_i = \log_{10} X_i$, $V_N := \log_{10} W_N$.

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$$V_N = \log_{10}(X_1 \cdot X_2 \cdots X_N)$$

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$$\begin{aligned} V_N &= \log_{10}(X_1 \cdot X_2 \cdots X_N) \\ &= \log_{10} X_1 + \log_{10} X_2 + \cdots + \log_{10} X_N \end{aligned}$$

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$$\begin{aligned} V_N &= \log_{10}(X_1 \cdot X_2 \cdots X_N) \\ &= \log_{10} X_1 + \log_{10} X_2 + \cdots + \log_{10} X_N \\ &= Y_1 + Y_2 + \cdots + Y_N. \end{aligned}$$

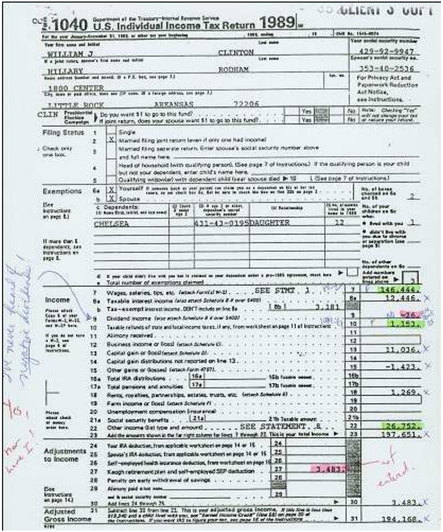
Need distribution of V_N mod 1, which by CLT becomes uniform, implying Benfordness!

Applications

Applications for the IRS: Detecting Fraud

A Tale of Two Steve Millers....

Applications for the IRS: Detecting Fraud



Form 1040 U.S. Individual Income Tax Return 1989

WILLIAM J CLINTON 429-92-9947

HILLARY RODHAM 353-40-2536

1800 CENTER

LITTLE ROCK ARKANSAS 72206

Filing Status

Exemptions

CHELSEA 431-43-0195 DAUGHTER 12

Income

7 Wages, salaries, tips, etc. SEE STMT 1 146,444.

8a Taxable interest income 12,446.

b Tax-exempt interest income 3,181

9 Dividend income 26.

10 Taxable refunds of state and local income taxes 1,153.

13 Capital gain or (loss) 11,036.

15 Other gains or (losses) -1,423.

18 Rents, royalties, partnerships, estates, trusts, etc. 1,269.

22 Other income SEE STATEMENT 8 26,752.

23 Add the amounts shown in the far right column for lines 7 through 22. This is your total income 197,651.

Adjustments to Income

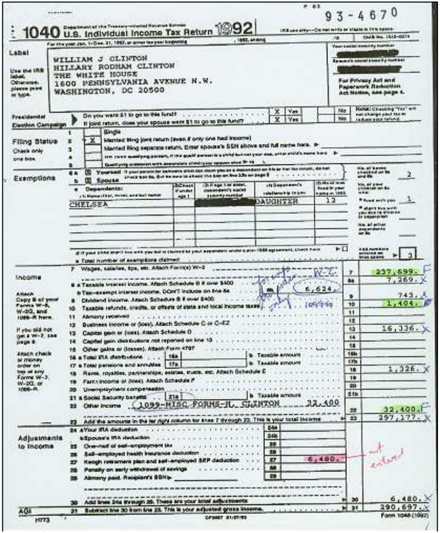
27 Keogh retirement plan and self-employed SEP deduction 3,483.

30 Add lines 24 through 29 3,483.

Adjusted Gross Income

31 Subtract line 30 from line 23. This is your adjusted gross income. 194,168.

Applications for the IRS: Detecting Fraud



93-4670

1040 U.S. Individual Income Tax Return 1992

Label

WILLIAM J CLINTON
HILLARY RODHAM CLINTON
THE WHITE HOUSE
1600 PENNSYLVANIA AVENUE N.W.
WASHINGTON, DC 20500

Your social security number [redacted]

Spouse's social security number [redacted]

For Privacy Act and Paperwork Reduction Act Notice, see page 4.

Presidential Election Campaign

Filing Status

Exemptions

CHELSEA [redacted] DAUGHTER 12

Income

7 Wages, salaries, tips, etc. Attach Form(s) W-2 — 237,699.

8a Taxable interest income — 7,269.

