Combinatorics in Analyzing L-Function Coefficients and Applications to Low-Lying Zeros

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Special Session in Number Theory in Celebration of the 70th Birthday of Ram Murty
CMS Summer Meeting, 4 June 2023

Lessons Learned

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Fortunate to have learned many lessons from Ram (and many in this room!) over the years.

- Find the right combinatorial object to study.
- Carefully read anything you are using.

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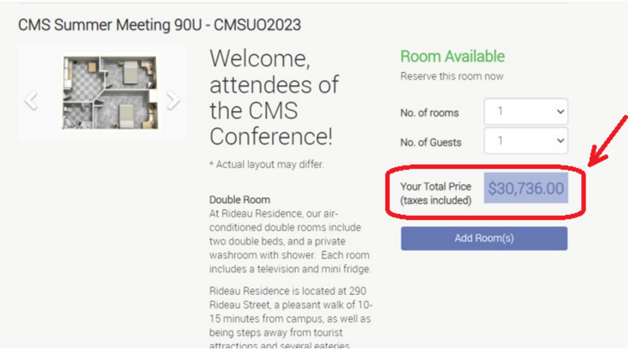

Effective Equidistribution

Effective equidistribution and the Sato-Tate law for families of elliptic curves (with M. Ram Murty), Journal of Number Theory **131** (2011), no. 1, 25–44.

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- Studied one-parameter families of elliptic curves: $a_{E_t}(p) = 2\sqrt{p}\cos\theta_{E_t}$.
- $D_{I,p}$: Difference between percent of t with $\theta_{E_t;p} \in I$ and $\mu_{\rm Sato-Tate}(I)$.

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Theorem (Miller-Murty)

There exists a computable C s.t. $D_{I,p} \leq Cp^{-1/4}$.

Proof Sketch

Suffices to study sums over family of $\cos(2m\theta_{E_t;p})$:

$$2\cos(2m\theta) \;=\; \sum_{r=0}^{m} c_{2m,2r}(2\cos\theta)^{2r},$$

where $c_{2r} = (2r)!/2$, $c_{0,0} = 0$, $c_{2m,0} = (-1)^m 2$ for $m \geq 1$, and for $1 \leq r \leq m$ set

$$c_{2m,2r} \;=\; \frac{(-1)^{r+m}}{c_{2r}} \prod_{j=0}^{r-1}(m^2 - j^2) \;=\; \frac{(-1)^{m+r}}{c_{2r}} \frac{m \cdot (m+r-1)!}{(m-r)!}.$$

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Reduced to combinatorial lemma: Let m be an integer greater than or equal to 1. Then

$$\sum_{r=0}^{m}(-1)^r\binom{m}{r}\binom{m+r}{r}\frac{1}{(r+1)(m+r)} \;=\; \begin{cases}1/2 & \text{if } m=1\\ 0 & \text{if } m\geq 2.\end{cases}$$

Random Matrix Theory and L-Functions

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Heavy nuclei (Uranium: 200$+$ protons / neutrons) worse!

Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation:

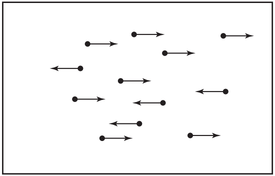
$$H\psi_n \;=\; E_n\psi_n$$

H : matrix, entries depend on system

E_n : energy levels

ψ_n : energy eigenfunctions

Origins of Random Matrix Theory



- Statistical Mechanics: for each configuration, calculate quantity (say pressure).
- Average over all configurations – most configurations close to system average.
- Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices (real Symmetric $A = A^T$, complex Hermitian $\overline{A}^T = A$).

Riemann Zeta Function

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s} \;=\; \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) \;=\; \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) \;=\; \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\text{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2}+i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

General L-functions

$$L(s,f) \;=\; \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} \;=\; \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \text{Re}(s) > 1.$$

Functional Equation:

$$\Lambda(s,f) \;=\; \Lambda_\infty(s,f) L(s,f) \;=\; \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (RH):

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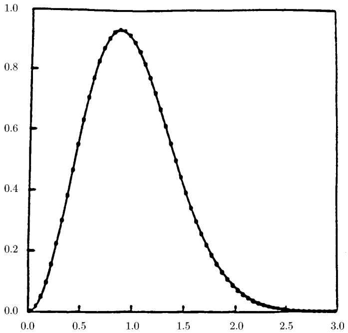
Zeros of $\zeta(s)$ vs GUE



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko).

Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} \quad = \quad -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) \;=\; -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

Explicit Formula (Contour Integration)

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\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{d}{ds}\log\zeta(s) = -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1} \\
&= \frac{d}{ds}\sum_p \log\left(1-p^{-s}\right) \\
&= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \text{Good}(s).
\end{aligned}
$$

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Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds \quad \text{vs} \quad \sum_p\log p\int\left(\frac{x}{p}\right)^s\frac{ds}{s}.$$

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Contour Integration (see Fourier Transform arising):

$$
\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s) e^{-\sigma \log p} e^{-it\log p} ds.
$$

Knowledge of zeros gives info on coefficients.

