Random Matrix Ensembles with Split Limiting Behavior

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Slides available at https://web.williams.edu/Mathematics/sjmiller/public_html

Introduction

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem intractable.

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Heavy nuclei (Uranium: 200+ protons / neutrons) worse!

Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation:

$$H\psi_n = E_n\psi_n$$

H : matrix, entries depend on system
E_n : energy levels
ψ_n : energy eigenfunctions

Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix p, define

$$\text{Prob}(A) = \prod_{1 \le i \le j \le N} p(a_{ij}).$$

This means

$$\text{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1 \le i \le j \le N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Want to understand eigenvalues of A.

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0:
$\int f(x)\delta(x - x_0)dx = f(x_0)$.

To each A, attach a probability measure:

$$\mu_{A,N}(x) = \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right)$$

$$\int_a^b \mu_{A,N}(x)dx = \frac{\#\left\{\lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a,b]\right\}}{N}$$

$$\text{k}^{\text{th}} \text{ moment} = \frac{\sum_{i=1}^{N}\lambda_i(A)^k}{2^k N^{\frac{k}{2}+1}} = \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}.$$

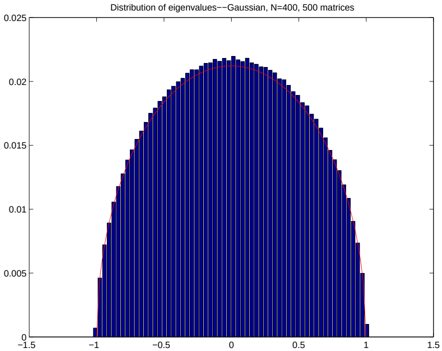
Wigner's Semi-Circle Law

Wigner's Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$ with mean 0, variance 1, and other moments finite. Then for almost all A, as $N \to \infty$

$$\mu_{A,N}(x) \longrightarrow \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

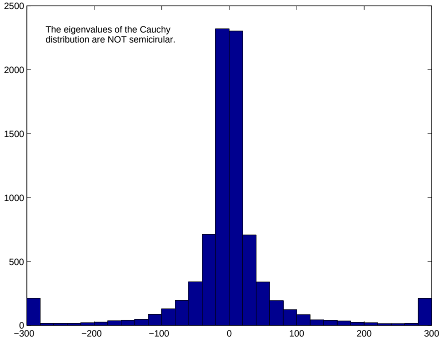
Numerical examples



500 Matrices: Gaussian 400×400

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Numerical examples



Cauchy Distribution: $p(x) = \frac{1}{\pi(1+x^2)}$

I. Zakharevich, *A generalization of Wigner's law*, Comm. Math. Phys. **268** (2006), no. 2, 403–414.

http://web.williams.edu/Mathematics/sjmiller/public_html/book/papers/innaz.pdf

SKETCH OF PROOF: Eigenvalue Trace Lemma

Want to understand the eigenvalues of A, but choose the matrix elements randomly and independently.

Eigenvalue Trace Lemma

Let A be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\mathrm{Trace}(A^k) \;=\; \sum_{n=1}^{N} \lambda_i(A)^k,$$

where

$$\mathrm{Trace}(A^k) \;=\; \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_N i_1}.$$

SKETCH OF PROOF: Correct Scale

$$\mathrm{Trace}(A^2) \;=\; \sum_{i=1}^{N} \lambda_i(A)^2.$$

By the Central Limit Theorem:

$$\mathrm{Trace}(A^2) \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}a_{ji} \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^2 \;\sim\; N^2$$

$$\sum_{i=1}^{N} \lambda_i(A)^2 \;\sim\; N^2$$

Gives $N\,\mathrm{Ave}(\lambda_i(A)^2) \sim N^2$ or $\mathrm{Ave}(\lambda_i(A)) \sim \sqrt{N}$.

SKETCH OF PROOF: Averaging Formula

Recall k-th moment of $\mu_{A,N}(x)$ is $\mathrm{Trace}(A^k)/2^k N^{k/2+1}$.

