Cookie Monster Meets the Fibonacci Numbers. Mmmmmm – Theorems!

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Difference Equations and Applications Session
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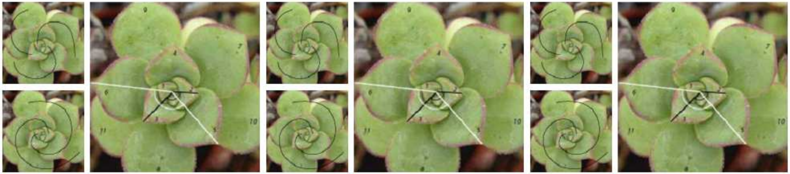
Introduction

Goals of the Talk

- Difference equations: special linear recursions.
- Chat about 'fun' properties of Fibonacci numbers.
- Answer questions on number of summands and gaps.
- Methods: Combinatorial vantage, generating functions.
- Some open problems.



Thanks to colleagues from the Williams College 2010 and 2011 SMALL REU programs, and Louis Gaudet.

Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
$F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \dots$.

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$$2012 = 1597 + 377 + 34 + 3 + 1 = F_{16} + F_{13} + F_8 + F_3 + F_1.$$

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Lekkerkerker's Theorem (1952)

The average number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ tends to $\frac{n}{\varphi^2+1} \approx .276n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden mean.

Previous Results

Central Limit Type Theorem

As $n \to \infty$, the distribution of the number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ is Gaussian (normal).

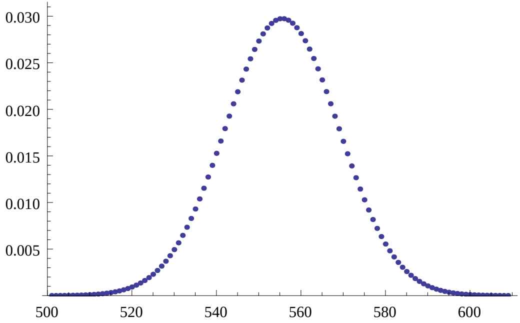

Figure: Number of summands in $[F_{2010}, F_{2011})$; $F_{2010} \approx 10^{420}$.

New Results

Theorem (Zeckendorf Gap Distribution (BM))

For Zeckendorf decompositions, $P(k) = \frac{\phi(\phi-1)}{\phi^k}$ for $k \geq 2$, with $\phi = \frac{1+\sqrt{5}}{2}$ the golden mean.

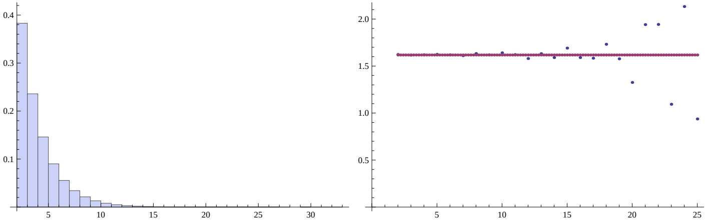

Figure: Distribution of gaps in $[F_{1000}, F_{1001})$; $F_{2010} \approx 10^{208}$.

Preliminaries: The Cookie Problem

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Preliminaries: The Cookie Problem: Reinterpretation

Reinterpreting the Cookie Problem

The number of solutions to $x_1 + \cdots + x_P = C$ with $x_i \geq 0$ is $\binom{C+P-1}{P-1}$.

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$$N = F_{i_1} + F_{i_2} + \cdots + F_{i_{k-1}} + F_n,$$
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$$d_1 := i_1 - 1,\ d_j := i_j - i_{j-1} - 2\ (j > 1).$$

$$d_1 + d_2 + \cdots + d_k = n - 2k + 1,\ d_j \geq 0.$$

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Cookie counting $\Rightarrow p_{n,k} = \binom{n-2k+1-k-1}{k-1} = \binom{n-k}{k-1}$.

An Erdos-Kac Type Theorem
(note slightly different notation)

Generalizing Lekkerkerker: Erdos-Kac type result

Theorem (KKMW 2010)

As $n \to \infty$, the distribution of the number of summands in Zeckendorf's Theorem is a Gaussian.

Sketch of proof: Use Stirling's formula,

$$n! \approx n^n e^{-n} \sqrt{2\pi n}$$

to approximates binomial coefficients, after a few pages of algebra find the probabilities are approximately Gaussian.

(Sketch of the) Proof of Gaussianity

The probability density for the number of Fibonacci numbers that add up to an integer in $[F_n, F_{n+1})$ is $f_n(k) = \binom{n-1-k}{k}/F_{n-1}$. Consider the density for the $n+1$ case. Then we have, by Stirling

$$\begin{aligned} f_{n+1}(k) &= \binom{n-k}{k}\frac{1}{F_n} \\ &= \frac{(n-k)!}{(n-2k)!k!}\frac{1}{F_n} = \frac{1}{\sqrt{2\pi}}\frac{(n-k)^{n-k+\frac{1}{2}}}{k^{(k+\frac{1}{2})}(n-2k)^{n-2k+\frac{1}{2}}}\frac{1}{F_n} \end{aligned}$$

plus a lower order correction term.

