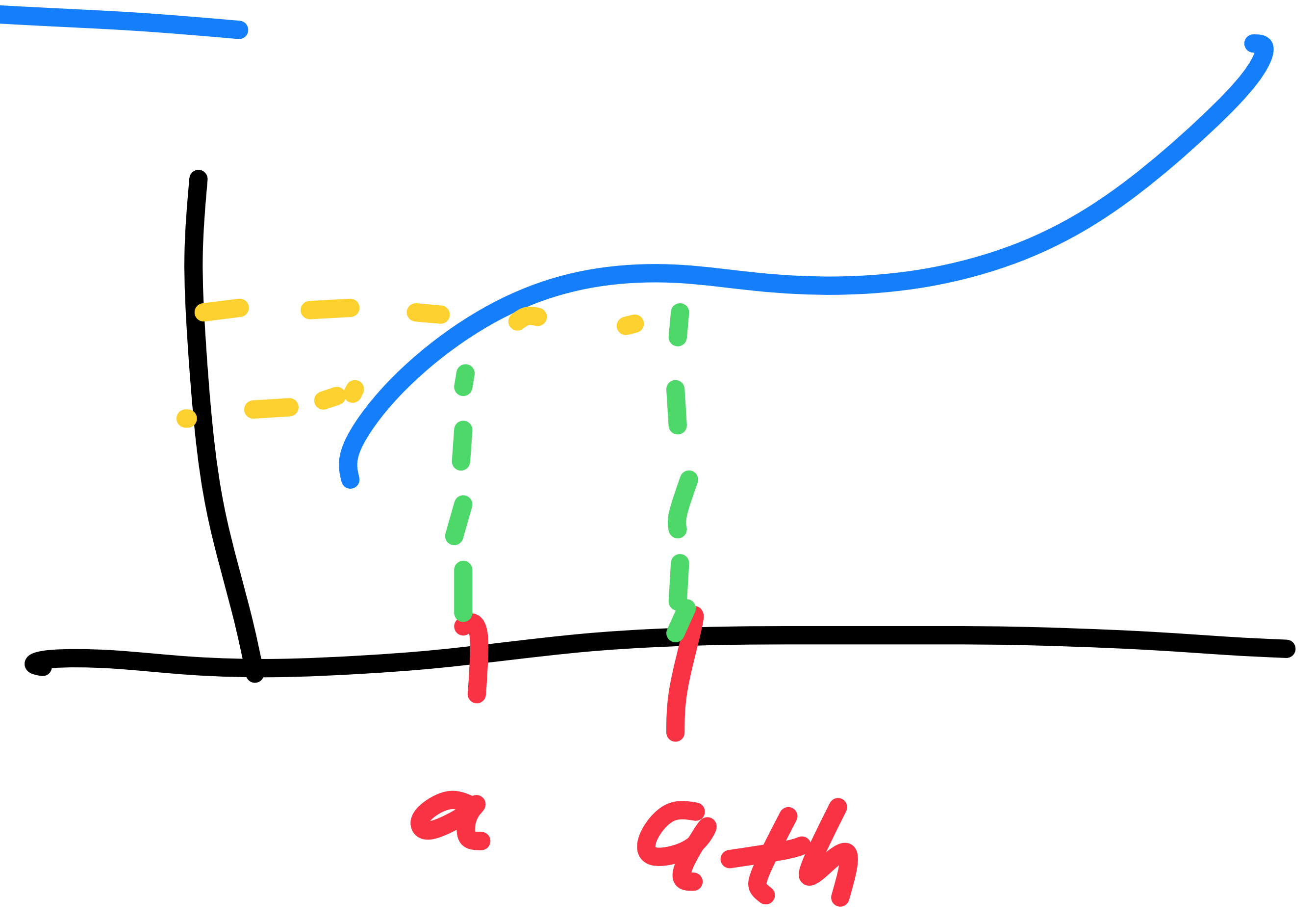
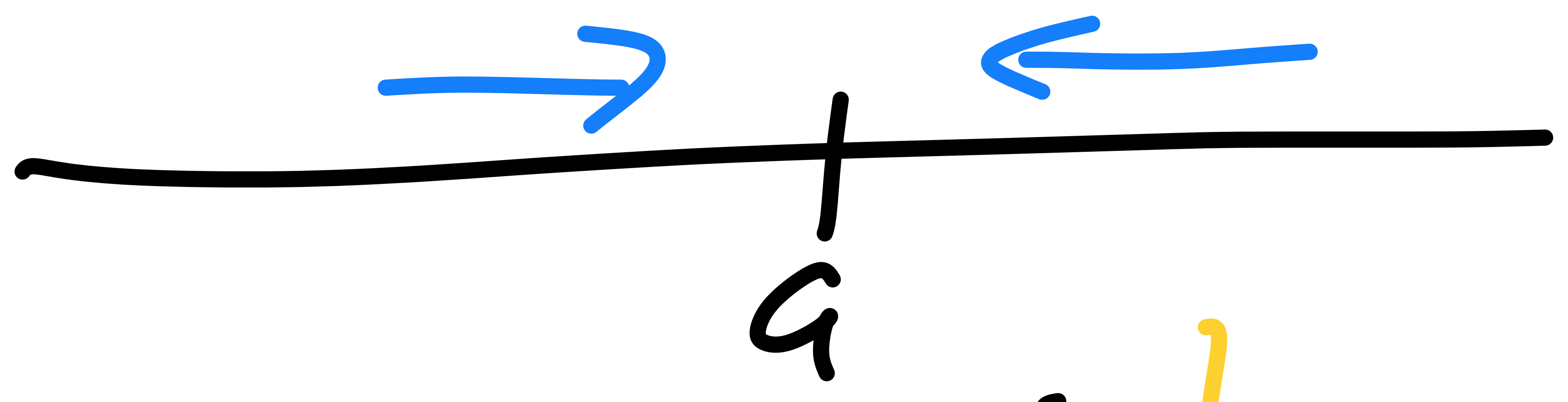
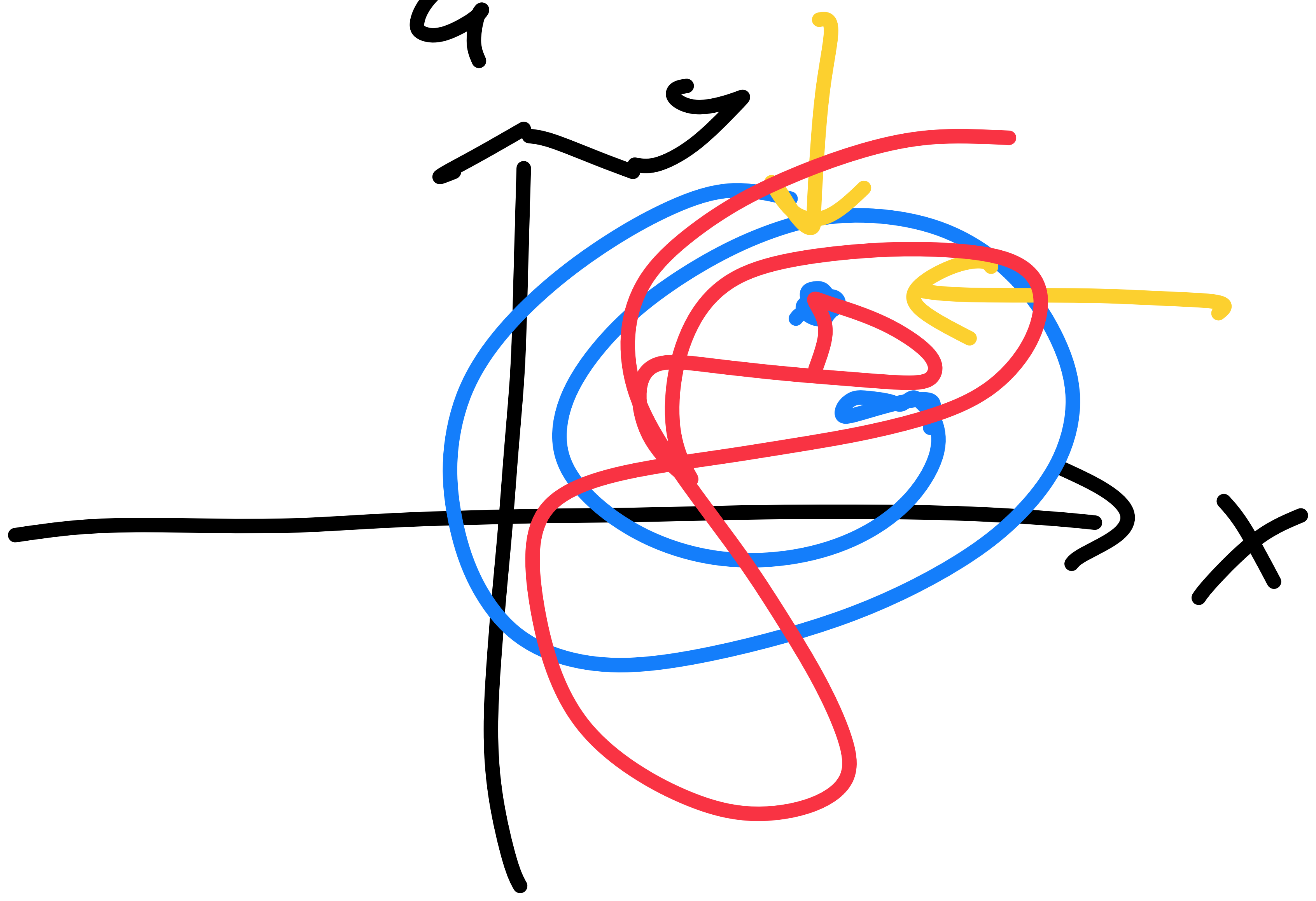
Calc III: Mult Var Calc Review