Average k-th moment is

$$\int \cdots \int \frac{\mathrm{Trace}(A^k)}{2^k N^{k/2+1}} \prod_{i \leq j} p(a_{ij}) da_{ij}.$$

Proof by method of moments: Two steps

- Show average of k-th moments converge to moments of semi-circle as $N \to \infty$;
- Control variance (show it tends to zero as $N \to \infty$).

SKETCH OF PROOF: Averaging Formula for Second Moment

Substituting into expansion gives

$$\frac{1}{2^2N^2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ji}^2\cdot p(a_{11})da_{11}\cdots p(a_{NN})da_{NN}$$

Integration factors as

$$\int_{a_{ij}=-\infty}^{\infty}a_{ij}^2 p(a_{ij})da_{ij}\cdot\prod_{\substack{(k,l)\neq(i,j)\\ k<l}}\int_{a_{kl}=-\infty}^{\infty}p(a_{kl})da_{kl} = 1.$$

Higher moments involve more advanced combinatorics (Catalan numbers).

SKETCH OF PROOF: Averaging Formula for Higher Moments

Higher moments involve more advanced combinatorics (Catalan numbers).

$$\frac{1}{2^k N^{k/2+1}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} \cdots a_{i_k i_1} \cdot \prod_{i \leq j} p(a_{ij}) da_{ij}.$$

Main contribution when the $a_{i_\ell i_{\ell+1}}$'s matched in pairs, not all matchings contribute equally (if did would get a Gaussian and not a semi-circle; this is seen in Real Symmetric Palindromic Toeplitz matrices).

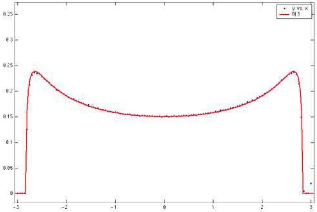
Distribution of eigenvalues of real symmetric palindromic Toeplitz matrices and circulant matrices (with Adam Massey and John Sinsheimer), Journal of Theoretical Probability **20** (2007), no. 3, 637–662.

http://arxiv.org/abs/math/0512146

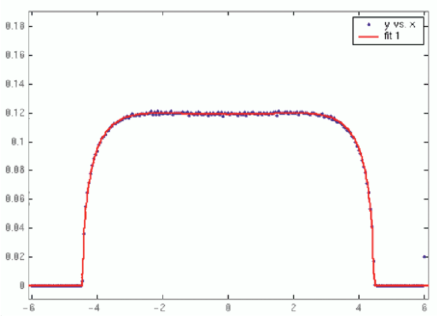
McKay's Law (Kesten Measure) with $d = 3$

Density of Eigenvalues for d-regular graphs

$$f(x) \;=\; \begin{cases} \frac{d}{2\pi(d^2-x^2)}\sqrt{4(d-1)-x^2} & |x| \le 2\sqrt{d-1} \\ 0 & \text{otherwise.} \end{cases}$$



McKay's Law (Kesten Measure) with $d = 6$



Fat Thin: fat enough to average, thin enough to get something different than semi-circle (though as $d \to \infty$ recover semi-circle).

The Ensemble of m-Block Circulant Matrices

Symmetric matrices periodic with period m on wrapped diagonals, i.e., symmetric block circulant matrices.

8-by-8 real symmetric 2-block circulant matrix:

$$\left(\begin{array}{cc|cc|cc|cc} c_0 & c_1 & {\color{red}c_2} & c_3 & c_4 & d_3 & c_2 & d_1 \\ c_1 & d_0 & d_1 & {\color{blue}d_2} & d_3 & d_4 & c_3 & d_2 \\ \hline c_2 & d_1 & c_0 & c_1 & {\color{red}c_2} & c_3 & c_4 & d_3 \\ c_3 & d_2 & c_1 & d_0 & d_1 & {\color{blue}d_2} & d_3 & d_4 \\ \hline c_4 & d_3 & c_2 & d_1 & c_0 & c_1 & {\color{red}c_2} & c_3 \\ d_3 & d_4 & c_3 & d_2 & c_1 & d_0 & d_1 & {\color{blue}d_2} \\ \hline {\color{red}c_2} & c_3 & c_4 & d_3 & c_2 & d_1 & c_0 & c_1 \\ d_1 & {\color{blue}d_2} & d_3 & d_4 & c_3 & d_2 & c_1 & d_0 \end{array}\right).$$

Choose distinct entries i.i.d.r.v.

