Eigenvalue Statistics of Toeplitz and Block *m*-Circulant Ensembles

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AMS Special Session on Random Processes
Holy Cross, April 9, 2011

Introduction

Goals

- See how the structure of the ensembles affects limiting behavior.
- Discuss the tools and techniques needed to prove the results.
- Due to time: only discussing eigenvalue density, not neighbor spacings.

Eigenvalue Distribution

To each A, attach a probability measure:

$$\mu_{A,N}(x) = \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right)$$

$$\int_a^b \mu_{A,N}(x)dx = \frac{\#\left\{\lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a,b]\right\}}{N}$$

$$\text{k}^{\text{th}}\text{ moment} = \frac{\sum_{i=1}^{N}\lambda_i(A)^k}{2^k N^{\frac{k}{2}+1}} = \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}.$$

Wigner's Semi-Circle Law

Wigner's Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$ with mean 0, variance 1, and other moments finite. Then for almost all A, as $N \to \infty$

$$\mu_{A,N}(x) \longrightarrow \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Fat-Thin Families

Need a family FAT enough to do averaging and THIN enough so that everything isn't averaged out.

Real Symmetric Matrices have $\frac{N(N+1)}{2}$ independent entries.

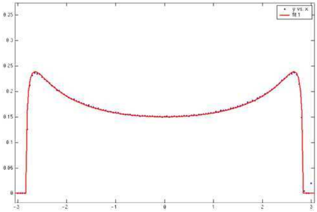
Examples of Fat-Thin sub-families:

- Band Matrices
- Random Graphs
- Special Matrices (Toeplitz, Circulant, ...)

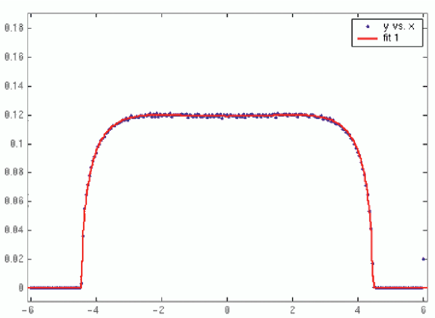
McKay's Law (Kesten Measure) with $d = 3$

Density of Eigenvalues for d-regular graphs

$$f(x) \;=\; \begin{cases} \frac{d}{2\pi(d^2-x^2)}\sqrt{4(d-1)-x^2} & |x| \le 2\sqrt{d-1} \\ 0 & \text{otherwise.} \end{cases}$$



McKay's Law (Kesten Measure) with $d = 6$



Fat-Thin: fat enough to average, thin enough to get something different than semi-circle (though as $d \to \infty$ recover semi-circle).

Toeplitz Ensembles

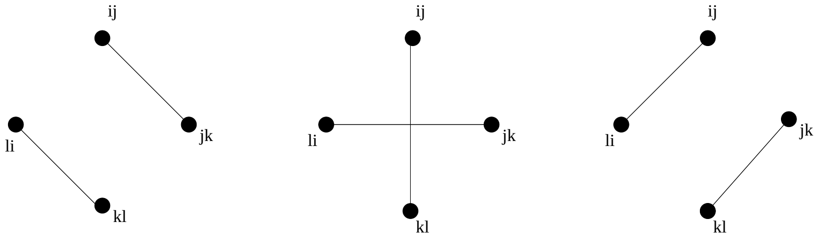
Toeplitz Ensembles

Toeplitz matrix is of the form

$$\begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{N-1} \\ b_{-1} & b_0 & b_1 & \cdots & b_{N-2} \\ b_{-2} & b_{-1} & b_0 & \cdots & b_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{1-N} & b_{2-N} & b_{3-N} & \cdots & b_0 \end{pmatrix}$$

- Will consider Real Symmetric Toeplitz matrices.
- Main diagonal zero, $N - 1$ independent parameters.
- Normalize Eigenvalues by $\sqrt{N}$.

The Fourth Moment



$$M_4(N) = \frac{1}{N^3} \sum_{1 \le i_1, i_2, i_3, i_4 \le N} \mathbb{E}(b_{|i_1 - i_2|} b_{|i_2 - i_3|} b_{|i_3 - i_4|} b_{|i_4 - i_1|})$$

Let $x_j = |i_j - i_{j+1}|$.