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

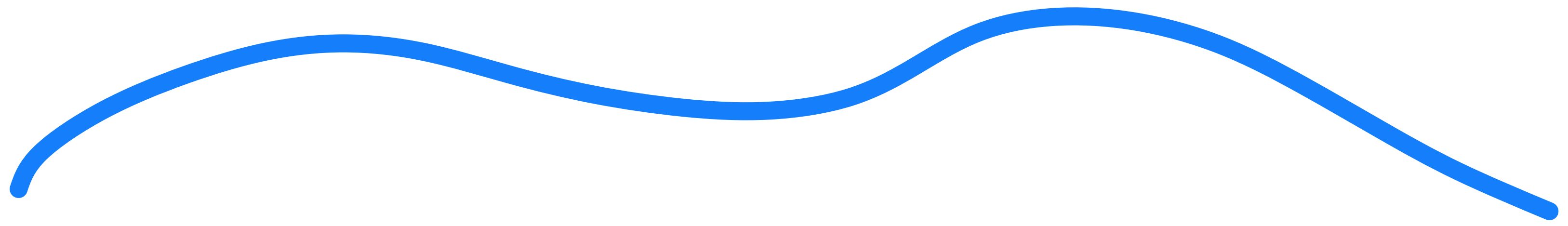






$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

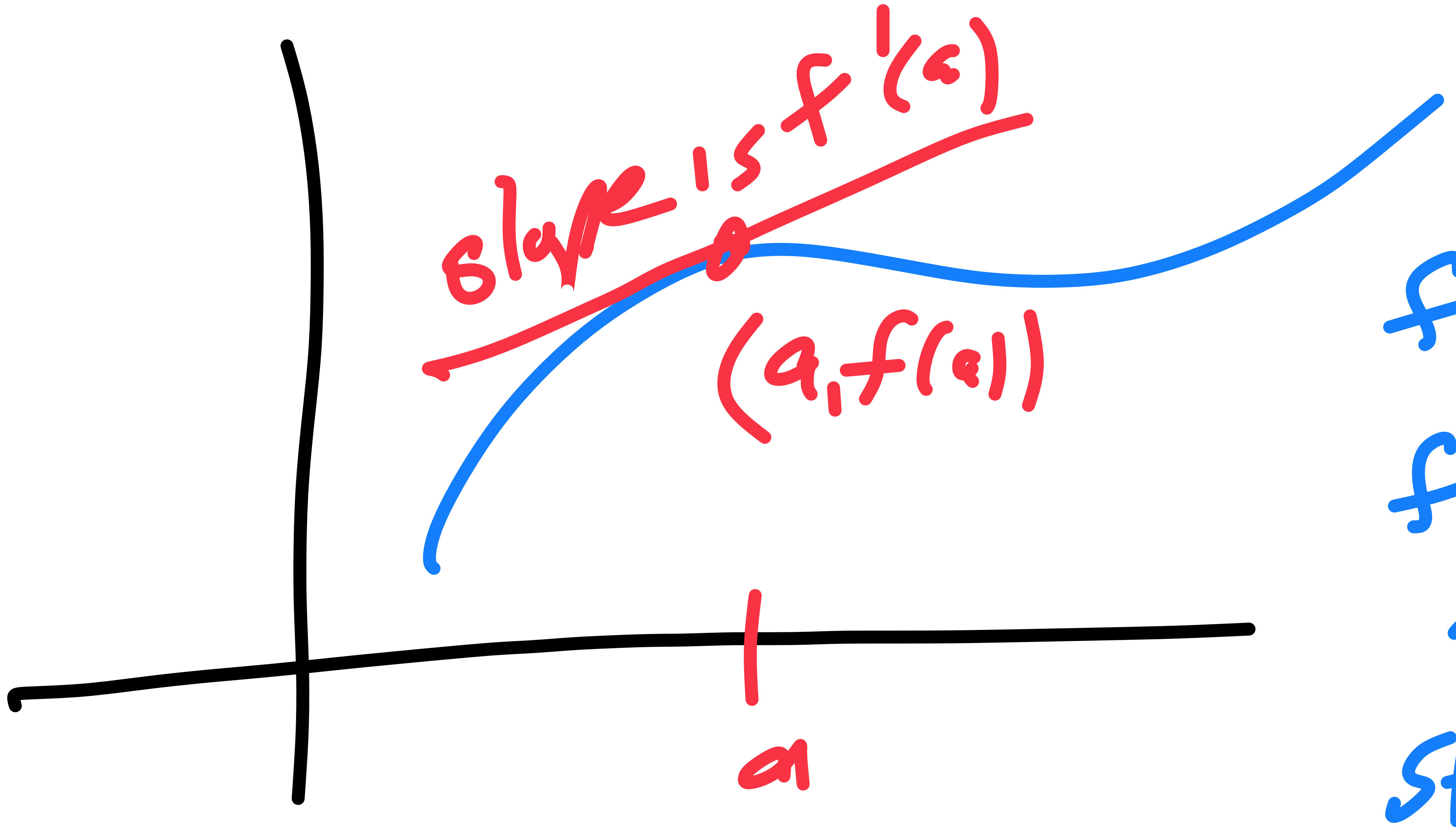
tangent line



$f'(a)$ such that

$$\lim_{x\to a} \frac{f(x)-f(a)-f'(a)(x-a)}{x-a}$$

$$= 0$$



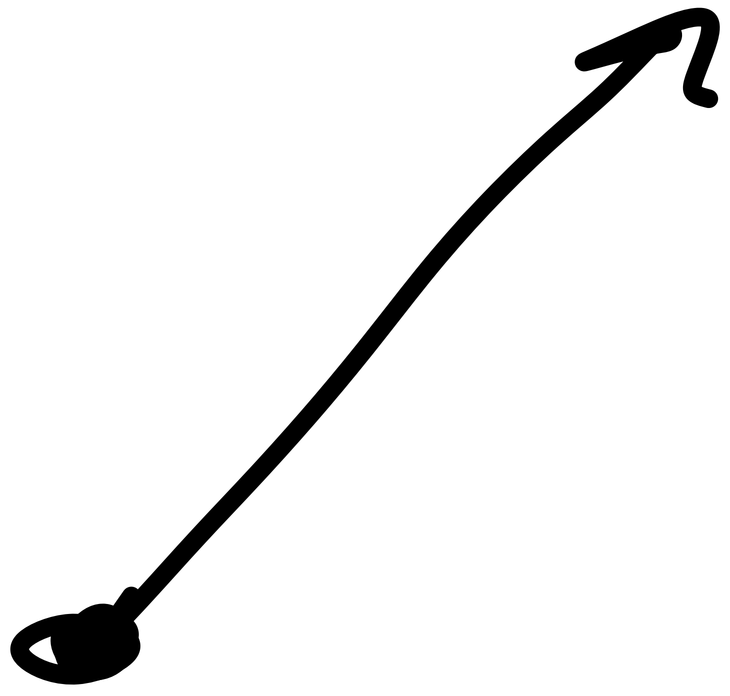
$f(x) \approx$

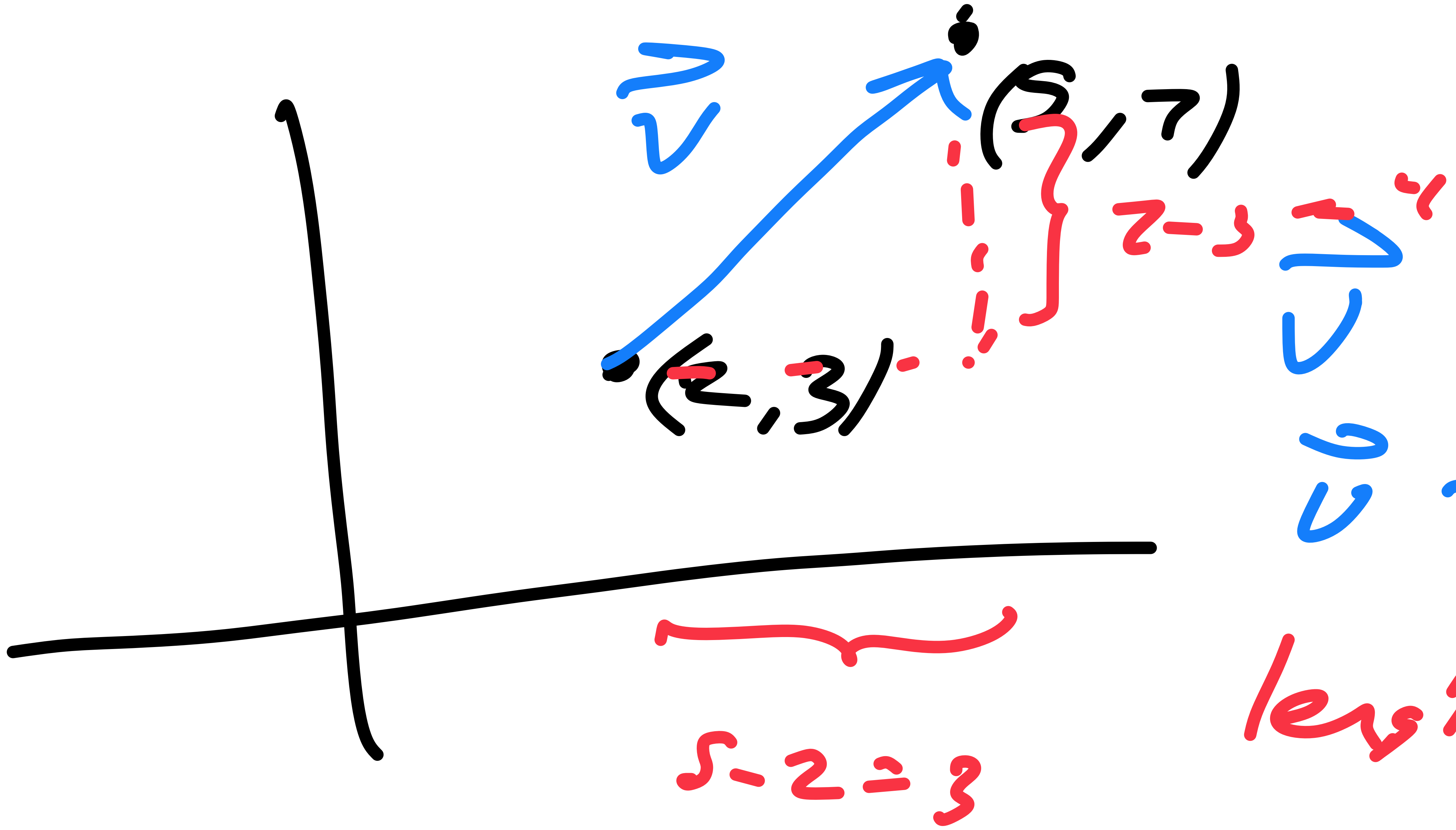
$f(a) + f'(a)(x-a)$

start + speed · time

Preliminaries

Vectors: magnitude and direction





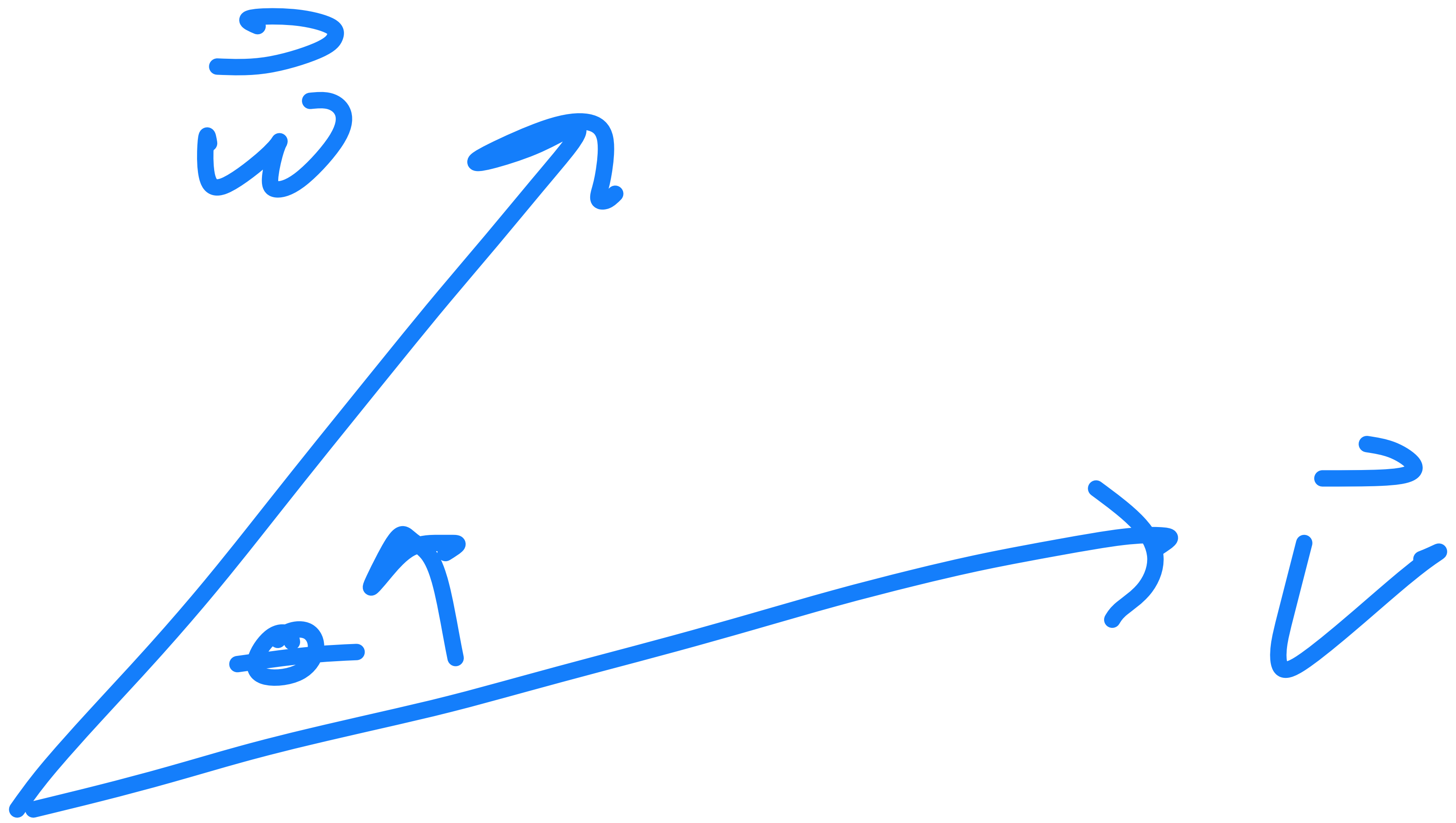
$\vec{v} = (5-2,\ 7-3)$

$\vec{v} = (3,4)$

length is 5

$\vec{v} = (x,y)$ then $\|\vec{v}\| = \sqrt{x^2+y^2}$ Pythagoras

$\vec{v} = (x_1,\dots,x_n)$ then $\|\vec{v}\| = \sqrt{x_1^2+\dots+x_n^2}$



Pythagoras

Law of Cosines

Dot Product

$$\vec{v} \cdot \vec{w} = v_1 w_1 + \cdots + v_n w_n$$

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta_{v,w}$$

Cross Product: only exists in some dimensions

$$\mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

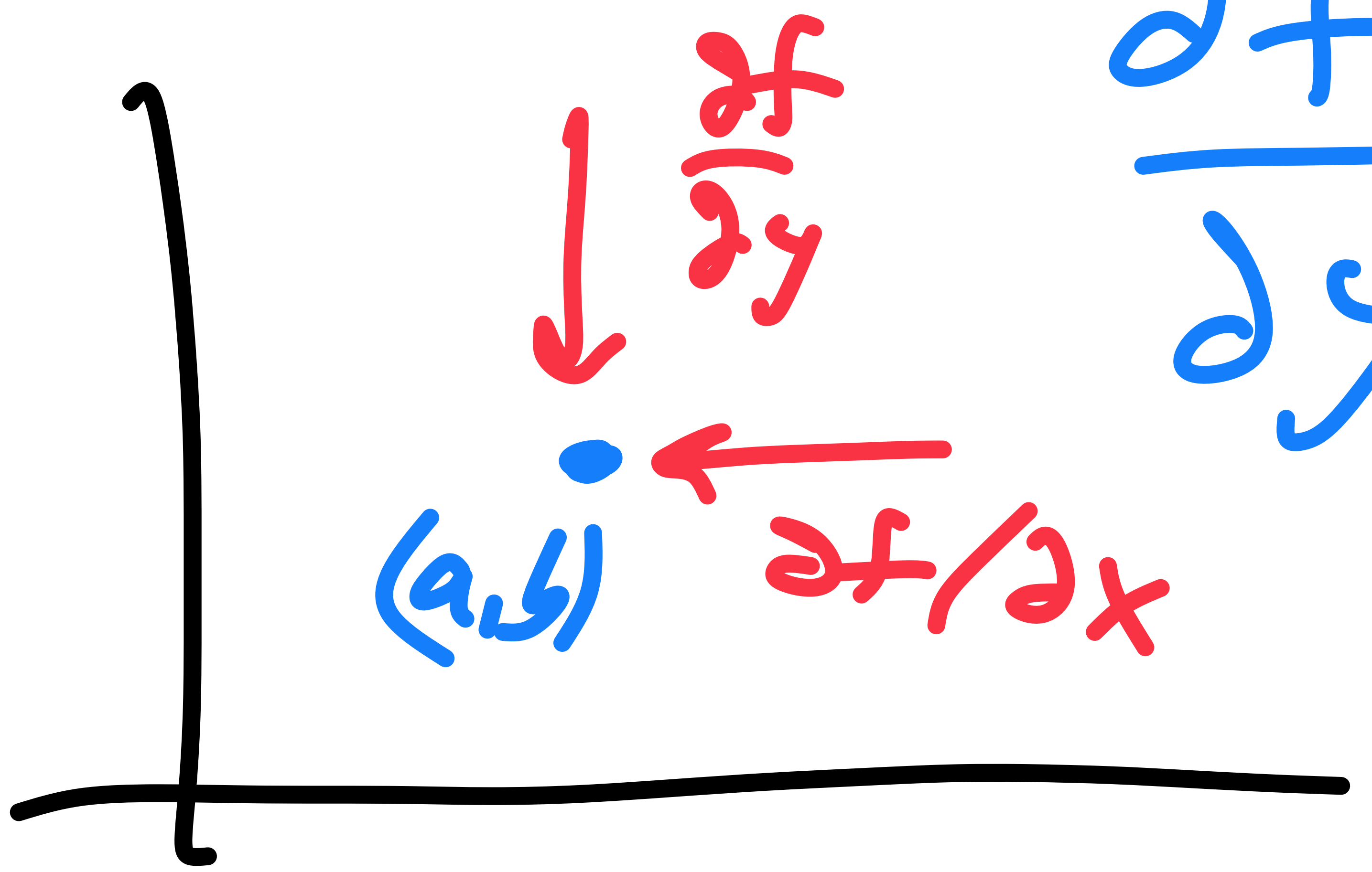
$$(x, y) \in \mathbb{R}^2 \longrightarrow (x, y, 0) \in \mathbb{R}^3$$

Partial Derivatives

$f(x)$ then $\dfrac{df}{dx}$ or $f'(x)$