Explicit Formula: Example

Cuspidal Newforms: Let $\mathcal{F}$ be a family of cupsidal newforms (say weight k, prime level N and possibly split by sign) $L(s,f) = \sum_n \lambda_f(n)/n^s$. Then

$$\begin{aligned} \frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{\gamma_f}\phi\left(\frac{\log R}{2\pi}\gamma_f\right) &= \widehat{\phi}(0)+\frac{1}{2}\phi(0)-\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}P(f;\phi) \\ &\quad + O\left(\frac{\log\log R}{\log R}\right) \\ P(f;\phi) &= \sum_{p\nmid N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}. \end{aligned}$$

Measures of Spacings: n-Level Correlations

1. Normalized spacings of $\zeta(s)$ starting at 10^{20} (Odlyzko).
2. 2 and 3-correlations of $\zeta(s)$ (Montgomery, Hejhal).
3. n-level correlations for all automorphic cupsidal L-functions (Rudnick-Sarnak).
4. n-level correlations for the classical compact groups (Katz-Sarnak).
5. Insensitive to any finite set of zeros.

Measures of Spacings: n-Level Density and Families

$\phi(x) := \prod_i \phi_i(x_i)$, ϕ_i even Schwartz functions whose Fourier Transforms are compactly supported.

n-level density

$$D_{n,f}(\phi) \quad = \quad \sum_{\substack{j_1,\dots,j_n \\ \text{distinct}}} \phi_1\left(L_f \gamma_f^{(j_1)}\right) \cdots \phi_n\left(L_f \gamma_f^{(j_n)}\right)$$

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1. Individual zeros contribute in limit.
2. Most of contribution is from low zeros.
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Katz-Sarnak Conjecture

For a 'nice' family of L-functions, the n-level density depends only on a symmetry group attached to the family.

Normalization of Zeros

Local (hard, use C_f) vs Global (easier, use $\log C = |\mathcal{F}_N|^{-1}\sum_{f\in\mathcal{F}_N}\log C_f$). Hope: ϕ a good even test function with compact support, as $|\mathcal{F}|\to\infty$,

$$\frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N} D_{n,f}(\phi) = \frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N}\sum_{\substack{j_1,\dots,j_n\\ j_i\neq\pm j_k}}\prod_i \phi_i\left(\frac{\log C_f}{2\pi}\gamma_E^{(j_i)}\right)$$

$$\to \int\cdots\int \phi(x) W_{n,\mathcal{G}(\mathcal{F})}(x)dx.$$

Katz-Sarnak Conjecture

As $C_f\to\infty$ the behavior of zeros near $1/2$ agrees with $N\to\infty$ limit of eigenvalues of a classical compact group.

Correspondences

Similarities between *L*-Functions and Nuclei:

Zeros	$\longleftrightarrow$	Energy Levels
Schwartz test function	$\longrightarrow$	Neutron
Support of test function	$\longleftrightarrow$	Neutron Energy.

Main Tools

1. Control of conductors: Usually monotone, gives scale to study low-lying zeros.
2. Explicit Formula: Relates sums over zeros to sums over primes.
3. Averaging Formulas: Orthogonality of characters for Dirichlet L-functions, Petersson formula for cusp forms.

Right Object to Study: 2-Level Density Example

$$\int_{x_1=2}^{R^\sigma}\int_{x_2=2}^{R^\sigma}\widehat{\phi}\left(\frac{\log x_1}{\log R}\right)\widehat{\phi}\left(\frac{\log x_2}{\log R}\right)J_{k-1}\left(4\pi\frac{\sqrt{m^2x_1x_2N}}{c}\right)\frac{dx_1dx_2}{\sqrt{x_1x_2}}$$

Change of variables and Jacobian:

$$\begin{array}{rclcrcl} u_2 & = & x_1x_2 & \qquad & x_2 & = & \frac{u_2}{u_1} \\ u_1 & = & x_1 & & x_1 & = & u_1 \end{array}$$

$$\left|\frac{\partial x}{\partial u}\right| = \begin{vmatrix} 1 & 0 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1} \end{vmatrix} = \frac{1}{u_1}.$$

Left with

$$\int\int\widehat{\phi}\left(\frac{\log u_1}{\log R}\right)\widehat{\phi}\left(\frac{\log\left(\frac{u_2}{u_1}\right)}{\log R}\right)\frac{1}{\sqrt{u_2}}J_{k-1}\left(4\pi\frac{\sqrt{m^2u_2N}}{c}\right)\frac{du_1du_2}{u_1}$$

Right Object to Study: 2-Level Density Example

Changing variables, u_1-integral is

$$\int_{w_1=\frac{\log u_2}{\log R}-\sigma}^{\sigma} \widehat{\phi}(w_1)\,\widehat{\phi}\left(\frac{\log u_2}{\log R}-w_1\right) dw_1.$$

Support conditions imply

$$\Psi_2\left(\frac{\log u_2}{\log R}\right) = \int_{w_1=-\infty}^{\infty} \widehat{\phi}(w_1)\,\widehat{\phi}\left(\frac{\log u_2}{\log R}-w_1\right) dw_1.$$

Substituting gives

$$\int_{u_2=0}^{\infty} J_{k-1}\left(4\pi\frac{\sqrt{m^2 u_2 N}}{c}\right)\Psi_2\left(\frac{\log u_2}{\log R}\right)\frac{du_2}{\sqrt{u_2}}.$$

Applications of *n*-level density

Bounding the order of vanishing at the central point:
$\text{Average rank} \cdot \phi(0) \ \leq \ \int \phi(x) W_{G(\mathcal{F})}(x) dx$ if ϕ non-negative.