9 Dividend income — 743.

10 Taxable refunds, credits, or offsets of state and local income taxes — 1,404.

13 Capital gain or (loss). Attach Schedule D — 16,336.

18 Rents, royalties, partnerships, estates, trusts, etc. Attach Schedule E — 1,326.

22 Other income 1099-MISC FORMS-HL CLINTON 32,400 — 32,400.

23 Add the amounts in the far right column for lines 7 through 22. This is your total income — 297,177.

Adjustments to Income

27 Keogh retirement plan and self-employed SEP deduction — 6,480.

30 Add lines 24a through 29. These are your total adjustments — 6,480.

AGI

31 Subtract line 30 from line 23. This is your adjusted gross income — 290,697.

Form 1040 (1992)

Detecting Fraud

Bank Fraud

- Audit of a bank revealed huge spike of numbers starting with

Detecting Fraud

Bank Fraud

- Audit of a bank revealed huge spike of numbers starting with 4

Detecting Fraud

Bank Fraud

- Audit of a bank revealed huge spike of numbers starting with 48 and 49, most due to one person.

Detecting Fraud

Bank Fraud

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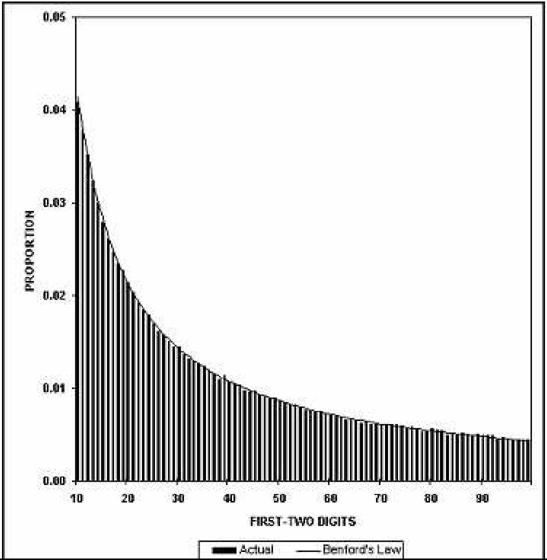
- Write-off limit of $5,000. Officer had friends applying for credit cards, ran up balances just under $5,000 then he would write the debts off.

Data Integrity: Stream Flow Statistics: 130 years, 457,440 records



Election Fraud: Iran 2009

Numerous questions over Iran's 2009 elections.

Lot of analysis; data moderately suspicious:

- First and second leading digits;
- Last two digits (should almost be uniform);
- Last two digits differing by at least 2.

Warning: enough tests, even if nothing wrong will find a suspicious result (but when all tests are on the boundary...).

The Riemann Zeta Function $\zeta(s)$ and Benford's Law

Riemann Zeta Function (for real part of s greater than 1)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

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$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1.$$

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Geometric Series Formula: $(1-x)^{-1} = 1 + x + x^2 + \cdots$.
Unique Factorization: $n = p_1^{r_1} \cdots p_m^{r_m}$.

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Unique Factorization: $n = p_1^{r_1} \cdots p_m^{r_m}$.

$$\prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \left[1 + \frac{1}{2^s} + \left(\frac{1}{2^s}\right)^2 + \cdots\right]\left[1 + \frac{1}{3^s} + \left(\frac{1}{3^s}\right)^2 + \cdots\right]\cdots$$

$$= \sum_n \frac{1}{n^s}.$$

Riemann Zeta Function (cont)

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1$$

$$\pi(x) = \#\{p : p \text{ is prime}, p \leq x\}$$

Properties of $\zeta(s)$ and Primes:

Riemann Zeta Function (cont)

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- $\lim_{s \to 1^+} \zeta(s) = \infty$, $\pi(x) \to \infty$.

Riemann Zeta Function (cont)

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1$$

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Properties of $\zeta(s)$ and Primes:

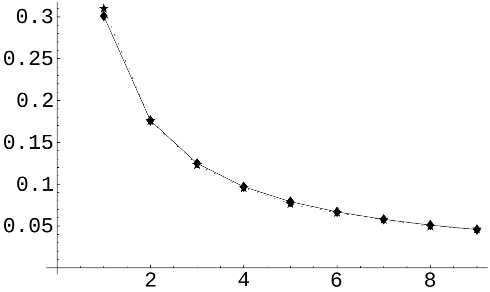
- $\lim_{s \to 1^+} \zeta(s) = \infty$, $\pi(x) \to \infty$.
- $\zeta(2) = \frac{\pi^2}{6}$, $\pi(x) \to \infty$.

The Riemann Zeta Function and Benford's Law

$$\left|\zeta\left(\frac{1}{2}+\mathrm{i}\frac{k}{4}\right)\right|,\ k \in \{0, 1, \dots, 65535\}.$$

The Riemann Zeta Function and Benford's Law

$$\left|\zeta\left(\frac{1}{2} + i\frac{k}{4}\right)\right|, k \in \{0, 1, \dots, 65535\}.$$



First digits of $\left|\zeta\left(\frac{1}{2} + i\frac{k}{4}\right)\right|$ versus Benford's law.