Results

Theorem: Koloğlu, Kopp and Miller

The limiting spectral density function $f_m(x)$ of the real symmetric m-block circulant ensemble is given by

$$f_m(x) = \frac{e^{-\frac{mx^2}{2}}}{\sqrt{2\pi m}} \sum_{r=0}^{m} \frac{1}{(2r)!} \sum_{s=0}^{m-r} \binom{m}{r+s+1} \frac{(2r+2s)!}{(r+s)!s!} \left(-\frac{1}{2}\right)^s (mx^2)^r.$$

Fixed m equals $m \times m$ GOE, as $m \to \infty$ converges to the semicircle distribution.

Results (continued)

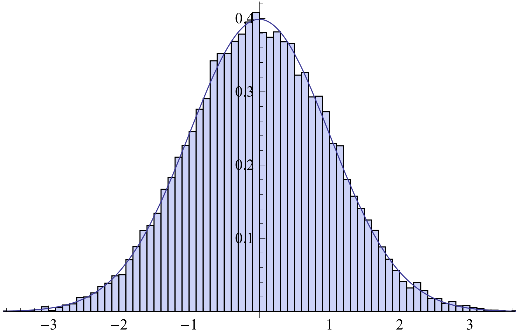


Figure: Plot for f_1 and histogram of eigenvalues of 100 circulant matrices of size 400×400.

Results (continued)

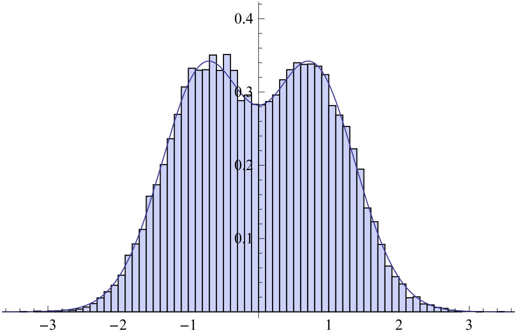


Figure: Plot for f_2 and histogram of eigenvalues of 100 2-block circulant matrices of size 400×400.

Results (continued)

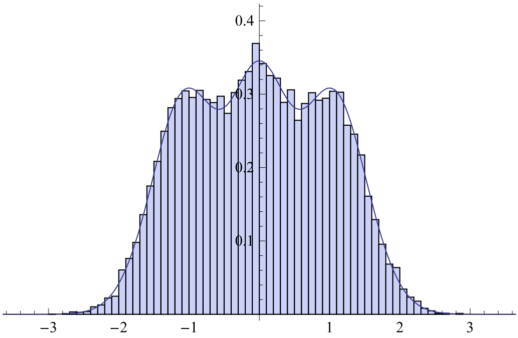


Figure: Plot for f_3 and histogram of eigenvalues of 100 3-block circulant matrices of size 402×402.

Results (continued)

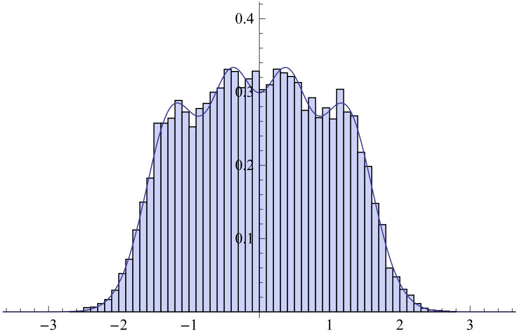


Figure: Plot for f_4 and histogram of eigenvalues of 100 4-block circulant matrices of size 400×400.