Applications of n-level density

Bounding the order of vanishing at the central point:
$\text{Average rank} \cdot \phi(0) \ \leq \ \int \phi(x) W_{G(\mathcal{F})}(x) dx$ if ϕ non-negative.
Can also use to bound the percentage that vanish to order r for any r.

Theorem (Miller, Hughes-Miller)

Using n-level arguments, for the family of cuspidal newforms of prime level $N \to \infty$ (split or not split by sign), for any r there is a c_r such that probability of at least r zeros at the central point is at most $c_n r^{-n}$.

Better results using 2-level than Iwaniec-Luo-Sarnak using the 1-level for $r \geq 5$.

Bounding Vanishing

With Elżbieta Bołdyriew, Fangu Chen, Charles Devlin VI, Justine Dell, Sohom Dutta, Simran Khunger, Stella Li, Alexander E. Shashkov, Alicia G. Smith Reina, Stephen D. Willis, Yingzi Yang, Jason Zhao.

Results from Iwaniec-Luo-Sarnak

- Orthogonal: Iwaniec-Luo-Sarnak: 1-level density for holomorphic even weight k cuspidal newforms of square-free level N (SO(even) and SO(odd) if split by sign) in $(-2, 2)$.

- Symplectic: Iwaniec-Luo-Sarnak: 1-level density for $\text{sym}^2(f)$, f holomorphic cuspidal newform.

Extended with Chris Hughes to n-level with support up to $1/(n-1)$, then up to $2/n$ with SMALL REUs.

Approaches for Bounding Vanishing

- Optimizing the test function.
- Increasing the support for the 1-level.
- Using n-level densities.

Previous Work: Optimizing the Test Function

Optimizing:

- *Determining Optimal Test Functions for Bounding the Average Rank in Families of L-Functions* (Jesse Freeman, Steven J. Miller), in SCHOLAR – a Scientific Celebration Highlighting Open Lines of Arithmetic Research, Conference in Honour of M. Ram Murty's Mathematical Legacy on his 60th Birthday (A. C. Cojocaru, C. David and F. Pappaardi, editors), Contemporary Mathematics **655**, AMS and CRM, 2015. https://arxiv.org/pdf/1507.03598.pdf

- *Determining optimal test functions for* 2*-level densities* (Elżbieta Bołdyriew, Fangu Chen, Charles Devlin VI, Steven J. Miller, Jason Zhao), Research in Number Theory **9** (2023), no. 32. https://arxiv.org/pdf/2011.10140.pdf

Previous Work: Increasing Support in n-Level Densities

Increasing support:

- *Extending support for the centered moments of the low lying zeroes of cuspidal newforms* (Peter Cohen, Justine Dell, Oscar E. Gonzalez, Geoffrey Iyer, Simran Khunger, Chung-Hang Kwan, Steven J. Miller, Alexander Shashkov, Alicia Smith Reina, Carsten Sprunger, Nicholas Triantafillou, Nhi Truong, Roger Van Peski, Stephen Willis, and Yingzi Yang), preprint: https://arxiv.org/pdf/2208.02625v1.pdf

- *Extending the support of* 1*- and* 2*-level densities for cusp form L-functions under square-root cancellation hypotheses* (Annika Mauro, Jack Miller, and Steven J. Miller), preprint: https://arxiv.org/pdf/2305.15293.pdf

New Work

$n \leq 4$: *Bounding Vanishing at the Central Point of Cuspidal Newforms* (Jiahui (Stella) Li, Steven J. Miller), Journal of Number Theory (Computational Section), **244** (2023), 279–307.
https://arxiv.org/pdf/2203.03061.pdf

$n \geq 2$: *Bounding Excess Rank of Cuspidal Newforms via Centered Moments* (Sohom Dutta, Steven J. Miller), preprint.
https://arxiv.org/pdf/2211.04945.pdf

Test Functions

Naive Test Function

The naive test function is the Fourier test function pair

$$\phi_{\text{naive}}(x) = \left(\frac{\sin(\pi v_n x)}{(\pi v_n x)}\right)^2 \quad , \quad \widehat{\phi}_{\text{naive}}(y) = \frac{1}{v_n}\left(y - \frac{|y|}{v_n}\right)$$

for $|y| < v_n$ where v_n is the support.