The Fourth Moment

Case One: $x_1 = x_2, x_3 = x_4$:

$$i_1 - i_2 = -(i_2 - i_3) \quad \text{and} \quad i_3 - i_4 = -(i_4 - i_1).$$

Implies

$$i_1 = i_3, \quad i_2 \text{ and } i_4 \text{ arbitrary.}$$

Left with $\mathbb{E}[b_{x_1}^2 b_{x_3}^2]$:

$$N^3 - N \text{ times get } 1, \; N \text{ times get } p_4 = \mathbb{E}[b_{x_1}^4].$$

Contributes 1 in the limit.

The Fourth Moment

$$M_4(N) = \frac{1}{N^3} \sum_{1 \le i_1, i_2, i_3, i_4 \le N} \mathbb{E}(b_{|i_1 - i_2|} b_{|i_2 - i_3|} b_{|i_3 - i_4|} b_{|i_4 - i_1|})$$

Case Two: Diophantine Obstruction: $x_1 = x_3$ and $x_2 = x_4$.

$$i_1 - i_2 \;=\; -(i_3 - i_4) \quad \text{and} \quad i_2 - i_3 \;=\; -(i_4 - i_1).$$

This yields

$$i_1 \;=\; i_2 + i_4 - i_3, \quad i_1, i_2, i_3, i_4 \in \{1, \dots, N\}.$$

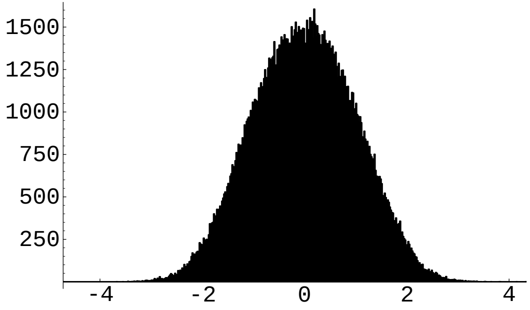
If $i_2, i_4 \ge \frac{2N}{3}$ and $i_3 < \frac{N}{3}$, $i_1 > N$: at most $(1 - \frac{1}{27})N^3$ valid choices.

The Fourth Moment

Theorem: Fourth Moment: Let p_4 be the fourth moment of p. Then

$$M_4(N) \;=\; 2\frac{2}{3} + O_{p_4}\left(\frac{1}{N}\right).$$

500 Toeplitz Matrices, 400×400.



Main Result

Theorem: Hammond and M– ’05

For real symmetric Toeplitz matrices, the limiting spectral measure converges in probability to a unique measure of unbounded support which is not the Gaussian. If p is even have strong convergence).

Theorem: Massey, M– and Sinsheimer ’07

For real symmetric palindromic matrices (first row a palindrome), converge in probability to the Gaussian (if p is even have strong convergence).

Results exist for highly palindromic (Jackson, M– and Pham ’11).

Block *m*-Circulant Matrices

Study circulant matrices periodic with period *m* on diagonals.

6-by-6 real symmetric period 2-circulant matrix:

$$\begin{pmatrix} c_0 & c_1 & c_2 & c_3 & c_2 & d_1 \\ c_1 & d_0 & d_1 & d_2 & c_3 & d_2 \\ c_2 & d_1 & c_0 & c_1 & c_2 & c_3 \\ c_3 & d_2 & c_1 & d_0 & d_1 & d_2 \\ c_2 & c_3 & c_2 & d_1 & c_0 & c_1 \\ d_1 & d_2 & c_3 & d_2 & c_1 & d_0 \end{pmatrix}.$$

Look at the *expected value* for the moments:

$$\begin{aligned} M_n(N) &:= \mathbb{E}(M_n(A,N)) \\ &= \frac{1}{N^{\frac{n}{2}+1}} \sum_{1\le i_1,\dots,i_n\le N} \mathbb{E}(a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_n i_1}). \end{aligned}$$

Matchings

Rewrite:

$$M_n(N) = \frac{1}{N^{\frac{n}{2}+1}} \sum_{\sim} \eta(\sim) m_{d_1(\sim)} \cdots m_{d_l(\sim)}.$$

where the sum is over equivalence relations on $\{(1,2),(2,3),...,(n,1)\}$. The $d_j(\sim)$ denote the sizes of the equivalence classes, and the m_d the moments of p. Finally, the coefficient $\eta(\sim)$ is the number of solutions to the system of Diophantine equations:
Whenever $(s, s+1) \sim (t, t+1)$,

- $i_{s+1} - i_s \equiv i_{t+1} - i_t \pmod N$ and $i_s \equiv i_t \pmod m$, or
- $i_{s+1} - i_s \equiv -(i_{t+1} - i_t) \pmod N$ and $i_s \equiv i_{t+1} \pmod m$.