Results (continued)

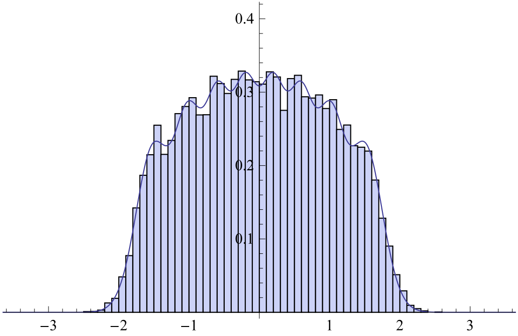


Figure: Plot for f_8 and histogram of eigenvalues of 100 8-block circulant matrices of size 400×400.

Results (continued)

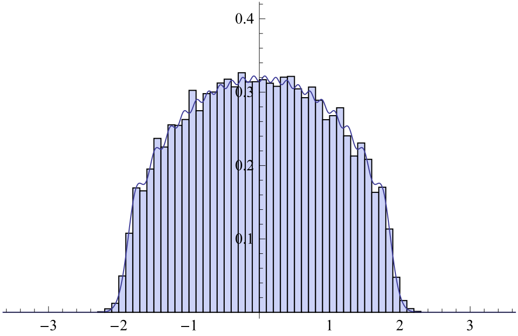


Figure: Plot for f_{20} and histogram of eigenvalues of 100 20-block circulant matrices of size 400×400.

Results (continued)

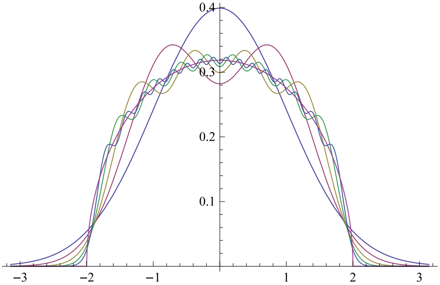


Figure: Plot of convergence to the semi-circle.

The Limiting Spectral Measure for Ensembles of Symmetric Block Circulant Matrices (with Murat Koloğlu, Gene S. Kopp, Frederick W. Strauch and Wentao Xiong), Journal of Theoretical Probability **26** (2013), no. 4, 1020–1060. http://arxiv.org/abs/1008.4812

k-Checkerboard Ensembles

Checkerboard Matrices: $N \times N$ (k, w)-checkerboard ensemble

Matrices $M = (m_{ij}) = M^T$ with a_{ij} iidrv, mean 0, variance 1, finite higher moments, w fixed and

$$m_{ij} = \begin{cases} a_{ij} & \text{if } i \not\equiv j \bmod k \\ w & \text{if } i \equiv j \bmod k. \end{cases}$$

Example: $(3, w)$-checkerboard matrix:

$$\begin{pmatrix} w & a_{0,1} & a_{0,2} & w & a_{0,4} & \cdots & a_{0,N-1} \\ a_{1,0} & w & a_{1,2} & a_{1,3} & w & \cdots & a_{1,N-1} \\ a_{2,0} & a_{2,1} & w & a_{2,3} & a_{2,4} & \cdots & w \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{0,N-1} & a_{1,N-1} & w & a_{3,N-1} & a_{4,N-1} & \cdots & w \end{pmatrix}$$

Split Eigenvalue Distribution

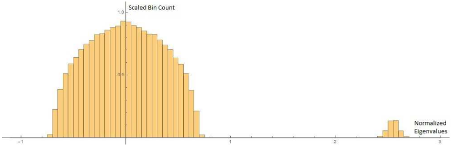


Figure: Histogram of normalized eigenvalues for 500 100×100 2-checkerboard matrices.

Eigenvalue Regimes

Theorem

Let $\{A_N\}_{N\in\mathbb{N}}$ be a sequence of (k, w)-checkerboard matrices. Then almost surely as $N \to \infty$ the eigenvalues of A_N fall into two regimes: $N - k$ of the eigenvalues are $O(N^{1/2+\epsilon})$ and k eigenvalues are of magnitude $Nw/k + O(N^{1/2+\epsilon})$.

Normalized Empirical Spectral Measure

Definition

Given an $N \times N$ Hermitian matrix M_N with eigenvalues $\{\lambda_i\}_{i=1}^N$, the **normalized empirical spectral measure** is

$$\nu_{\frac{1}{\sqrt{N}}M_N}(x) := \frac{1}{N}\sum_{i=1}^{N}\delta(x - \lambda_i/\sqrt{N})$$

Theorem

Let $\{M_N\}_{N\in\mathbb{N}}$ be a sequence of real $N \times N$ k-checkerboard matrices. Then, the normalized empirical spectral measures $\mu_{\frac{1}{\sqrt{N}}M_N}$ converge weakly almost surely to the semi-circle distribution.