Test Functions

Naive close but not optimal; optimal for 1-level is

$$\widehat{\phi}_{\text{optimal}}(y) := (f_0 * \overline{f_0})(y)$$

for

$$f_0(x) := \frac{\cos\left(\frac{|x|}{2} - \frac{\pi+1}{4}\right)}{\sqrt{2}\sin\left(\frac{1}{4}\right) + \sin\left(\frac{\pi+1}{4}\right)}, \quad 0 \le |x| \le 1$$

for $G = \mathrm{SO(even)}$ and

$$f_0(x) := \frac{\cos\left(\frac{|x|}{2} + \frac{\pi-1}{4}\right)}{3\sin\left(\frac{\pi+1}{4}\right) - 2\sin\left(\frac{\pi-1}{4}\right)}, \quad 0 < |x| < 1$$

for $G = \mathrm{SO(odd)}$.

1-Level Bound

$$D_1(\mathcal{F}_N, \phi) \;:=\; \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_{\gamma_f} \phi\left(\frac{\gamma_f}{2\pi} \log c_f\right).$$

1-Level Bound

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From [ILS]: Let ϕ be a non-negative, even Schwartz function with $\operatorname{supp}(\widehat{\phi}) \subset (-\sigma, \sigma)$ for some finite σ. Let $\mathcal{G}$ be the group associated to the family $\mathcal{F}_N$ (i.e., Unitary, Symplectic, Orthogonal, $\mathrm{SO(even)}$, $\mathrm{SO(odd)}$). *Set*

$$g_{\mathcal{F}}(\phi) \ := \ \int_{-\infty}^{\infty} \widehat{\phi}(y) \widehat{W}_{\mathcal{G}(\mathcal{F})}(y) dy.$$

As $N \to \infty$ the percent of forms in the family $\mathcal{F}_N$ that vanish to order exactly (or at least) r is bounded by

$$p_r \ \leq \ \frac{1}{r}(g_{\mathcal{F}}(\phi)).$$

Naive vs Optimal for 1-Level

Comparing Bounds from 1-Level for $G = \mathrm{SO(even)}$		
Rank	Naive test fn	Optimal test fn
2	0.43750000	0.43231300
4	0.21875000	0.21615700
6	0.14583333	0.14410400
8	0.10937500	0.10807800
10	0.08750000	0.08646260
12	0.07291670	0.07205220
14	0.06250000	0.06175900
16	0.05468750	0.05403910
18	0.04861110	0.04803848
20	0.04375000	0.04323130

4-Level Bounds (with J. Li, generalizing Hughes-Miller combinatorics)

Order vanish	1-level	2-level	4^{th} centered moment*
6	0.144090	0.01576870	0.00853841
8	0.108067	0.00788434	0.00081336
10	0.086454	0.00473060	0.00018684
20	0.043227	0.00105125	$4.49988 \cdot 10^{-6}$
50	0.017290	0.00015768	$7.13387 \cdot 10^{-8}$

TABLE 2. Comparison of order of vanishing bounds from various approaches. These are upper bounds for vanishing at least r (number in order vanishing column). For the 1-level column, we calculated the bound using the optimal 1-level bound from [ILS]. The support of the Fourier transform of the test function used is $(-2, 2)$. For the 2-level column, we calculated the bound using the optimal 2-level bound from [BCDMZ]. The support of the Fourier transform of the test functions used is $(-1, 1)$.
For the 4^{th} centered moment* column, the * signifies that we used the 4 copies of the naive test functions φ_{naive}. The support of the Fourier transform of the test function used is $(-1/3, 1/3)$.

Bounds from n-Level: Preliminaries: From Cohen et. al.

Let $n \geq 2$ and $\mathrm{supp}(\phi) \subset (-\frac{2}{n}, \frac{2}{n})$. Define

$$\sigma_\phi^2 \quad := \quad 2\int_{-\infty}^{\infty} |y|\widehat{\phi}(y)^2 dy$$

$$R(m,i;\phi) \quad := \quad 2^{m-1}(-1)^{m+1}\sum_{l=0}^{i-1}(-1)^l\binom{m}{l}$$

$$\left(-\frac{1}{2}\phi^m(0) + \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\widehat{\phi}(x_2)\cdots\widehat{\phi}(x_{l+1})\right.$$

$$\left.\int_{-\infty}^{\infty}\phi^{m-l}(x_1)\frac{\sin(2\pi x_1(1+|x_2|+\cdots+|x_{l+1}|))}{2\pi x_1}dx_1\cdots dx_{l+1}\right)$$

$$S(n,a,\phi) \quad := \quad \sum_{l=0}^{\lfloor\frac{a-1}{2}\rfloor}\frac{n!}{(n-2l)!l!}R(n-2l,a-2l,\phi)\left(\frac{\sigma_\phi^2}{2}\right)^l \quad \text{then}$$

$$\lim_{\substack{N\to\infty \\ N \text{ prime}}}\left\langle\left(D(f;\phi)-\langle D(f;\phi)\rangle_\pm\right)^n\right\rangle_\pm \;=\; (n-1)!!\sigma_\phi^n 1_{n \text{ even}} \pm S(n,a;\phi).$$