The $3x + 1$ Problem and Benford's Law

$3x+1$ Problem

- Kakutani (conspiracy), Erdös (not ready).
- x odd, $T(x) = \frac{3x+1}{2^k}$, $2^k || 3x+1$.
- Conjecture: for some $n = n(x)$, $T^n(x) = 1$.

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- 7

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- x odd, $T(x) = \frac{3x+1}{2^k}$, $2^k || 3x+1$.
- Conjecture: for some $n = n(x)$, $T^n(x) = 1$.
- $7 \to_1 11 \to_1 17 \to_2 13 \to_3 5 \to_4 1 \to_2 1$,
  2-path $(1, 1)$, 5-path $(1, 1, 2, 3, 4)$.
  m-path: $(k_1, \dots, k_m)$.

Heuristic Proof of $3x + 1$ Conjecture

$$\begin{aligned}
a_{n+1} &= T(a_n) \\
\mathbb{E}[\log a_{n+1}] &\approx \sum_{k=1}^{\infty} \frac{1}{2^k} \log\left(\frac{3a_n}{2^k}\right) \\
&= \log a_n + \log 3 - \log 2 \sum_{k=1}^{\infty} \frac{k}{2^k} \\
&= \log a_n + \log\left(\frac{3}{4}\right).
\end{aligned}$$

Geometric Brownian Motion, drift $\log(3/4) < 1$.

$3x + 1$ and Benford

Theorem (Kontorovich and M–, 2005)

As $m \to \infty$, $x_m/(3/4)^m x_0$ is Benford.

Theorem (Lagarias-Soundararajan, 2006)

$X \geq 2^N$, for all but at most $c(B)N^{-1/36}X$ initial seeds the distribution of the first N iterates of the $3x+1$ map are within $2N^{-1/36}$ of the Benford probabilities.

$3x+1$ Data: random 10,000 digit number, $2^k || 3x+1$

80,514 iterations ($(4/3)^n = a_0$ predicts 80,319);
$\chi^2 = 13.5$ (5% 15.5).

Digit	Number	Observed	Benford
1	24251	0.301	0.301
2	14156	0.176	0.176
3	10227	0.127	0.125
4	7931	0.099	0.097
5	6359	0.079	0.079
6	5372	0.067	0.067
7	4476	0.056	0.058
8	4092	0.051	0.051
9	3650	0.045	0.046

$3x+1$ Data: random 10,000 digit number, $2|3x+1$

241,344 iterations, $\chi^2 = 11.4$ (5% 15.5).

Digit	Number	Observed	Benford
1	72924	0.302	0.301
2	42357	0.176	0.176
3	30201	0.125	0.125
4	23507	0.097	0.097
5	18928	0.078	0.079
6	16296	0.068	0.067
7	13702	0.057	0.058
8	12356	0.051	0.051
9	11073	0.046	0.046

$5x+1$ Data: random 10,000 digit number, $2^k || 5x+1$

27,004 iterations, $\chi^2 = 1.8$ (5% 15.5).

Digit	Number	Observed	Benford
1	8154	0.302	0.301
2	4770	0.177	0.176
3	3405	0.126	0.125
4	2634	0.098	0.097
5	2105	0.078	0.079
6	1787	0.066	0.067
7	1568	0.058	0.058
8	1357	0.050	0.051
9	1224	0.045	0.046

$5x+1$ Data: random 10,000 digit number, $2|5x+1$

241,344 iterations, $\chi^2 = 3 \cdot 10^{-4}$ (5% 15.5).

Digit	Number	Observed	Benford
1	72652	0.301	0.301
2	42499	0.176	0.176
3	30153	0.125	0.125
4	23388	0.097	0.097
5	19110	0.079	0.079
6	16159	0.067	0.067
7	13995	0.058	0.058
8	12345	0.051	0.051
9	11043	0.046	0.046

Stick Decomposition

Fixed Proportion Decomposition Process

Decomposition Process

1. Consider a stick of length $\mathcal{L}$.

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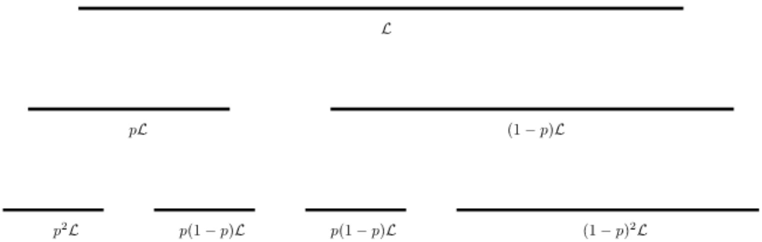
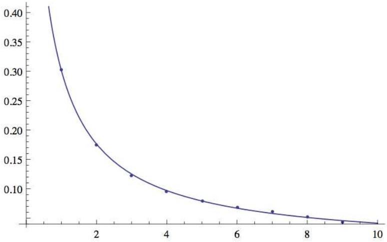
3. Break the stick into two pieces—lengths $p\mathcal{L}$ and $(1-p)\mathcal{L}$.
4. Repeat N times (using the same proportion).

