Lower-Order Biases in Elliptic Curve Fourier Coefficients

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Elliptic Curves

An *elliptic curve* E over $\mathbb{Q}$ is the set of solutions $(x, y) \in \mathbb{Q}^2$ to an equation of the form

$$E : y^2 = x^3 + ax^2 + bx + c$$

with $a, b, c \in \mathbb{Z}$. For primes $p > 3$ the *elliptic curve Fourier coefficients* are

$$a_E(p) = p - \#\{(x, y) \in \mathbb{F}_p^2 : y^2 = x^3 + ax^2 + bx + c\}.$$

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The associated Dirichlet series

$$L(E, s) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s}, \quad \Re(s) > \frac{3}{2}$$

can be analytically continued to an L-function on all of $\mathbb{C}$.

Families and Moments

A *one-parameter family* of elliptic curves is given by

$$\mathcal{E} : y^2 = x^3 + A[T]x^2 + B[T]x + C[T]$$

with $A[T], B[T], C[T] \in \mathbb{Z}[T]$.

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- Each specialization of T to an integer t gives an elliptic curve $\mathcal{E}(t)$ over $\mathbb{Q}$.
- The r^{th} *moment* of the Fourier coefficients is

$$A_{r,\mathcal{E}}(p) = \sum_{t=0}^{p-1} a_{\mathcal{E}(t)}(p)^r.$$

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Bias Conjecture

Second Moment Asymptotic [Michel]

For "nice" families $\mathcal{E}$, the second moment of the Fourier coefficients is equal to

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In every family we have studied, we have observed:

Bias Conjecture

The largest lower term in the second moment expansion which does not average to 0 is on average **negative**.

One Interpretation

Sato-Tate Law for Families without CM

For large primes p, the distribution of $\frac{a_{\mathcal{E}(t)}(p)}{\sqrt{p}}$, $t \in \{0, \dots, p-1\}$, approaches the semicircular density $F(x) = \frac{1}{2\pi} \int_{-2}^{x} \sqrt{4 - u^2} du$ on $[-2, 2]$.

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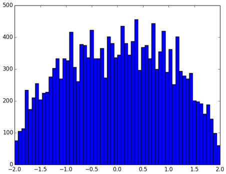

Figure: $a_{\mathcal{E}(t)}(p)$ for $y^2 = x^3 + Tx + 1$ at the 2014th prime

Implications for Excess Rank

- Katz-Sarnak's one-level density statistic is used to measure the average rank of curves over a family.
- More curves with rank than expected have been observed, though this excess average rank vanishes in the limit.
- Lower-order biases in the moments of families explain a small fraction of this excess rank phenomenon.

Negative Bias in the First Moment

The First Moment $A_{1,\mathcal{E}}(p)$ and Family Rank [Rosen-Silverman]

$$\lim_{X\to\infty}\frac{1}{X}\sum_{p\le X}\frac{A_{1,\mathcal{E}}(p)\log p}{p} = -\mathrm{rank}(\mathcal{E}(\mathbb{Q}[T]))$$

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- We can use this to study families of varying rank and understand the relationship between $A_{2,\mathcal{E}}(p)$ and $\mathrm{rank}(\mathcal{E}(\mathbb{Q}[T]))$.

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Methods for Obtaining Explicit Formulas

For a family $\mathcal{E} : y^2 = x^3 + A[T]x^2 + B[T]x + C[T]$, we can write

$$a_{\mathcal{E}(t)}(p) = -\sum_{x=0}^{p-1} \left(\frac{x^3 + A(t)x^2 + B(t)x + C(t)}{p} \right)$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol, given by

$$\left(\frac{x}{p}\right) = \begin{cases} 1 & \text{if } x \text{ is a nonzero square in } \mathbb{F}_p \\ 0 & \text{if } x = 0 \text{ in } \mathbb{F}_p \\ -1 & \text{if } x \text{ is not a square in } \mathbb{F}_p \end{cases}$$

Lemmas on Legendre Symbols

Linear and Quadratic Legendre Sums

$$\sum_{x=0}^{p-1} \left(\frac{ax+b}{p}\right) = 0 \quad \text{if } p \nmid a$$

$$\sum_{x=0}^{p-1} \left(\frac{ax^2+bx+c}{p}\right) = \begin{cases} -\left(\frac{a}{p}\right) & \text{if } p \nmid b^2-4ac \\ (p-1)\left(\frac{a}{p}\right) & \text{if } p \mid b^2-4ac \end{cases}$$

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Average Values of Legendre Symbols

The value of $\left(\frac{x}{p}\right)$ for $x \in \mathbb{Z}$, when averaged over all primes p, is 1 if x is a non-zero square, and 0 otherwise.

Rank 0 Families

Theorem [MMRW'14]: Rank 0 Families Obeying the Bias Conjecture

For families of the form $\mathcal{E} : y^2 = x^3 + ax^2 + bx + cT + d$,

$$A_{2,\mathcal{E}}(p) = p^2 - p\left(1 + \left(\frac{-3}{p}\right) + \left(\frac{a^2 - 3b}{p}\right)\right).$$

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- The average bias in the size p term is -2 or -1, according to whether $a^2 - 3b \in \mathbb{Z}$ is a non-zero square.

Families with Rank

Theorem [MMRW'14]: Families with Rank

For families of the form $\mathcal{E} : y^2 = x^3 + aT^2x + bT^2$,

$$A_{2,\mathcal{E}}(p) = p^2 - p\left(1 + \left(\frac{-3}{p}\right) + \left(\frac{-3a}{p}\right)\right) - \left(\sum_{x(p)} \left(\frac{x^3+ax}{p}\right)\right)^2$$

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- These include families of rank 0, 1, and 2.
- The average bias in the size p terms is -3 or -2, according to whether $-3a \in \mathbb{Z}$ is a non-zero square.