Also we can write $F_n = \frac{1}{\sqrt{5}}\phi^{n+1} = \frac{\phi}{\sqrt{5}}\phi^n$ for large n, where ϕ is the golden ratio (we are using relabeled Fibonacci numbers where $1 = F_1$ occurs once to help dealing with uniqueness and $F_2 = 2$). We can now split the terms that exponentially depend on n.

$$f_{n+1}(k) = \left(\frac{1}{\sqrt{2\pi}}\sqrt{\frac{(n-k)}{k(n-2k)}}\frac{\sqrt{5}}{\phi}\right)\left(\phi^{-n}\frac{(n-k)^{n-k}}{k^k(n-2k)^{n-2k}}\right).$$

Define

$$N_n = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{(n-k)}{k(n-2k)}}\frac{\sqrt{5}}{\phi}, \quad S_n = \phi^{-n}\frac{(n-k)^{n-k}}{k^k(n-2k)^{n-2k}}.$$

Thus, write the density function as

$$f_{n+1}(k) = N_n S_n$$

where N_n is the first term that is of order $n^{-1/2}$ and S_n is the second term with exponential dependence on n.

(Sketch of the) Proof of Gaussianity (cont)

Model the distribution as centered around the mean by the change of variable $k = \mu + x\sigma$ where μ and σ are the mean and the standard deviation, and depend on n. The discrete weights of $f_n(k)$ will become continuous. This requires us to use the change of variable formula to compensate for the change of scales:

$$f_n(k)dk = f_n(\mu + \sigma x)\sigma dx.$$

Using the change of variable, we can write N_n as

$$\begin{aligned} N_n &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{n-k}{k(n-2k)}}\frac{\phi}{\sqrt{5}} \\ &= \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-k/n}{(k/n)(1-2k/n)}}\frac{\sqrt{5}}{\phi} \\ &= \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-(\mu+\sigma x)/n}{((\mu+\sigma x)/n)(1-2(\mu+\sigma x)/n)}}\frac{\sqrt{5}}{\phi} \\ &= \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-C-y}{(C+y)(1-2C-2y)}}\frac{\sqrt{5}}{\phi} \end{aligned}$$

where $C = \mu/n \approx 1/(\phi+2)$ (note that $\phi^2 = \phi + 1$) and $y = \sigma x/n$. But for large n, the y term vanishes since $\sigma \sim \sqrt{n}$ and thus $y \sim n^{-1/2}$. Thus

$$N_n \approx \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-C}{C(1-2C)}}\frac{\sqrt{5}}{\phi} = \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{(\phi+1)(\phi+2)}{\phi}}\frac{\sqrt{5}}{\phi} = \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{5(\phi+2)}{\phi}} = \frac{1}{\sqrt{2\pi\sigma^2}}$$

since $\sigma^2 = n\frac{\phi}{5(\phi+2)}$.

(Sketch of the) Proof of Gaussianity (cont)

For the second term S_n, take the logarithm and once again change variables by $k = \mu + x\sigma$,

$$\begin{aligned}
\log(S_n) &= \log\left(\phi^{-n}\frac{(n-k)^{(n-k)}}{k^k(n-2k)^{(n-2k)}}\right) \\
&= -n\log(\phi) + (n-k)\log(n-k) - (k)\log(k) \\
&\quad - (n-2k)\log(n-2k) \\
&= -n\log(\phi) + (n-(\mu+x\sigma))\log(n-(\mu+x\sigma)) \\
&\quad - (\mu+x\sigma)\log(\mu+x\sigma) \\
&\quad - (n-2(\mu+x\sigma))\log(n-2(\mu+x\sigma)) \\
&= -n\log(\phi) \\
&\quad + (n-(\mu+x\sigma))\left(\log(n-\mu)+\log\left(1-\frac{x\sigma}{n-\mu}\right)\right) \\
&\quad - (\mu+x\sigma)\left(\log(\mu)+\log\left(1+\frac{x\sigma}{\mu}\right)\right) \\
&\quad - (n-2(\mu+x\sigma))\left(\log(n-2\mu)+\log\left(1-\frac{x\sigma}{n-2\mu}\right)\right) \\
&= -n\log(\phi) \\
&\quad + (n-(\mu+x\sigma))\left(\log\left(\frac{n}{\mu}-1\right)+\log\left(1-\frac{x\sigma}{n-\mu}\right)\right) \\
&\quad - (\mu+x\sigma)\log\left(1+\frac{x\sigma}{\mu}\right) \\
&\quad - (n-2(\mu+x\sigma))\left(\log\left(\frac{n}{\mu}-2\right)+\log\left(1-\frac{x\sigma}{n-2\mu}\right)\right).
\end{aligned}$$