Moment convergence theorem

Theorem (Moment Convergence Theorem)

Let μ be a measure on $\mathbb{R}$ with finite moments $\mu^{(m)}$ for all $m \in \mathbb{Z}_{\geq 0}$, and $\mu_1, \mu_2, \ldots$ a sequence of measures with finite moments $\mu_n^{(m)}$ such that $\lim_{n\to\infty} \mu_n^{(m)} = \mu^{(m)}$ for all $m \in \mathbb{Z}_{\geq 0}$. If in addition the moments $\mu^{(m)}$ uniquely characterize a measure (Carleman's condition), then the sequence μ_n converges weakly to μ.

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Remark

If the moments converge almost-surely, then the measures almost-surely converge weakly.

Standard arguments

We wish to show m^{th} moments $X_{m,N}$ of empirical spectral measure of $N \times N$ ensemble converge a.s. to desired M_m as $N \to \infty$.
Show

$$|X_{m,N} - M_m| \;\leq\; |X_{m,N} - \mathbb{E}[X_{m,N}]| + |\mathbb{E}[X_{m,N}] - M_m|\,.$$

converges a.s. to 0 as $N \to \infty$.

Bulk Distribution: Obstructions

- There are $N-k$ eigenvalues of order $O(N^{1/2+\epsilon})$ in the bulk.

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- There are $N - k$ eigenvalues of order $O(N^{1/2+\epsilon})$ in the bulk.
- Recall that there are k eigenvalues of magnitude $Nw/k + O(N^{1/2+\epsilon})$.
- Because of these high magnitude eigenvalues, the limiting expected moments of the normalized ESD do not exist.
- This obstructs the standard application of the method of moments.

Perturbation Theorem

Theorem (Tao)

Let $\{\mathcal{A}_N\}_{N\in\mathbb{N}}$ be a sequence of random Hermitian matrix ensembles such that $\{\nu_{\mathcal{A}_N,N}\}_{N\in\mathbb{N}}$ converges weakly almost surely to a limit ν. Let $\{\tilde{\mathcal{A}}_N\}_{N\in\mathbb{N}}$ be another sequence of random matrix ensembles such that $\frac{1}{N}\operatorname{rank}(\tilde{\mathcal{A}}_N)$ converges almost surely to zero. Then $\{\nu_{\mathcal{A}_N+\tilde{\mathcal{A}}_N,N}\}_{N\in\mathbb{N}}$ converges weakly almost surely to ν.

Examining the Blip I

- To understand the limiting distribution of the blip, we localize our measure to the blip regime.

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- To understand the limiting distribution of the blip, we localize our measure to the blip regime.

- To do this, define a new empirical spectral measure by

$$\mu_{A,N} := \frac{1}{k} \sum_{\lambda \text{ eigenvalue of } A} f\left(\frac{k\lambda}{N}\right) \delta\left(x - \left(\lambda - \frac{N}{k}\right)\right)$$

with f a function ≈ 0 on the bulk and ≈ 1 on the blip.

Examining the Blip II

- Candidates for f must be amenable to Eigenvalue-Trace Lemma arguments (so we must either choose a polynomial or deal with Taylor series convergence).

Examining the Blip II

- Candidates for f must be amenable to Eigenvalue-Trace Lemma arguments (so we must either choose a polynomial or deal with Taylor series convergence).
- Any given polynomial does not vanish to a high enough order at $x = 0$ as $N \to \infty$, so we choose family of polynomials.

The Weighting Function

Use weighting function $f_n(x) = x^{2n}(x-2)^{2n}$.