Key Expansion

Using binomial theorem and Cauchy-Schwartz, can replace the mean from finite N with the limit:

$$\lim_{\substack{N\to\infty \\ N \text{ prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} \left(\sum_j \phi\left(\gamma_{f,j} c_n\right) - \mu(\phi,\mathcal{F}) \right)^n$$
$$= 1_{n \text{ even}}(n-1)!!\sigma_\phi^n \pm S(n,a;\phi),$$

and main term of the mean of the 1-level density of $\mathcal{F}_N$ is

$$\mu(\phi,\mathcal{F}) := \widehat{\phi}(0) + \frac{1}{2}\int_{-1}^{1} \widehat{\phi}(y)dy.$$

Bound from n-Level for r Sufficiently Large

Theorem (Dutta-Miller)

For an even n with $r \geq \mu(\phi, \mathcal{F})/\phi(0)$,

$$p_r(\mathcal{F}) \ \leq \ \frac{(n-1)!!\sigma_\phi^n \pm S(n, \frac{n}{2}; \phi)}{(r\phi(0) - \mu(\phi, \mathcal{F}))^n}.$$

Sketch of Proof: $r \geq \mu(\phi, \mathcal{F})/\phi(0)$

For even-level densities, contributions always positive, drop forms with fewer than r zeros:

$$\lim_{\substack{N\to\infty \\ N \text{ prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_{N,r}} \left(r\phi(0) + B_f(\phi) - \mu(\phi,\mathcal{F})\right)^n \;\leq\; \cdots .$$

Have $(r\phi(0) + B_f(\phi) - \mu(\phi,\mathcal{F}_N))^n$; dropping $B_f(\phi)$ can increase sum if first two terms are less than the third.

By assumption on r, sum with and without $B_f(\phi)$ is positive; dropping gives bound

$$\lim_{\substack{N\to\infty \\ N \text{ prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_{N,r}} \left(r\phi(0) - \mu(\phi,\mathcal{F})\right)^n \leq 1_{n \text{ even}}(n-1)!!\sigma_\phi^n \pm S(n,a;\phi)$$

Comparison of different *n* from Dutta-Miller: Naive Test Function

Using support up to $2/n$:

	1-level	2-level	4-th centered moment	6-th centered moment	8-th centered moment	10-th centered moment
Mean	1.5	2.	3.	4.	5.	6.
2	0.4375	Invalid Bound	Invalid Bound	Invalid Bound	Invalid Bound	Invalid Bound
4	0.21875	0.0666667	0.0733686	Invalid Bound	Invalid Bound	Invalid Bound
6	0.145833	0.0205761	0.00247516	0.00234651	0.0509282	Invalid Bound
8	0.109375	0.00986193	0.000405905	0.0000689901	0.0000579621	0.000408389
10	0.0875	0.00576701	0.00011739	7.59594×10^{-6}	1.55879×10^{-6}	1.1438×10^{-6}
12	0.0729167	0.00377929	0.0000456017	1.51897×10^{-6}	1.30376×10^{-7}	2.89295×10^{-8}
14	0.0625	0.00266667	0.0000212365	4.2749×10^{-7}	1.96743×10^{-8}	1.97827×10^{-9}
16	0.0546875	0.00198177	0.0000111825	1.50176×10^{-7}	4.26683×10^{-9}	2.39101×10^{-10}
18	0.0486111	0.00153046	6.435×10^{-6}	6.16387×10^{-8}	1.18309×10^{-9}	4.18191×10^{-11}
20	0.04375	0.00121743	3.96025×10^{-6}	2.83895×10^{-8}	3.91773×10^{-10}	9.47969×10^{-12}

FIGURE 1. Approximate Bounds for the Percent of Vanishing to exact order r for the case G=SO(even) with support $v = 2$ for the 1-level and $v = 2/n$ for the n-level with n going from 1 to 10 and r from 2 through 20 obtained using the naive test function.

Comparison of different n from Dutta-Miller: Optimal Test Function

Using support up to $2/n$:

	1-level	2-level	4-th centered moment	6-th centered moment	8-th centered moment	10-th centered moment
Mean	1.01055	1.5211	2.54219	3.56329	4.58439	5.60548
2	0.432313	1.90775	Invalid Bound	Invalid Bound	Invalid Bound	Invalid Bound
4	0.216157	0.0712029	0.0857764	84.2903	Invalid Bound	Invalid Bound
6	0.144104	0.0218109	0.00270998	0.0027933	0.0818933	43 045.8
8	0.108078	0.0104235	0.000436612	0.0000766611	0.0000712948	0.000634483
10	0.0864626	0.00608608	0.000125234	8.22163×10^{-6}	1.78495×10^{-6}	1.46382×10^{-6}
12	0.0720522	0.0039846	0.000048418	1.62145×10^{-6}	1.44421×10^{-7}	3.43972×10^{-8}
14	0.061759	0.00280973	0.0000224782	4.52439×10^{-7}	2.13806×10^{-8}	2.26279×10^{-9}
16	0.0540391	0.00208711	0.0000118106	1.58019×10^{-7}	4.57946×10^{-9}	2.67029×10^{-10}
18	0.0480385	0.00161124	6.78542×10^{-6}	6.45851×10^{-8}	1.25871×10^{-9}	4.59521×10^{-11}
20	0.0432313	0.00128134	4.17071×10^{-6}	2.96519×10^{-8}	4.14116×10^{-10}	1.02951×10^{-11}

FIGURE 5. Approximate Bounds for the Percent of Vanishing to exact order r for the case G=SO(even) with support $v = 2$ for the 1-level and $v = 2/n$ for the n-level with n going from 1 to 10 and r from 2 through 20 obtained using the optimal test function.