- $i_{s+1} - i_s \equiv i_{t+1} - i_t \pmod{N}$ and $i_s \equiv i_t \pmod{m}$, or
- $i_{s+1} - i_s \equiv -(i_{t+1} - i_t) \pmod{N}$ and $i_s \equiv i_{t+1} \pmod{m}$.

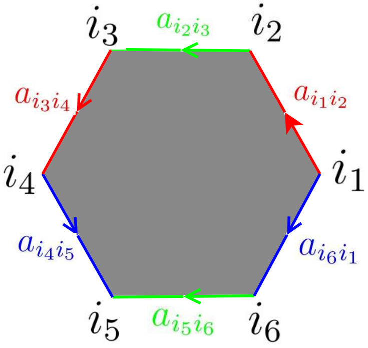
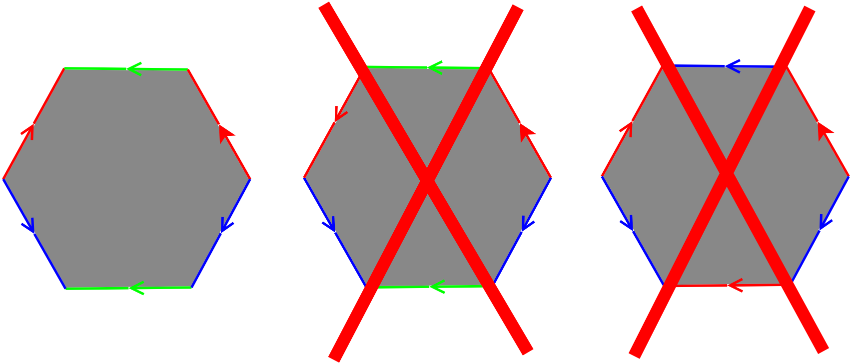


Figure: Red edges same orientation and blue, green opposite.

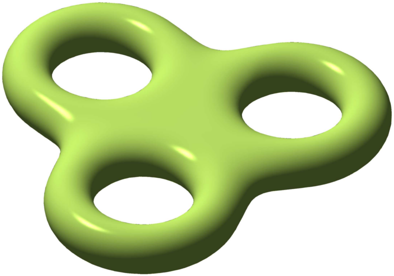
Contributing Terms

As $N \to \infty$, the only terms that contribute to this sum are those in which the entries are matched in pairs and with opposite orientation.



Algebraic Topology

Think of pairings as topological identifications, the contributing ones give rise to orientable surfaces.



Contribution from such a pairing is m^{-2g}, where g is the genus (number of holes) of the surface. Proof: combinatorial argument involving Euler characteristic.

Computing the Even Moments

Theorem: Even Moment Formula

$$M_{2k} = \sum_{g=0}^{\lfloor k/2 \rfloor} \varepsilon_g(k) m^{-2g} + O_k\left(\frac{1}{N}\right),$$

with $\varepsilon_g(k)$ the number of pairings of the edges of a $(2k)$-gon giving rise to a genus g surface.

J. Harer and D. Zagier (1986) gave generating functions for the $\varepsilon_g(k)$.

Harer and Zagier

$$\sum_{g=0}^{\lfloor k/2 \rfloor} \varepsilon_g(k) r^{k+1-2g} \;=\; (2k-1)!!\, c(k,r)$$

where

$$1 + 2\sum_{k=0}^{\infty} c(k,r) x^{k+1} \;=\; \left(\frac{1+x}{1-x}\right)^r.$$

Thus, we write

$$M_{2k} \;=\; m^{-(k+1)}(2k-1)!!\, c(k,m).$$

A multiplicative convolution and Cauchy's residue formula yields the *characteristic function* of the distribution (inverse Fourier transform of the density).

