The Limiting Eigenvalue Density for the Ensemble of m-Circulant Matrices

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Classical
Random Matrix Theory

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Heavy nuclei (Uranium: $200+$ protons / neutrons) worse!

Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation:

$$H\psi_n \;=\; E_n\psi_n$$

H : matrix, entries depend on system

E_n : energy levels

ψ_n : energy eigenfunctions

Origins of Random Matrix Theory

- Statistical Mechanics: for each configuration, calculate quantity (say pressure).
- Average over all configurations – most configurations close to system average.
- Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices (real Symmetric $A = A^T$, complex Hermitian $\overline{A}^T = A$).

Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix p, define

$$\text{Prob}(A) = \prod_{1 \le i \le j \le N} p(a_{ij}).$$

This means

$$\text{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1 \le i \le j \le N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Want to understand eigenvalues of A.

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0:
$\int_{\mathbb{R}} f(x)\delta(x - x_0)dx = f(x_0)$.

To *each* matrix A, attach a probability measure:

$$\mu_{A,N}(x) := \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{\sqrt{N}}\right)$$

$$\int_{\mathbb{R}} f(x)\mu_{A_N}(x)dx = \sum_{i=1}^{N} f\left(\frac{\lambda_i(A)}{\sqrt{N}}\right)$$

$$M_n(A, N) := \text{n}^{\text{th}} \text{ moment} = \frac{1}{N^{\frac{n}{2}+1}}\sum_{i=1}^{N}\lambda_i(A)^n = \frac{\text{Trace}(A^n)}{N^{\frac{n}{2}+1}}.$$

Eigenvalue Trace Formula

We want to understand the eigenvalues of A, but it is the matrix elements that are chosen randomly and independently.

Eigenvalue Trace Lemma

Let A be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\mathrm{Trace}(A^k) \ = \ \sum_{n=1}^{N} \lambda_i(A)^k,$$

where

$$\mathrm{Trace}(A^k) \ = \ \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}.$$

Our ensemble:
m-Circulant Matrices

m-Circulant Ensemble

We look at matrices A of the following form, which we call m-circulant or m-doped palindromic Toeplitz.

$$\left(\begin{array}{ccccccccccc}
b_{1,0} & b_{1,1} & b_{1,2} & & \cdots & & b_{1,\frac{N}{2}-1} & b_{1,\frac{N}{2}} & b_{1,\frac{N}{2}-1} & \cdots & b_{1,1} \\
b_{1,1} & b_{2,0} & b_{2,1} & & \cdots & & & b_{2,\frac{N}{2}-1} & b_{2,\frac{N}{2}} & & b_{2,2} \\
b_{1,2} & b_{2,1} & b_{3,0} & & \cdots & & & & \ddots & \ddots & b_{3,3} \\
 & & & \ddots & & & & & & & \vdots \\
 & & & & b_{m,0} & b_{m,1} & b_{m,2} & & \cdots & & b_{m,m+1} \\
\vdots & \vdots & \vdots & & b_{m,1} & b_{1,0} & b_{1,1} & & \cdots & & b_{1,m+2} \\
 & & & & b_{m,2} & b_{1,1} & b_{2,0} & & \cdots & & b_{2,m+3} \\
 & & & & \vdots & \vdots & \vdots & & \ddots & & \vdots \\
b_{1,1} & b_{2,2} & b_{3,3} & \cdots & b_{m,m+1} & b_{1,m+2} & b_{2,m+3} & & \cdots & & b_{m,0}
\end{array}\right)$$

Matrices are real symmetric, becomes a probability space when we choose the red entries independently from a fixed distribution p of mean 0 and variance 1, and fill in the rest of the matrix as per the structure defined.

Averaging

Look at the *expected value* for the moments:

$$\begin{aligned} M_n(N) &:= \mathbb{E}(M_n(A,N)) \\ &= \frac{1}{N^{\frac{n}{2}+1}} \mathbb{E}(\mathrm{Trace}(A^n)) \\ &= \frac{1}{N^{\frac{n}{2}+1}} \sum_{1 \le i_1,\dots,i_n \le N} \mathbb{E}(a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_n i_1}). \end{aligned}$$

As $N \to \infty$, these moments converge to the moments of the limiting spectral distribution. A bounding arguments involving Chebyshev's inequality and the Borel-Cantelli lemma shows that a “typical" *m*-circulant matrix of large dimension has an eigenvalue distribution “close" to this limiting density.

Matchings

We rewrite our formula for the moments as

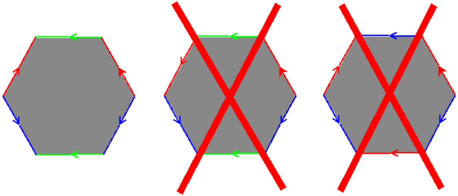
$$M_n(N) = \frac{1}{N^{\frac{n}{2}+1}} \sum_{\sim} \eta(\sim) m_{d_1(\sim)} \cdots m_{d_l(\sim)}.$$

where the sum is over equivalence relations on $\{(1,2),(2,3),...,(n,1)\}$. The $d_j(\sim)$ denote the sizes of the equivalence classes, and the m_d the moments of p. Finally, the coefficient $\eta(\sim)$ is the number of solutions to the system of Diophantine equations:
Whenever $(s, s+1) \sim (t, t+1)$,

- $i_{s+1} - i_s \equiv i_{t+1} - i_t \pmod N$ and $i_s \equiv i_t \pmod m$, or
- $i_{s+1} - i_s \equiv -(i_{t+1} - i_t) \pmod N$ and $i_s \equiv i_{t+1} \pmod m$.