Best Results

Lowest Bounds for Each Rank for G=SO(even)		
Rank	Level Used	Bound
2	1	0.43231300
4	2	0.066666667
6	6	0.003346510
8	8	0.000579210
10	10	1.14380×10^{-6}
12	12	1.85901×10^{-8}
14	14	2.59310×10^{-10}
16	16	3.09185×10^{-12}
18	18	3.26332×10^{-14}
20	20	3.08920×10^{-16}

Best Results

Lowest Bounds for Each Rank for G=SO(odd)		
Rank	Level Used	Bound
1	N/A	1.0000000
3	2	0.111111111
5	2	0.020408300
7	6	0.000292790
9	8	7.65596×10^{-6}
11	10	1.53302×10^{-7}
13	12	2.50956×10^{-9}
15	16	3.03362×10^{-11}
17	18	3.10549×10^{-13}
19	20	4.18402×10^{-17}

Refs

References

- E. Bołdyriew, F. Chen, C. Devlin VI, S. J. Miller, J. Zhao, *Determining optimal test functions for 2-level densities*, Research in Number Theory **9** (2023), no. 32. https://arxiv.org/pdf/2011.10140.pdf
- P. Cohen, J. Dell, O. E. Gonzalez, G. Iyer, S. Khunger, C.-H. Kwan, S. J. Miller, A. Shashkov, A. Smith Reina, C. Sprunger, N. Triantafillou, N. Truong, R. Van Peski, S. Willis and Y. Yang, *Extending support for the centered moments of the low lying zeroes of cuspidal newforms*, preprint. https://arxiv.org/pdf/2208.02625v1.pdf
- S. Dutta and S. J. Miller, *Bounding Excess Rank of Cuspidal Newforms via Centered Moments*, preprint. https://arxiv.org/pdf/2211.04945.pdf
- J. Freeman and S. J. Miller, *Determining Optimal Test Functions for Bounding the Average Rank in Families of L-Functions*, in SCHOLAR – a Scientific Celebration Highlighting Open Lines of Arithmetic Research, Conference in Honour of M. Ram Murty's Mathematical Legacy on his 60th Birthday (A. C. Cojocaru, C. David and F. Pappaardi, editors), Contemporary Mathematics **655**, AMS and CRM, 2015. https://arxiv.org/abs/1507.03598
- C. Hughes and S. J. Miller, *Low lying zeros of L–functions with orthogonal symmetry*, Duke Mathematical Journal **136** (2007), no. 1, 115–172. https://arxiv.org/abs/math/0507450v1
- H. Iwaniec, W. Luo, and P. Sarnak, *Low lying zeros of families of L-functions*, Inst. Hautes Études Sci. Publ. Math. **91** (2000), 55–131. https://arxiv.org/abs/math/9901141
- J. Li and S. J. Miller, *Bounding Vanishing at the Central Point of Cuspidal Newforms*, to appear in the Journal of Number Theory. https://arxiv.org/pdf/2203.03061.pdf
- A. Mauro, J. Miller, and S. J. Miller, *Extending the support of 1- and 2-level densities for cusp form L-functions under square-root cancellation hypotheses*, preprint: https://arxiv.org/pdf/2305.15293.pdf.

Cuspidal Newforms

Results from Iwaniec-Luo-Sarnak

- Orthogonal: Iwaniec-Luo-Sarnak: 1-level density for holomorphic even weight k cuspidal newforms of square-free level N (SO(even) and SO(odd) if split by sign) in $(-2, 2)$.

- Symplectic: Iwaniec-Luo-Sarnak: 1-level density for $\text{sym}^2(f)$, f holomorphic cuspidal newform.

Will review Orthogonal case and talk about extensions (joint with Chris Hughes, then much further with SMALL REUs).

Modular Form Preliminaries

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{c} ad - bc = 1 \\ c \equiv 0(N) \end{array} \right\}$$

f is a weight k holomorphic cuspform of level N if

$$\forall \gamma \in \Gamma_0(N), \quad f(\gamma z) = (cz + d)^k f(z).$$

- Fourier Expansion: $f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i z}$, $L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}$.
- Petersson Norm: $\langle f, g \rangle = \int_{\Gamma_0(N) \backslash \mathbf{H}} f(z) \overline{g(z)} y^{k-2} dx dy$.
- Normalized coefficients:

$$\psi_f(n) = \sqrt{\frac{\Gamma(k-1)}{(4\pi n)^{k-1}} \frac{1}{||f||}} a_f(n).$$