$$\begin{aligned}\phi(t) &= \sum_{k=0}^{\infty} \frac{(it)^{2k} M_{2k}}{(2k)!} \\ &= \frac{1}{2\pi i m} \oint_{|z|=2} \frac{1}{2z^{-1}} \left(\left(\frac{1+z^{-1}}{1-z^{-1}} \right)^m - 1 \right) e^{-t^2 z/2m} \frac{dz}{z} \\ &= \frac{1}{m} e^{\frac{-t^2}{2m}} \sum_{l=1}^{m} \binom{m}{l} \frac{1}{(l-1)!} \left(\frac{-t^2}{m} \right)^{l-1}\end{aligned}$$

Results

Fourier transform and algebra yields

Theorem: Kopp, Koloğlu and M–

The limiting spectral density function $f_m(x)$ of the real symmetric m-circulant ensemble is given by the formula

$$f_m(x) = \frac{e^{-\frac{mx^2}{2}}}{\sqrt{2\pi m}} \sum_{r=0}^{m} \frac{1}{(2r)!} \sum_{s=0}^{m-r} \binom{m}{r+s+1} \frac{(2r+2s)!}{(r+s)!s!} \left(-\frac{1}{2}\right)^s (mx^2)^r.$$

As $m \to \infty$, the limiting spectral densities approach the semicircle distribution.

Proof of $m \to \infty$ Convergence

The characteristic function for the spectral measures of the period m-circulant matrices can be written in terms of the Laguerre polynomial

$$\phi_m(t) \;=\; \frac{1}{m} e^{-\frac{t^2}{2m}} L_{m-1}^{(1)}(t^2/m),$$

or equivalently in terms of the confluent hypergeometric function

$$\phi_m(t) \;=\; \exp(-t^2/2m) M(m+1, 2, -t^2/m).$$

From 13.2.2 of [AS] we have $\lim_{m\to\infty} \phi_m(t) = \phi(t) = J_1(2t)/t$; however, we need some control on the rate of convergence.

Proof of $m \to \infty$ Convergence (cont)

Let $r > 1/3$ and $\beta = \frac{2}{3}(1-r)$. For all m and all t we have

$$|\phi_m(t) - \phi(t)| \ll_r \begin{cases} m^{-(1-r)} & \text{if } |t| \le m^{\beta} \\ t^{-\frac{3}{2}} + m^{-\frac{5}{4}} \exp(-\frac{t^2}{2m}) & \text{otherwise,} \end{cases}$$

$$|f_m(x) - f_{\mathrm{S.C.}}(x)| \le \int_{-\infty}^{\infty} |\phi_m(t) - \phi(t)|\, dt \ll m^{-\frac{1-r}{3}}.$$

$\epsilon > 0$ and $r = \frac{1}{3} + 3\epsilon$ bound integral with $O(m^{-\frac{2}{9}+\epsilon})$.
Key idea in proof: 13.3.7 of [AS]

$$\phi_m(t) = e^{-\frac{t^2}{2m}} M(m+1, 2, -\frac{t^2}{m}) = \frac{J_1(2t)}{t} - \frac{1}{2m} \sum_{n=1}^{\infty} A_n \left(\frac{-t}{2m}\right)^{n-1} J_{n+1}(2t),$$

where $A_0 = 1$, $A_1 = 0$, $A_2 = 1$ and $A_{n+1} = A_{n-1} + \frac{2m}{n+1} A_{n-2}$ for $n \ge 2$.
For any $r > \frac{1}{3}$ have $A_n \ll_r m^{rn}$.

Results (continued)

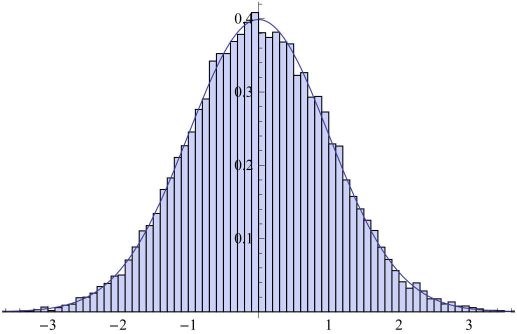


Figure: Plot for f_1 and histogram of eigenvalues of 100 circulant matrices of size 400×400.