$f(x,y)$ $\quad \dfrac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \dfrac{f(a+h,b) - f(a,b)}{h}$

$$\frac{\partial f}{\partial y}(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$



Hope: if know $\partial f/\partial x_i$ for each variable, we know the "derivative"

Ex! $f(x,y) = (xy)^{1/3}$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = 0$$

$$g(x) = f(x,x^2) = (x\cdot x^2)^{1/3} = x$$

Take $y = x^2$

$$g'(x) = \frac{dg}{dx} = 1$$

If partials exist and are continuous, fn is diff

Gradient of f to be $(\nabla f)(\vec{a})$ is

$$\left(\frac{\partial f}{\partial x_1}(\vec{a}), \dots, \frac{\partial f}{\partial x_n}(\vec{a})\right) = (\nabla f)(\vec{a})$$

Ex: $f(x,y,z) = x^2 y + xe^y$

$$\frac{\partial f}{\partial x} = 2xy + e^y \qquad \frac{\partial f}{\partial y} = x^2 + xe^y \qquad \frac{\partial f}{\partial z} = 0$$

Say f is diff at $\vec{a}$ if

$$\lim_{h\to 0} \frac{f(\vec{x}) - f(\vec{a}) - (\nabla f)(\vec{a}) \cdot (\vec{x} - \vec{a})}{\|\vec{x} - \vec{a}\|} = 0$$

$$(\nabla f)(\vec{a}) \cdot (\vec{x} - \vec{a})$$

$$= \frac{\partial f}{\partial x_1}(\vec{a})(x_1 - a_1) + \cdots + \frac{\partial f}{\partial x_n}(\vec{a})(x_n - a_n)$$