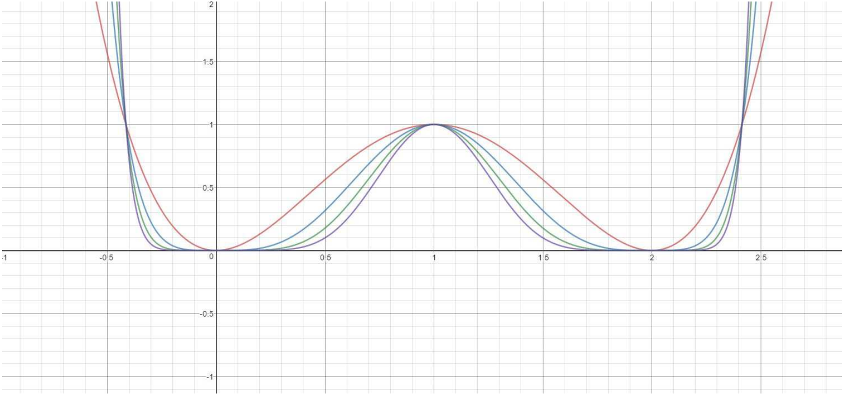

Figure: $f_n(x)$ plotted for $n = 1$ to $n = 4$.

The New Spectral Measure I

Using the weighting function $f_n(x)$ we form a new empirical spectral measure.

Definition

The **empirical blip spectral measure** associated to an $N \times N$ k-checkerboard matrix A is

$$\mu_{A,N} := \frac{1}{k} \sum_{\lambda \text{ eigenvalue of } A} f_{n(N)}\left(\frac{k\lambda}{N}\right) \delta\left(x - \left(\lambda - \frac{N}{k}\right)\right)$$

where $n(N)$ is a function for which there exists some ϵ so that $N^{\epsilon} \ll n(N) \ll N^{1-\epsilon}$.

Main theorem

Definition

The **hollow Gaussian Orthogonal Ensemble** is given by $B = (b_{ij}) = B^T$ with

$$b_{ij} = \mathcal{N}_{\mathbb{R}}(0,1)(1-\delta_{ij})$$

Theorem

We have

$$\lim_{N\to\infty} \mathbb{E}[\overline{\mu}_{A,N}^{(m)}] \;=\; \frac{1}{k}\mathbb{E}_k \;\text{Tr}\; B^m,$$

where $\overline{\mu}_{A,N}^{(m)}$ is the centered moments of the empirical blip spectral measure of the $N \times N$ k-checkerboard ensemble and B is in the hollow GOE.

Main Result

Issue: Can't look at blip of just one matrix as only fixed number eigenvalues; average over $g(N)$ such matrices.

Theorem

Let $g : \mathbb{N} \to \mathbb{N}$ be such that there exists an $\delta > 0$ for which $g(N) = \omega(N^\delta)$. Then, as $N \to \infty$, the averaged empirical spectral measures $\mu_{N,g,\overline{A}}$ of the k-checkerboard ensemble converge weakly almost-surely to the measure with moments $M_{k,m} = \frac{1}{k}\mathbb{E}_k \operatorname{Tr}[B^m]$.

Spectral distribution of hollow GOE

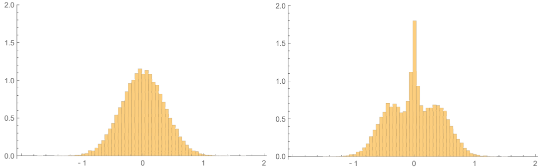

Figure: Hist. of eigenvals of 32000 (Left) 2×2 hollow GOE matrices, (Right) 3×3 hollow GOE matrices.

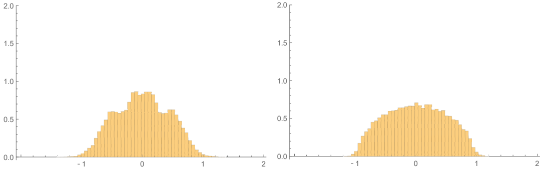

Figure: Hist. of eigenvals of 32000 (Left) 4×4 hollow GOE matrices, (Right) 16×16 hollow GOE matrices.

References / Acknowledgements

Acknowledgments

- Full paper available on arXiv: https://arxiv.org/abs/1609.03120

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Other Random Matrix Theory Papers

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