Families with Complex Multiplication

Theorem [MMRW'14]: Families with Complex Multiplication

For families of the form $\mathcal{E} : y^2 = x^3 + (aT + b)x$,

$$A_{2,\mathcal{E}}(p) = (p^2 - p)\left(1 + \left(\frac{-1}{p}\right)\right).$$

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- The average bias in the size p term is -1.
- The size p^2 term is not constant, but is on average p^2, and an analogous Bias Conjecture holds.

Families with Unusual Distributions of Signs

Theorem [MMRW'14]: Families with Unusual Signs

For the family $\mathcal{E} : y^2 = x^3 + Tx^2 - (T+3)x + 1$,

$$A_{2,\mathcal{E}}(p) = p^2 - p\left(2 + 2\left(\frac{-3}{p}\right)\right) - 1.$$

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- The average bias in the size p term is -2.

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- The average bias in the size p term is -2.
- The family has an unusual distribution of signs in the functional equations of the corresponding L-functions.

The Size $p^{3/2}$ Term

Theorem [MMRW'14]: Families with a Large Error

For families of the form
$\mathcal{E} : y^2 = x^3 + (T + a)x^2 + (bT + b^2 - ab + c)x - bc,$

$$A_{2,\mathcal{E}}(p) = p^2 - 3p - 1 + p \sum_{x=0}^{p-1} \left(\frac{-cx(x+b)(bx-c)}{p} \right)$$

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- The size $p^{3/2}$ term is given by an elliptic curve coefficient and is thus on average 0.
- The average bias in the size p term is -3.

General Structure of the Lower Order Terms

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- have no size $p^{3/2}$ term or a size $p^{3/2}$ term that is on average 0;
- exhibit their negative bias in the size p term;
- be determined by polynomials in p, elliptic curve coefficients, and values of Legendre symbols.

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Largest Error Term

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- Possible approach: numerically measure the average value of the lower-order terms by averaging

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  over large ranges of primes.
- Problem: the $p^{3/2}$ normalization averages to 0; the p normalization does not appear to converge.

Higher Genus Sato-Tate

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Higher Genus Sato-Tate

- We believe that when Michel's estimate is sharp, the size $p^{3/2}$ term is given by Fourier coefficients of some L-function.
- A generalized Sato-Tate conjecture due to Sutherland predicts the limiting distributions of hyperelliptic curve coefficients.
- We can compute an approximate distribution for $\frac{A_{2,\mathcal{E}}(p)-p^2}{p^{3/2}}$ and compare it with the Fourier coefficient distribution of some hyperelliptic curve.

Distribution of Error Terms: Example 1

Denote $c_{3/2}(p) = \frac{A_{2,\mathcal{E}}(p) - p^2}{p^{3/2}}$. Consider the family

$$\mathcal{E} : y^2 = 4x^3 + 5x^2 + (4T - 2)x + 1$$

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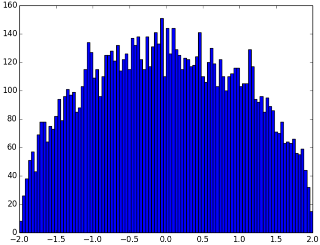
Figure: Distribution of $c_{3/2}(p)$ over the first 10000 primes



Approx. moments: 1, 0, 1, 0, 2, 0, 5, 0, 14, ...
Hyperelliptic curve: $y^2 = x^3 + x + 1$

Distribution of Error Terms: Example 2

Denote $c_{3/2}(p) = \frac{A_{2,\mathcal{E}}(p) - p^2}{p^{3/2}}$. Consider the family

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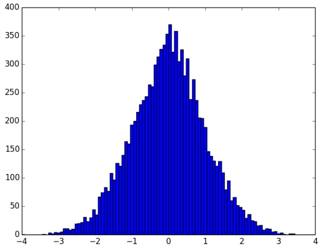

Approx. moments: 1, 0, 1, 0, 3, 0, 14, 0, 84, ...
Hyperelliptic curve: $y^2 = x^5 - x + 1$

Distribution of Error Terms: Example 3

Denote $c_{3/2}(p) = \frac{A_{2,\mathcal{E}}(p) - p^2}{p^{3/2}}$. Consider the family

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Distribution of Error Terms: Example 3

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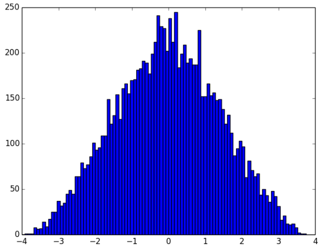

Approx. moments: 1, 0, 2, 0, 6, 0, 10, 0, 70, ...
Hyperelliptic curve: $y^2 = x^6 + x^2 + 1$

Summary of Error Term Investigations

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- Larger error terms that average to 0 prevent us from numerically measuring average biases that arise in the size p terms.
- In every case we studied, the size $p^{3/2}$ error term appeared to be governed by (hyper)elliptic curve coefficients.
- We do not have a general way of identifying the hyperelliptic curve coefficient associated to the error term of a given family.

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- What other (families of) objects obey the Bias Conjecture? Kloosterman sums? Cusp forms of a given weight and level? Higher genus curves?
- How does the second moment bias relate to other properties of the family?

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- University of Michigan Computing Resources
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