and $(\nabla f)(\vec{a})$ is the derivative

Rules for Deriv

$$D(f+g) = Df + Dg$$

$$(\nabla h)(\vec{x}) = (\nabla f)(\vec{x}) + (\nabla g)(\vec{x})$$

when $h = f+g$

Chain Rule

one-dim

$f(g(x))$

$g: \mathbb{R} \to \mathbb{R}$

$f: \mathbb{R} \to \mathbb{R}$

$f \circ g: \mathbb{R} \to \mathbb{R}$

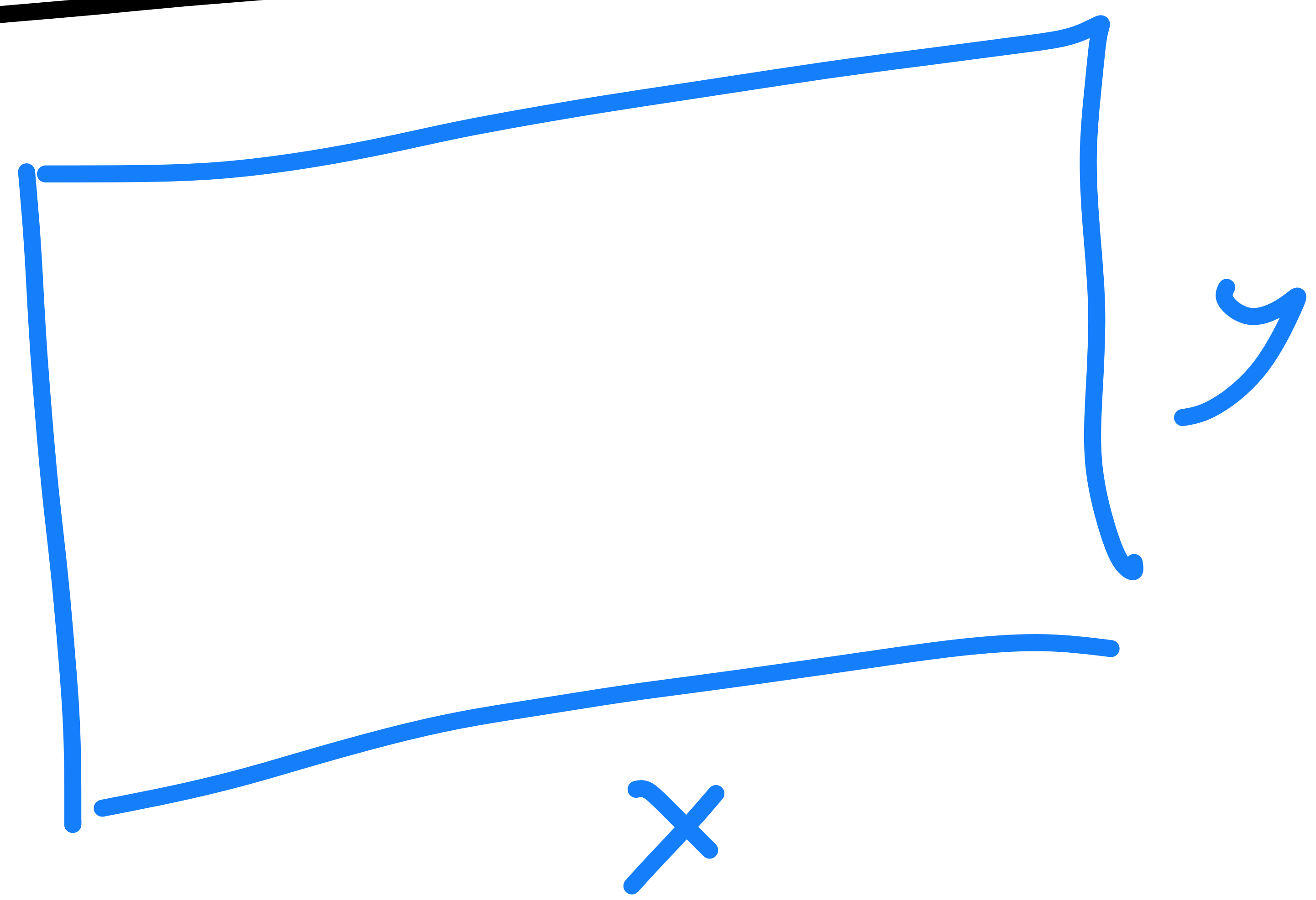
Several vars

$g: \mathbb{R}^2 \to \mathbb{R}^3$

$f: \mathbb{R}^3 \to \mathbb{R}^5$

$f \circ g: \mathbb{R}^2 \to \mathbb{R}^5$

Optimization



Perim of 40

max area

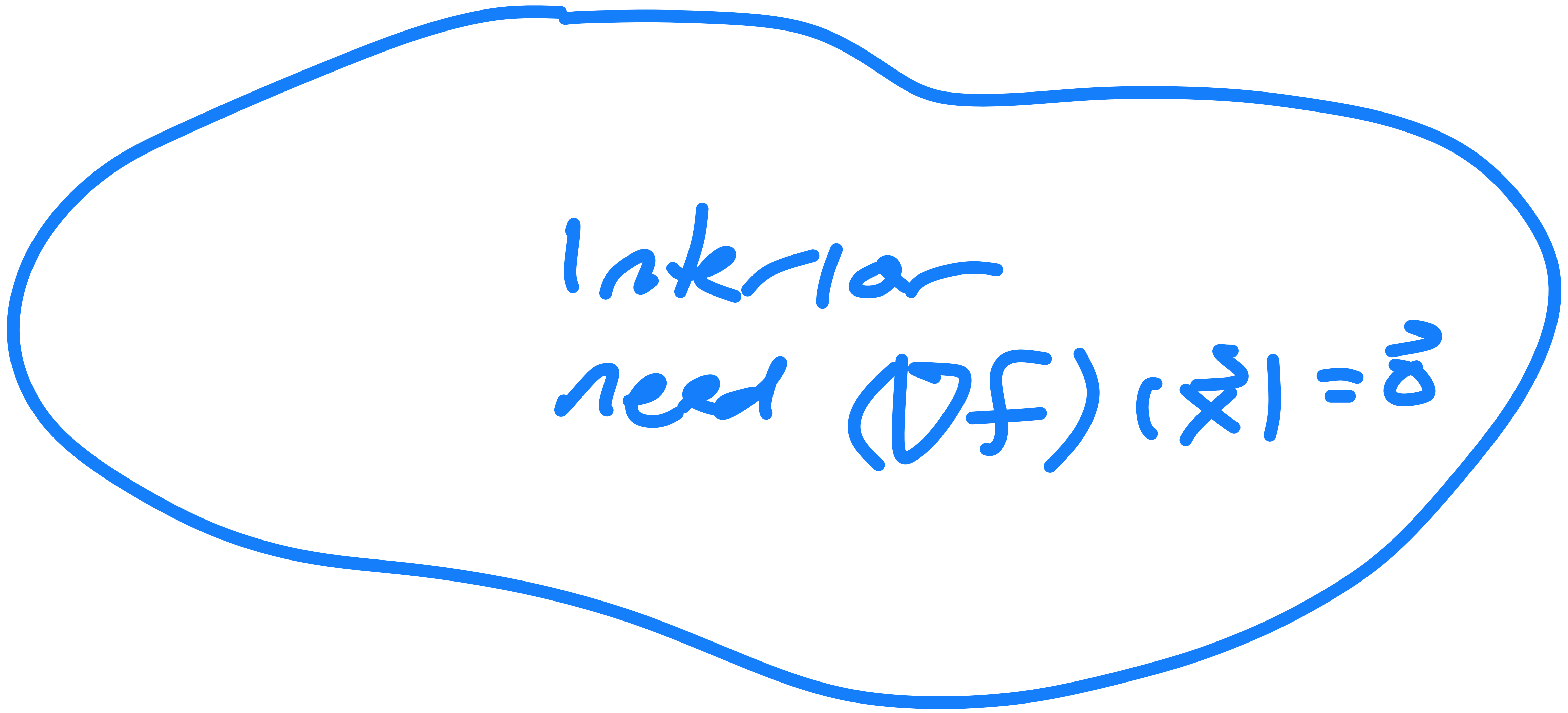
$$Area(x,y) = xy$$

Constraint: $2x + 2y = 40$

$$y = 20 - x$$

$$A(x) = x \cdot (20 - x)$$

Lagrange Multipliers



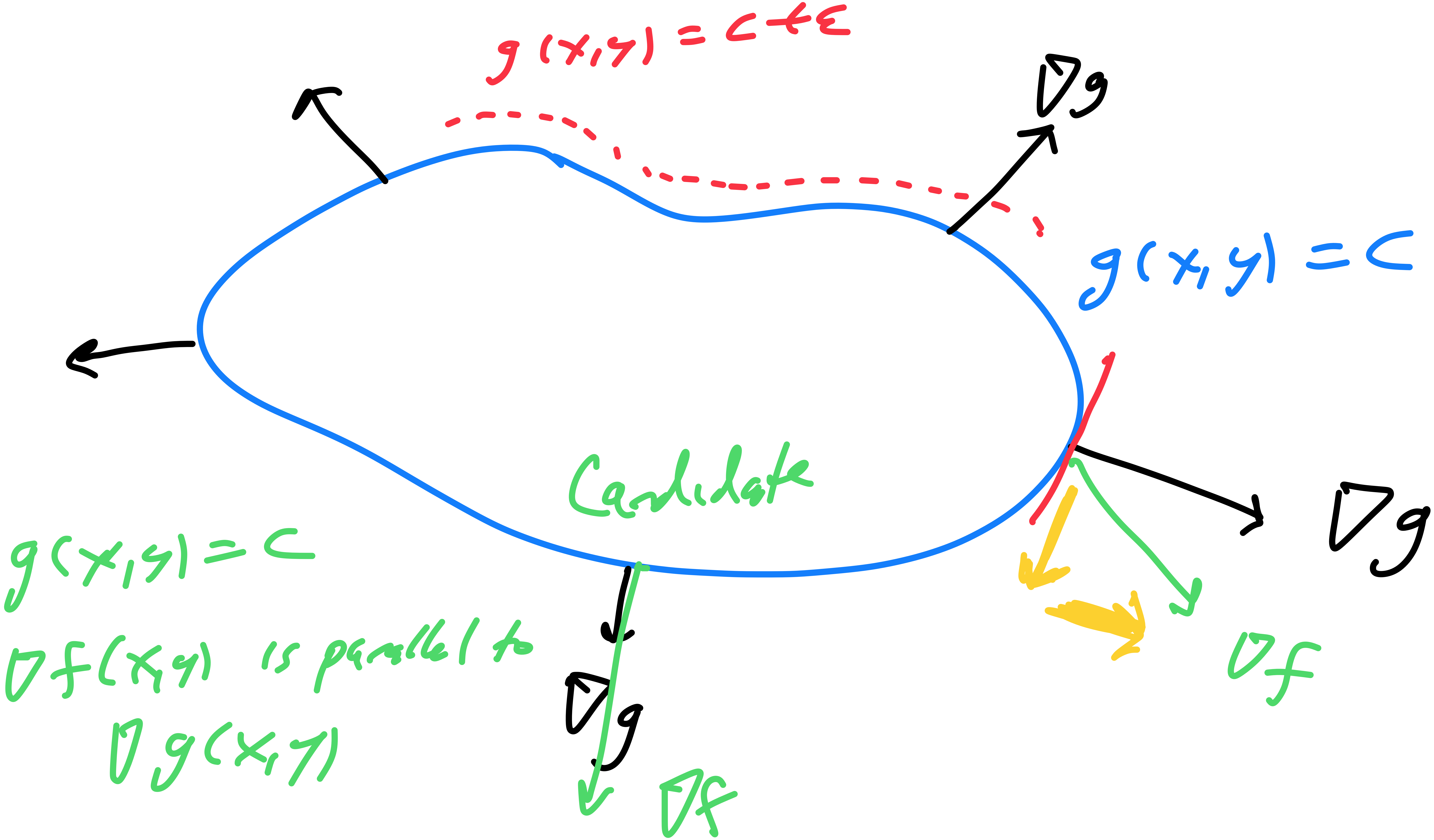
Looking for candidates for max/min

Interior: need $(\nabla f)(\vec{x}) = \vec{0}$

critical points

infinitely many endpoints

↳ boundary



g(x,y) = c+ε
∇g
g(x,y) = c
Candidate
∇g
∇f
∇g
∇f
g(x,y) = c
∇f(x,y) is parallel to
∇g(x,y)