Results (continued)

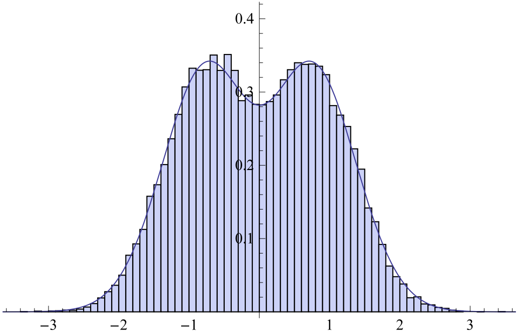


Figure: Plot for f_2 and histogram of eigenvalues of 100 2-circulant matrices of size 400×400.

Results (continued)

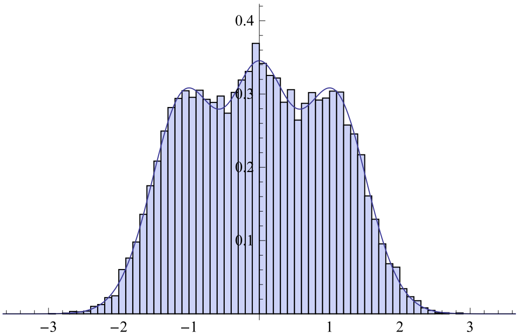


Figure: Plot for f_3 and histogram of eigenvalues of 100 3-circulant matrices of size 402×402.

Results (continued)

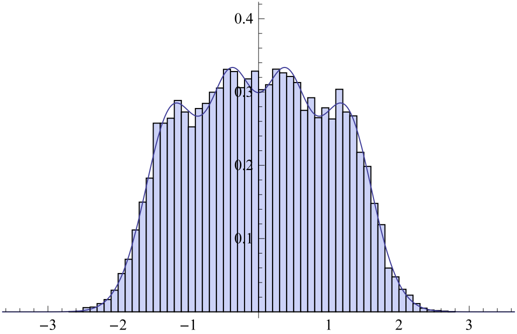


Figure: Plot for f_4 and histogram of eigenvalues of 100 4-circulant matrices of size 400×400.

Results (continued)

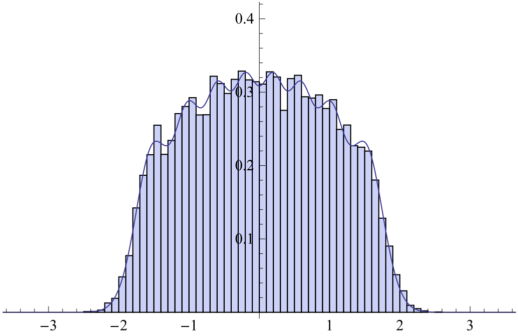


Figure: Plot for f_8 and histogram of eigenvalues of 100 8-circulant matrices of size 400×400.

Results (continued)

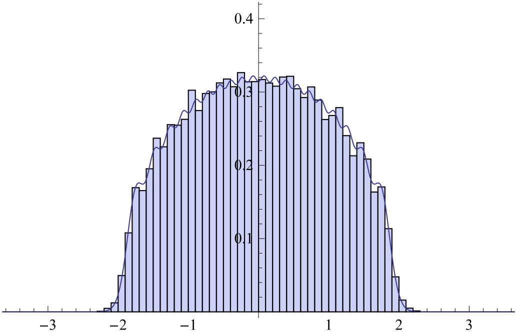


Figure: Plot for f_{20} and histogram of eigenvalues of 100 20-circulant matrices of size 400×400.

Results (continued)

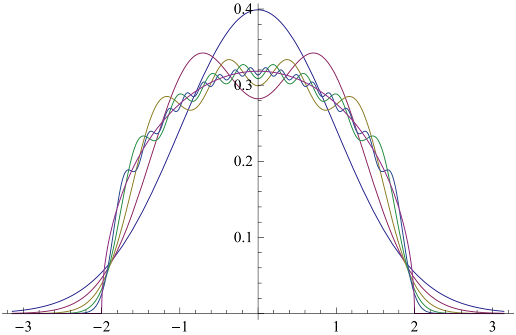


Figure: Plot of convergence to the semi-circle.

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