Fixed Proportion Decomposition Process

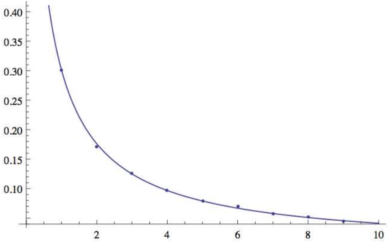


Fixed Proportion Conjecture (Joy Jing '13)

Conjecture: The above decomposition process is Benford as $N \to \infty$ for any $p \in (0,1)$, $p \neq \frac{1}{2}$.



(B) $p = 0.51$ and $N = 10000$.



(B) $p = 0.99$ and $N = 50000$. Benford distribution overlaid.

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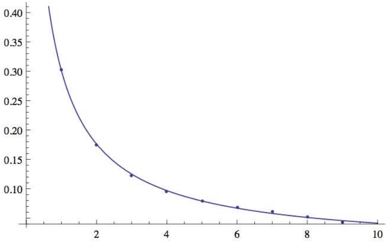

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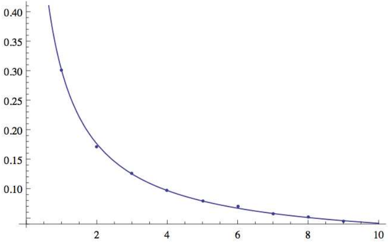

(B) $p = 0.99$ and $N = 50000$. Benford distribution overlaid.

Counterexample (SMALL '13): $p = \frac{1}{11}$, $1 - p = \frac{10}{11}$.

Benford Analysis

At N^{th} level,

- 2^N sticks
- $N+1$ distinct lengths:

$$p^N\left(\frac{1-p}{p}\right)^j, \ j \in \{0,\dots,N\}, \text{ have } \binom{N}{j} \text{ times.}$$

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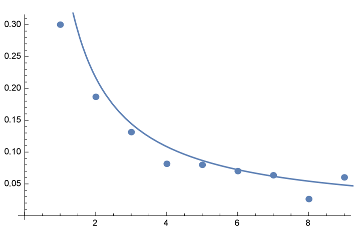
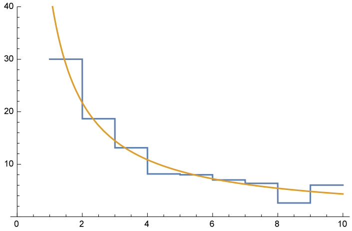
(Weighted) Geometric with ratio $\frac{1-p}{p}=10^y$;
behavior depends on irrationality of y!

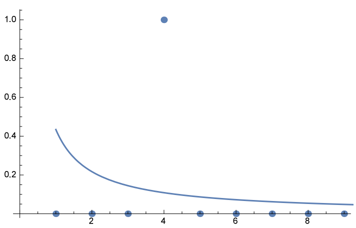
Examples

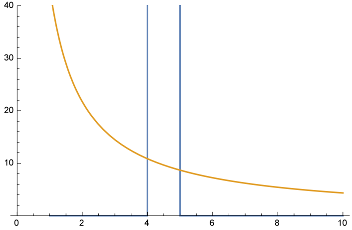




$p = 3/11$, 1000 levels; $y = \log_{10}(8/3) \notin \mathbb{Q}$ (irrational)

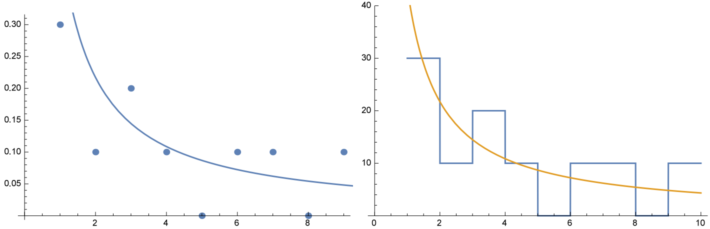
Examples





$p = 1/11$, 1000 levels; $y = 1 \in \mathbb{Q}$ (rational)

Examples



$p = 1/(1+10^{33/10})$, 1000 levels; $y = 33/10 \in \mathbb{Q}$ (rational)

Conclusions

Current / Future Investigations

- Develop more sophisticated tests for fraud.

- Study digits of other systems.
  ◇ Break rod of fixed length a variable number of times.
  ◇ Break rods of variable length a variable number of times.
  ◇ Break rods of variable length, each piece then breaks with given probability.
  ◇ Break rods of variable integer length, each piece breaks until is a prime, or a square, ...

Conclusions and Future Investigations

- See many different systems exhibit Benford behavior.
- Ingredients of proofs (logarithms, equidistribution).
- Applications to fraud detection / data integrity.

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