$$g(x,y) = c$$

$$(\nabla f)(x,y) = \lambda (\nabla g)(x,y)$$

$$g(x,y) = c$$

$$\frac{\partial f}{\partial x}(x,y) = \lambda \frac{\partial g}{\partial x}(x,y)$$

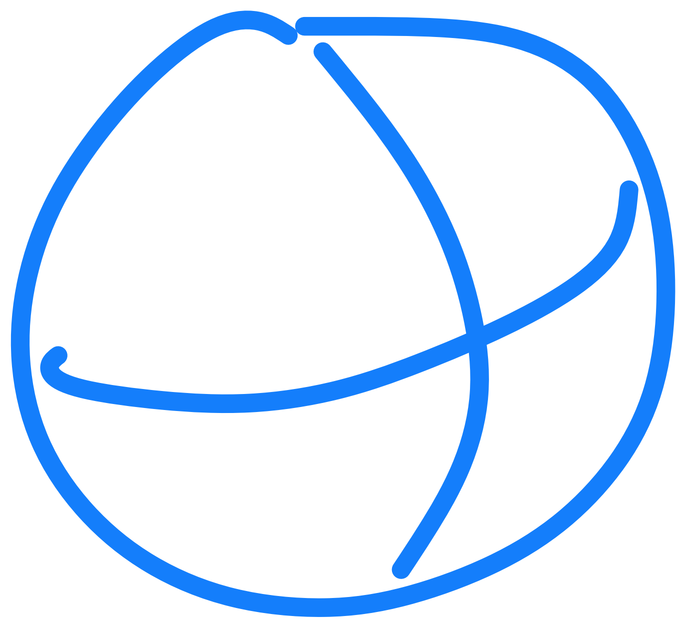
$$\frac{\partial f}{\partial y}(x,y) = \lambda \frac{\partial g}{\partial y}(x,y)$$

3 vars

3 unknowns

$\Longrightarrow$ solvable!

Ex: $g(x,y,z) \Rightarrow 8^2 = x^2+y^2+z^2$



$f(x,y,z) = x^2+2y^3+3z^4$

$\nabla g = (2x, 2y, 2z)$

$\nabla f = (2x, 6y^2, 12z^3)$

Solve $2x = \lambda 2x \rightarrow x=0$ or $\lambda=1$

$6y^2 = \lambda 2y$

$12z^3 = \lambda 2z$

and $8^2 = x^2+y^2+z^2$

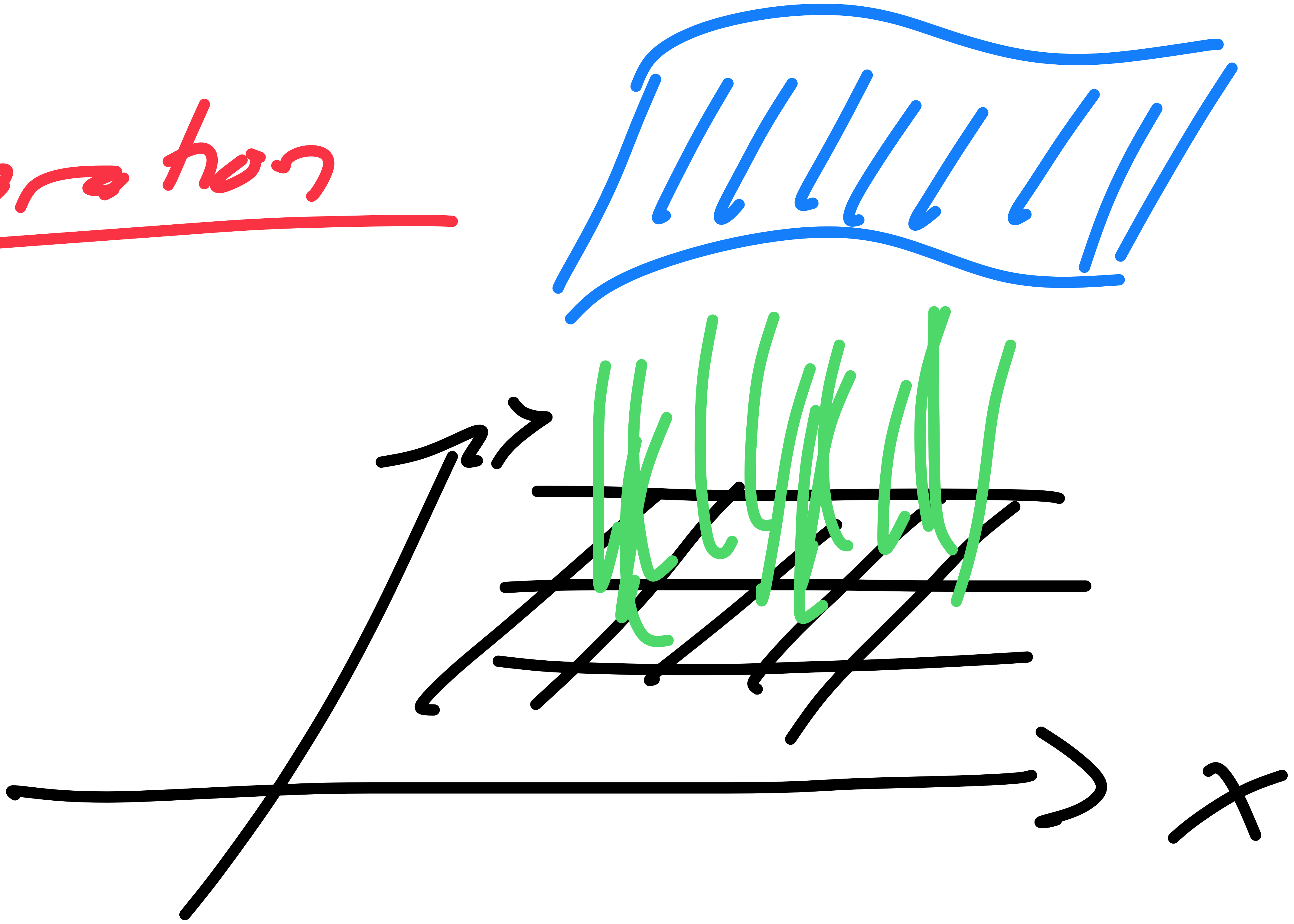
$$\frac{d^2f}{dx^2} = f''(x) = \frac{d}{dx}\left(\frac{d}{dx} f\right)$$

$\frac{\partial^2 f}{\partial x \partial y}$ is this $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f\right)$ or $\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f\right)$?

denote f_{xy}

Thm $f_{xy} = f_{yx}$ if continuous!