Modular Form Preliminaries: Petersson Formula

$B_k(N)$ an orthonormal basis for weight k level N. Define

$$\Delta_{k,N}(m,n) \ = \ \sum_{f \in B_k(N)} \psi_f(m)\overline{\psi}_f(n).$$

Petersson Formula

$$\Delta_{k,N}(m,n) \ = \ 2\pi i^k \sum_{c \equiv 0(N)} \frac{S(m,n,c)}{c} J_{k-1}\left(4\pi \frac{\sqrt{mn}}{c}\right) + \ \delta(m,n).$$

Modular Form Preliminaries: Explicit Formula

Let $\mathcal{F}$ be a family of cupsidal newforms (say weight k, prime level N and possibly split by sign) $L(s,f) = \sum_n \lambda_f(n)/n^s$. Then

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{\gamma_f}\phi\left(\frac{\log R}{2\pi}\gamma_f\right) = \widehat{\phi}(0) + \frac{1}{2}\phi(0) - \frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}} P(f;\phi) + O\left(\frac{\log\log R}{\log R}\right)$$

$$P(f;\phi) = \sum_{p\nmid N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}.$$

Modular Form Preliminaries:Fourier Coefficient Review

$$\begin{aligned}\lambda_f(n) &= a_f(n)n^{\frac{k-1}{2}} \\ \lambda_f(m)\lambda_f(n) &= \sum_{\substack{d|(m,n)\\(d,M)=1}} \lambda_f\left(\frac{mn}{d}\right).\end{aligned}$$

For a newform of level N, $\lambda_f(N)$ is trivially related to the sign of the form:

$$\epsilon_f = i^k\mu(N)\lambda_f(N)\sqrt{N}.$$

The above will allow us to split into even and odd families: $1 \pm \epsilon_f$.

Key Kloosterman-Bessel integral from ILS

Ramanujan sum:

$$R(n,q) \;=\; \sum_{a \bmod q}^{*} e(an/q) \;=\; \sum_{d|(n,q)} \mu(q/d)d,$$

where $*$ restricts the summation to be over all a relatively prime to q.

Theorem (ILS)

Let Ψ be an even Schwartz function with $\mathrm{supp}(\widehat{\Psi}) \subset (-2,2)$. Then

$$\sum_{m\le N^\epsilon} \frac{1}{m^2} \sum_{(b,N)=1} \frac{R(m^2,b)R(1,b)}{\varphi(b)} \int_{y=0}^{\infty} J_{k-1}(y)\widehat{\Psi}\left(\frac{2\log(by\sqrt{N}/4\pi m)}{\log R}\right)\frac{dy}{\log R}$$
$$= -\frac{1}{2}\left[\int_{-\infty}^{\infty}\Psi(x)\frac{\sin 2\pi x}{2\pi x}dx - \frac{1}{2}\Psi(0)\right] + O\left(\frac{k\log\log kN}{\log kN}\right),$$

where $R = k^2N$ and φ is Euler's totient function.

Limited Support ($\sigma < 1$): Sketch of proof

- Estimate Kloosterman-Bessel terms trivially.
  $\diamond$ Kloosterman sum: $d\bar{d} \equiv 1 \bmod q$, $\tau(q)$ is the number of divisors of q,

$$S(m, n; q) \;=\; \sum_{d \bmod q}^{*} e\left(\frac{md}{q} + \frac{n\bar{d}}{q}\right)$$

$$|S(m, n; q)| \;\leq\; (m, n, q)\sqrt{\min\left\{\frac{q}{(m, q)}, \frac{q}{(n, q)}\right\}}\;\tau(q).$$

  $\diamond$ Bessel function: integer $k \geq 2$,
  $J_{k-1}(x) \ll \min\left(x, x^{k-1}, x^{-1/2}\right)$.

- Use Fourier Coefficients to split by sign: N fixed:
  $\pm \sum_f \lambda_f(N) * (\cdots)$.

Increasing Support ($\sigma < 2$): Sketch of the proof

- Using Dirichlet Characters, handle Kloosterman terms.

- Have terms like

$$\int_0^\infty J_{k-1}\left(4\pi\frac{\sqrt{m^2yN}}{c}\right)\widehat{\phi}\left(\frac{\log y}{\log R}\right)\frac{dy}{\sqrt{y}}$$

  with arithmetic factors to sum outside.

- Works for support up to $(-2, 2)$.

Increasing Support ($\sigma < 2$): Kloosterman-Bessel details

Stating in greater generality for later use.