Contributing Terms

As $N \to \infty$, the only terms that contribute to this sum are those in which the entries are matched in pairs and with opposite orientation.



Therefore, the odd moments go to zero as $N^{-1/2}$.

Algebraic Topology

If we think of these pairings as topological identifications, the contributing ones are precisely the ones that give rise to orientable surfaces.

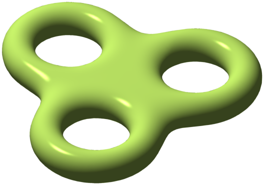


Figure: A three holed torus.

It turns out that the contribution from such a pairing is m^{-2g}, where g is the genus (number of holes) of the surface. The proof is a combinatorial argument involving Euler characteristic.

Computing the Even Moments

Our formula for the even moments becomes

$$M_{2k} = \sum_{g=0}^{\lfloor k/2 \rfloor} \varepsilon_g(k) m^{-2g} + O_k\left(\frac{1}{N}\right),$$

with $\varepsilon_g(k)$ the number of pairings of the edges of a $(2k)$-gon giving rise to a genus g surface. J. Harer and D. Zagier (1986) gave generating functions for the $\varepsilon_g(k)$. Their results and a bit of analysis yield explicit formulas for the limiting spectral density.

Results

Theorem

The limiting spectral density function $f_m(x)$ of the real symmetric m-circulant ensemble is given by the formula

$$f_m(x) = \frac{e^{-\frac{mx^2}{2}}}{\sqrt{2\pi m}} \sum_{l=1}^{m} \sum_{s=0}^{l-1} \binom{m}{l} \frac{(2s-1)!!}{(l-1)!} \binom{2(l-1)}{2s} \cdot \left(mx^2\right)^{l-1-s} (-1)^s.$$

Results (continued)

Theorem

As $m \to \infty$, the limiting spectral densities approach the semicircle distribution.

Results (continued)

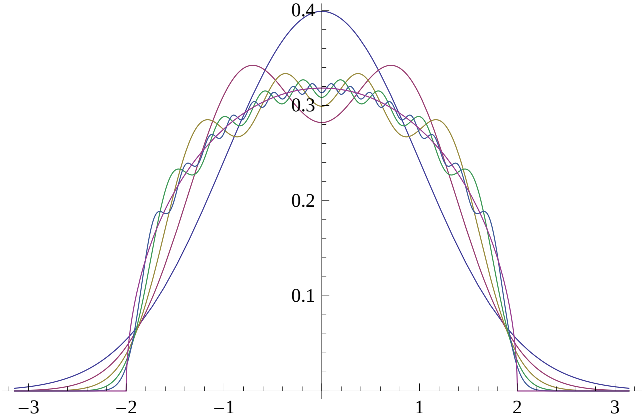


Figure: Plots for f_1, f_2, f_4, f_8, f_{16} and the semicircle.

Now that we have an explicit formula for the moments

$$M_{2k} = \sum_{g=0}^{k/2} \varepsilon_g(k) m^{-2g}$$

$$\varepsilon_g(k) = \frac{(2k)!}{(k+1)!(k-2g)!} \times \left(\text{Coefficientof } x^{2g} \text{ in } \left(\frac{x/2}{\tanh(x/2)}\right)^{k+$$

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construct char. fn of limiting spectral distribution:

$$\phi(t) \;=\; \sum_{k=0}^{\infty} \frac{1}{(2k)!} (it)^{2k} M_{2k}$$

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$$\begin{aligned}
\phi(t) &= \sum_{k=0}^{\infty} \frac{1}{(2k)!}(it)^{2k} M_{2k} \\
&= \sum_{k=0}^{\infty} \sum_{g=0}^{k/2} \varepsilon_g(k) m^{-2g} (-t^2)^k/(2k)!.
\end{aligned}$$

Using $(2k-1)!! = \frac{(2k)!}{2^k k!}$, rewrite the characteristic function:

$$\phi(t) = m^{-1} \sum_{k=0}^{\infty} \sum_{g=0}^{k/2} \frac{\varepsilon_g(k) m^{k+1-2g}}{(2k-1)!!} \frac{(-t^2/2m)^k}{k!}$$

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Now consider two functions

$$F(y) := \sum_{k=0}^{\infty} \sum_{g=0}^{k/2} \frac{\varepsilon_g(k) m^{k+1-2g}}{(2k-1)!!} y^k$$

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$$= \frac{1}{2y} \left(\left(\frac{1+y}{1-y} \right)^m - 1 \right)$$

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$$G(y) := e^y = \sum_{k=0}^{\infty} y^k / k!.$$

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