Integration



Riemann Sum

Fubini's Thm
true for fns
"good"

Iterated integrals

$$\int_{y=c}^{d}\left[\int_{x=a}^{b} f(x,y)\,dx\right]dy$$

here is

$$= \int_{x=a}^{b}\left[\int_{y=c}^{d} \cdots\right]$$

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} \neq \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn}$$

$m \uparrow$

$$\begin{matrix} 0 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & -1 & \cdot & \cdot \\ 0 & -1 & +1 & \cdot & \cdot \\ -1 & +1 & 0 & \cdot & \cdot \\ +1 & 0 & 0 & \cdot & \cdot \end{matrix} \quad \rightarrow$$

Sum(verticals) = 0

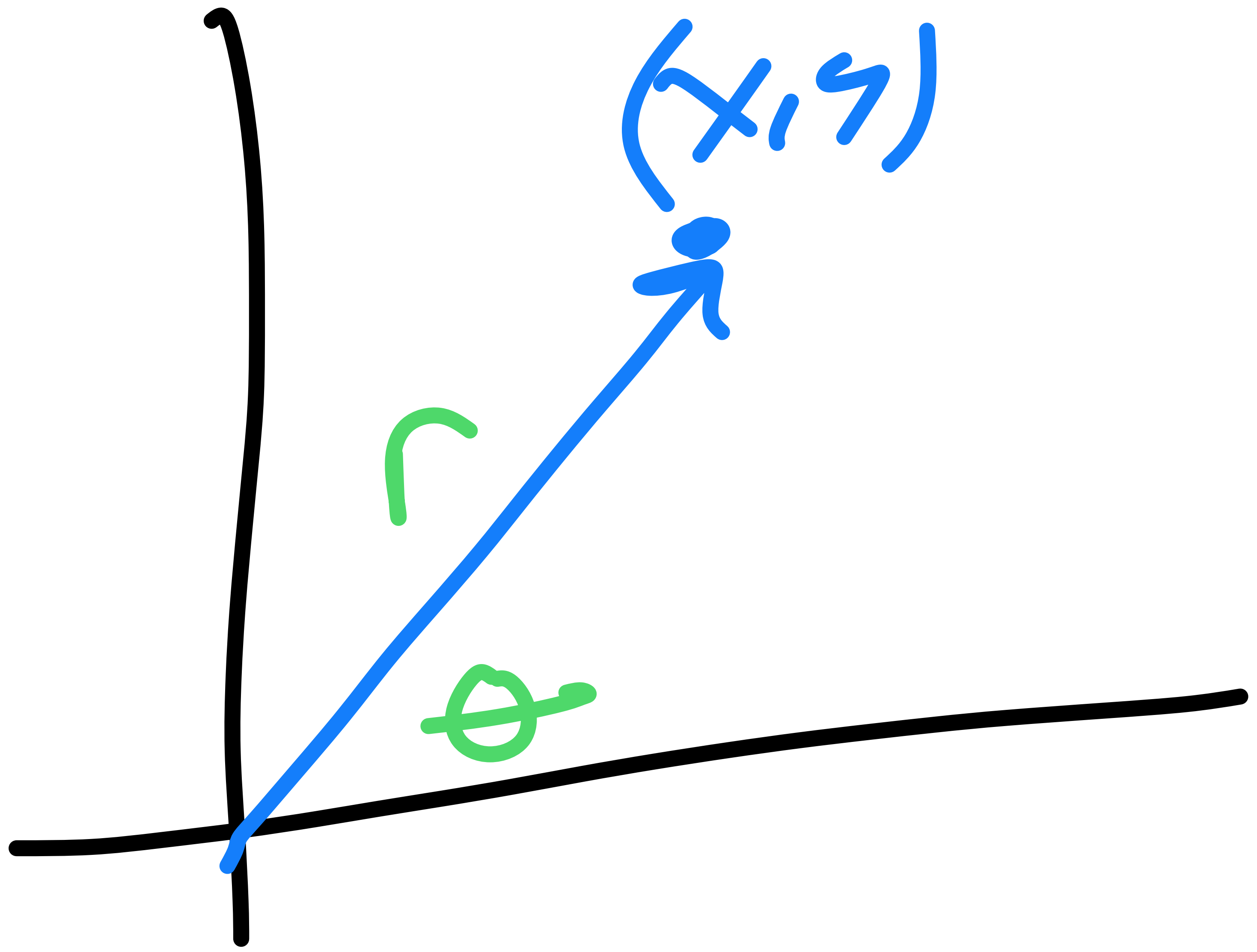
Sum(horizontals)

$= 1 + 0 + 0 + 0 + \dots = 1$

BAD! $\sum \sum |a_{m,n}| = \infty$

Change of Variables

Polar (also cylindrical spherical)