Gauss sum: χ a character modulo q: $|G_\chi(n)| \le \sqrt{q}$ with

$$G_\chi(n) \;=\; \sum_{a \bmod q} \chi(a) \exp(2\pi i a n/q).$$

Increasing Support ($\sigma < 2$): Kloosterman-Bessel details

Kloosterman expansion:

$$S(m^2, p_1 \cdots p_n N; Nb) = \frac{-1}{\varphi(b)} \sum_{\chi (\mathrm{mod}\ b)} \chi(N) G_\chi(m^2) G_\chi(1) \overline{\chi}(p_1 \cdots p_n).$$

Lemma: Assuming GRH for Dirichlet L-functions, $\mathrm{supp}(\widehat{\phi}) \subset \left(-\frac{2}{n}, \frac{2}{n}\right)$, non-principal characters negligible.
Proof: use $J_{k-1}(x) \ll x$ and see

$$\ll \frac{1}{\sqrt{N}} \sum_{m \le N^\epsilon} \frac{1}{m} \sum_{\substack{(b,N)=1 \\ b < N^{2006}}} \frac{1}{b} \frac{1}{\varphi(b)} \sum_{\substack{\chi (\mathrm{mod}\ b) \\ \chi \neq \chi_0}} \left| G_\chi(m^2) G_\chi(1) \right| \times \frac{m}{b\sqrt{N}} \prod_{j=1}^{n} \left| \sum_{p_j \neq N} \overline{\chi}(p_j) \log p_j \cdot \frac{1}{\log R} \widehat{\phi}\left(\frac{\log p_j}{\log R}\right) \right|.$$

2-Level Density

$$\int_{x_1=2}^{R^\sigma}\int_{x_2=2}^{R^\sigma} \widehat{\phi}\left(\frac{\log x_1}{\log R}\right)\widehat{\phi}\left(\frac{\log x_2}{\log R}\right) J_{k-1}\left(4\pi\frac{\sqrt{m^2 x_1 x_2 N}}{c}\right)\frac{dx_1\,dx_2}{\sqrt{x_1 x_2}}$$

Change of variables and Jacobian:

$$\begin{array}{rclcrcl} u_2 & = & x_1 x_2 & \qquad & x_2 & = & \frac{u_2}{u_1} \\ u_1 & = & x_1 & & x_1 & = & u_1 \end{array}$$

$$\left|\frac{\partial x}{\partial u}\right| = \left|\begin{array}{cc} 1 & 0 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1}\end{array}\right| = \frac{1}{u_1}.$$

Left with

$$\int\int \widehat{\phi}\left(\frac{\log u_1}{\log R}\right)\widehat{\phi}\left(\frac{\log\left(\frac{u_2}{u_1}\right)}{\log R}\right)\frac{1}{\sqrt{u_2}} J_{k-1}\left(4\pi\frac{\sqrt{m^2 u_2 N}}{c}\right)\frac{du_1\,du_2}{u_1}$$

2-Level Density

Changing variables, u_1-integral is

$$\int_{w_1=\frac{\log u_2}{\log R}-\sigma}^{\sigma} \widehat{\phi}(w_1)\,\widehat{\phi}\left(\frac{\log u_2}{\log R}-w_1\right) dw_1.$$

Support conditions imply

$$\Psi_2\left(\frac{\log u_2}{\log R}\right) \;=\; \int_{w_1=-\infty}^{\infty} \widehat{\phi}(w_1)\,\widehat{\phi}\left(\frac{\log u_2}{\log R}-w_1\right) dw_1.$$

Substituting gives

$$\int_{u_2=0}^{\infty} J_{k-1}\left(4\pi\frac{\sqrt{m^2u_2N}}{c}\right)\Psi_2\left(\frac{\log u_2}{\log R}\right)\frac{du_2}{\sqrt{u_2}}.$$

3-Level Density

$$\int_{x_1=2}^{R^\sigma}\int_{x_2=2}^{R^\sigma}\int_{x_3=2}^{R^\sigma}\widehat{\phi}\left(\frac{\log x_1}{\log R}\right)\widehat{\phi}\left(\frac{\log x_2}{\log R}\right)\widehat{\phi}\left(\frac{\log x_3}{\log R}\right)$$

$$* \quad J_{k-1}\left(4\pi\frac{\sqrt{m^2x_1x_2x_3N}}{c}\right)\frac{dx_1dx_2dx_3}{\sqrt{x_1x_2x_3}}$$

Change variables as below and get Jacobian:

$$\begin{aligned} u_3 &= x_1x_2x_3 & \qquad x_3 &= \frac{u_3}{u_2} \\ u_2 &= x_1x_2 & x_2 &= \frac{u_2}{u_1} \\ u_1 &= x_1 & x_1 &= u_1 \end{aligned}$$

$$\left|\frac{\partial x}{\partial u}\right| = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1} & 0 \\ 0 & -\frac{u_3}{u_2^2} & \frac{1}{u_2} \end{vmatrix} = \frac{1}{u_1u_2}.$$

n-Level Density: Sketch of proof

Expand Bessel-Kloosterman piece, use GRH to drop non-principal characters, change variables, main term is

$$\frac{b\sqrt{N}}{2\pi m}\int_0^\infty J_{k-1}(x)\widehat{\Phi_n}\left(\frac{2\log(bx\sqrt{N}/4\pi m)}{\log R}\right)\frac{\mathrm{d}x}{\log R}$$

with $\Phi_n(x) = \phi(x)^n$.

Main Idea

Difficulty in comparison with classical RMT is that instead of having an n-dimensional integral of $\phi_1(x_1)\cdots\phi_n(x_n)$ we have a 1-dimensional integral of a new test function. This leads to harder combinatorics but allows us to appeal to the result from ILS.