$x = r\cos\theta$

$y = r\sin\theta$

$r = \sqrt{x^2+y^2}$

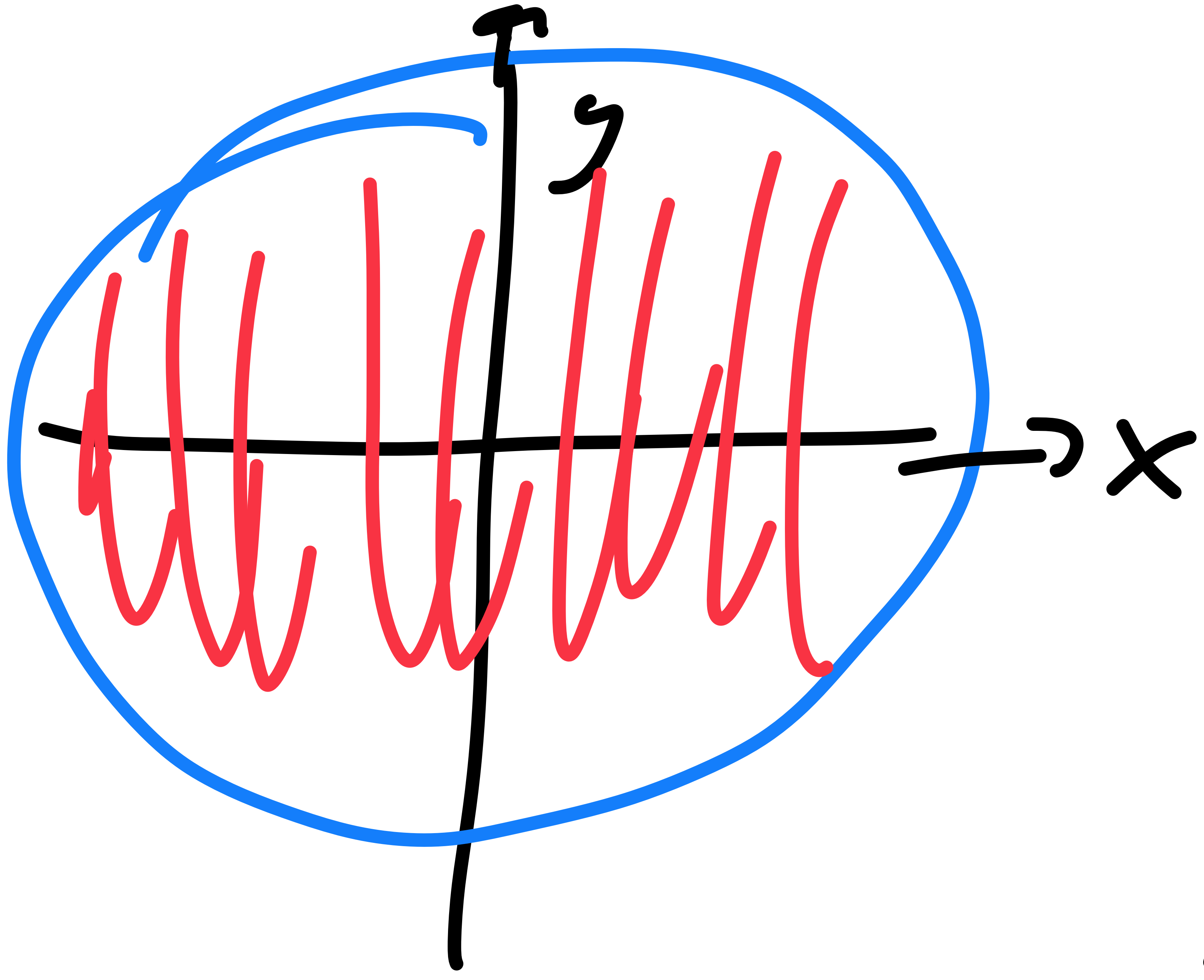
$\theta = \arctan(y/x)$

$0 \le r$

$0 \le \theta < 2\pi$

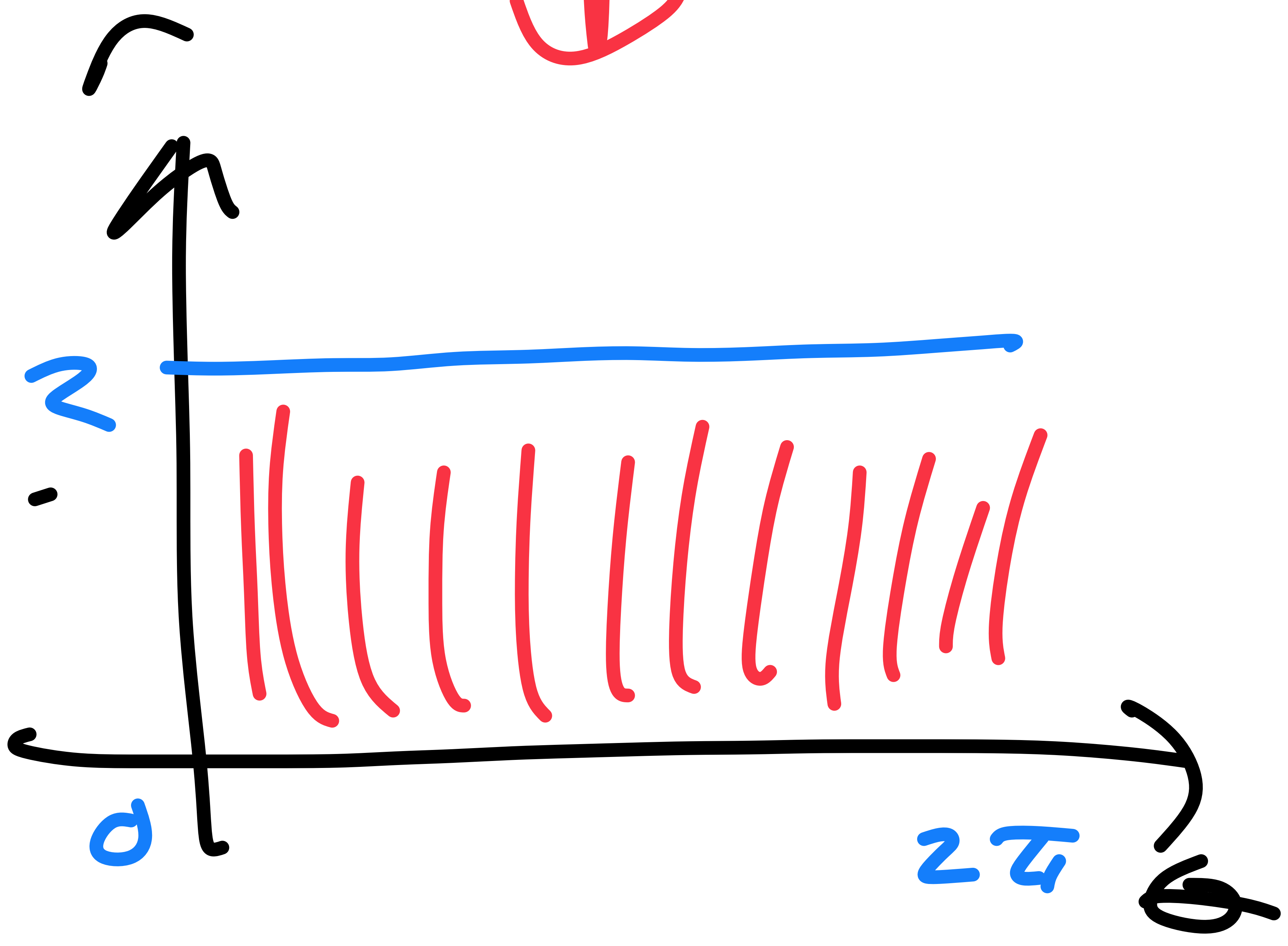
$x = r\cos\theta$

$y = r\sin\theta$



rectangle

$x^2 + y^2 \leq 2^2$



$dx\,dy \longleftrightarrow r\,dr\,d\theta$

$r\,d\theta$

$$\iint_{x^2+y^2\le 2^2} f(x,y)\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

$f(x,y)=1$

expect

4π

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{2} 1\, r\,dr\,d\theta = \int_{\theta=0}^{2\pi} \frac{r^2}{2}\Big|_0^2 d\theta$$

$$= \int_{\theta=0}^{2\pi} 2\,d\theta = 2\theta\Big|_0^{2\pi} = 4\pi \checkmark$$

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

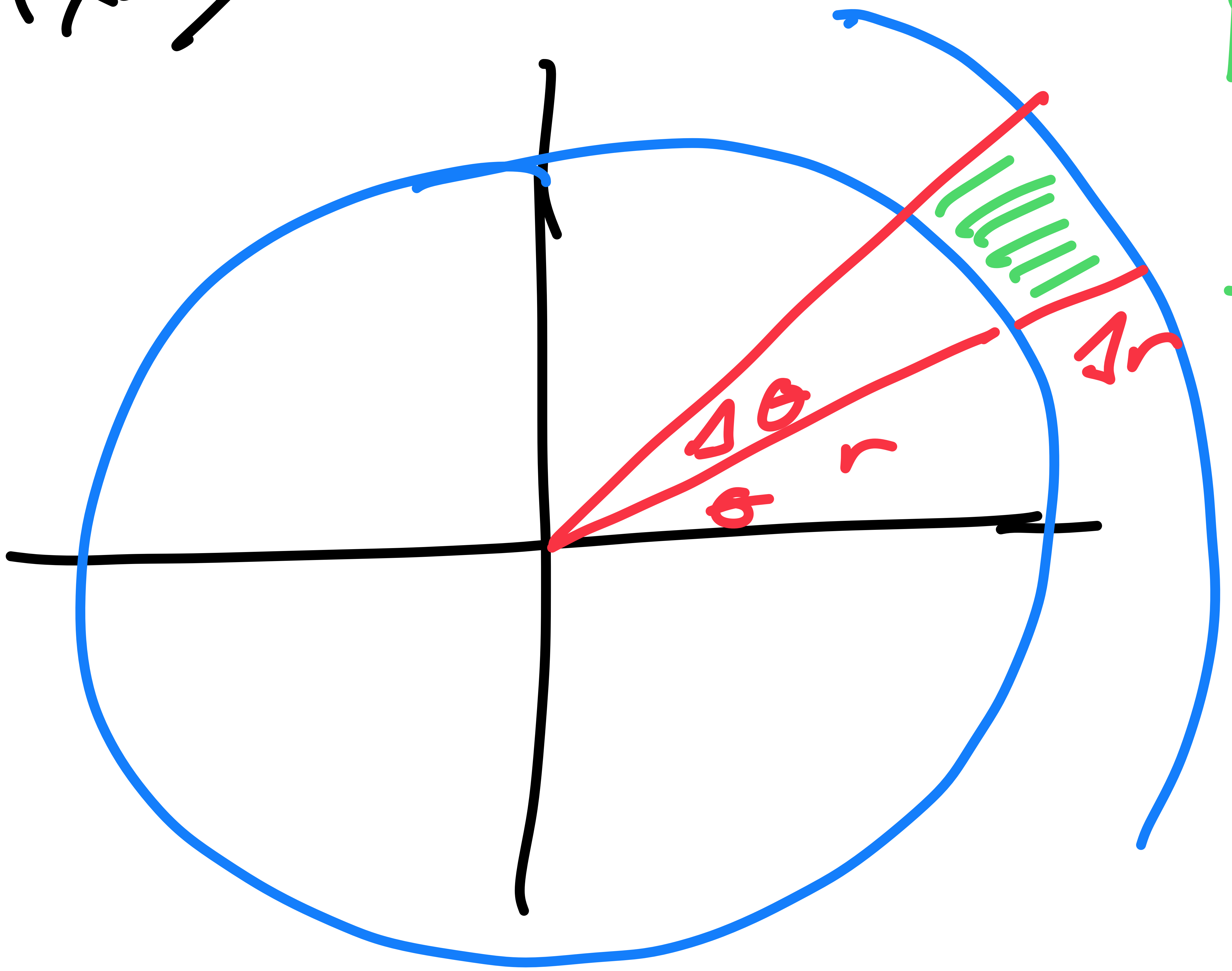
$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

$$= \left[\int_{\theta=0}^{2\pi} d\theta\right]\left[\frac{1}{2}\int_{r=0}^{\infty} e^{-r^2}\, 2r\,dr\right]$$

$$= 2\pi \cdot \frac{1}{2}\, e^{-r^2}\Big|_{\infty}^{0} = \pi \Rightarrow I = \sqrt{\pi}$$

$dx\,dy = r\,dr\,d\theta$



Big Circle Area: $\pi(r+\Delta r)^2$

Small Circle Area: πr^2

Area of ring is difference

have $\frac{\Delta\theta}{2\pi}$ of the ring

Area Patch is $\frac{\Delta\theta}{2\pi}\left(\pi r^2 + 2\pi r\Delta r + \pi(\Delta r)^2 - \pi r^2\right)$

$= r\Delta r\Delta\theta + \frac{1}{2}\Delta\theta\Delta r\Delta r \rightarrow r\,dr\,d\theta$