(Sketch of the) Proof of Gaussianity (cont)

Note that, since $n/\mu = \phi + 2$ for large n, the constant terms vanish. We have $\log(S_n)$

$$\begin{aligned}
= \quad & -n\log(\phi) + (n-k)\log\left(\frac{n}{\mu} - 1\right) - (n-2k)\log\left(\frac{n}{\mu} - 2\right) + (n-(\mu+x\sigma))\log\left(1 - \frac{x\sigma}{n-\mu}\right) \\
& - (\mu + x\sigma)\log\left(1 + \frac{x\sigma}{\mu}\right) - (n - 2(\mu+x\sigma))\log\left(1 - \frac{x\sigma}{n-2\mu}\right) \\
= \quad & -n\log(\phi) + (n-k)\log(\phi+1) - (n-2k)\log(\phi) + (n-(\mu+x\sigma))\log\left(1 - \frac{x\sigma}{n-\mu}\right) \\
& - (\mu + x\sigma)\log\left(1 + \frac{x\sigma}{\mu}\right) - (n - 2(\mu+x\sigma))\log\left(1 - \frac{x\sigma}{n-2\mu}\right) \\
= \quad & n(-\log(\phi) + \log\left(\phi^2\right) - \log(\phi)) + k(\log(\phi^2) + 2\log(\phi)) + (n-(\mu+x\sigma))\log\left(1 - \frac{x\sigma}{n-\mu}\right) \\
& - (\mu + x\sigma)\log\left(1 + \frac{x\sigma}{\mu}\right) - (n - 2(\mu+x\sigma))\log\left(1 - 2\frac{x\sigma}{n-2\mu}\right) \\
= \quad & (n-(\mu+x\sigma))\log\left(1 - \frac{x\sigma}{n-\mu}\right) - (\mu + x\sigma)\log\left(1 + \frac{x\sigma}{\mu}\right) \\
& - (n - 2(\mu+x\sigma))\log\left(1 - 2\frac{x\sigma}{n-2\mu}\right).
\end{aligned}$$

(Sketch of the) Proof of Gaussianity (cont)

Finally, we expand the logarithms and collect powers of $x\sigma/n$.

$$\begin{aligned}
\log(S_n) &= (n-(\mu+x\sigma))\left(-\frac{x\sigma}{n-\mu}-\frac{1}{2}\left(\frac{x\sigma}{n-\mu}\right)^2+\dots\right)\\
&\quad -(\mu+x\sigma)\left(\frac{x\sigma}{\mu}-\frac{1}{2}\left(\frac{x\sigma}{\mu}\right)^2+\dots\right)\\
&\quad -(n-2(\mu+x\sigma))\left(-2\frac{x\sigma}{n-2\mu}-\frac{1}{2}\left(2\frac{x\sigma}{n-2\mu}\right)^2+\dots\right)\\
&= (n-(\mu+x\sigma))\left(-\frac{x\sigma}{n\frac{(\phi+1)}{(\phi+2)}}-\frac{1}{2}\left(\frac{x\sigma}{n\frac{(\phi+1)}{(\phi+2)}}\right)^2+\dots\right)\\
&\quad -(\mu+x\sigma)\left(\frac{x\sigma}{\frac{n}{\phi+2}}-\frac{1}{2}\left(\frac{x\sigma}{\frac{n}{\phi+2}}\right)^2+\dots\right)\\
&\quad -(n-2(\mu+x\sigma))\left(-\frac{2x\sigma}{n\frac{\phi}{\phi+2}}-\frac{1}{2}\left(\frac{2x\sigma}{n\frac{\phi}{\phi+2}}\right)^2+\dots\right)\\
&= \frac{x\sigma}{n}n\left(-\left(1-\frac{1}{\phi+2}\right)\frac{(\phi+2)}{(\phi+1)}-1+2\left(1-\frac{2}{\phi+2}\right)\frac{\phi+2}{\phi}\right)\\
&\quad -\frac{1}{2}\left(\frac{x\sigma}{n}\right)^2 n\left(-2\frac{\phi+2}{\phi+1}+\frac{\phi+2}{\phi+1}+2(\phi+2)-(\phi+2)+4\frac{\phi+2}{\phi}\right)\\
&\quad +O\left(n\left(x\sigma/n\right)^3\right)
\end{aligned}$$

(Sketch of the) Proof of Gaussianity (cont)

$$
\begin{aligned}
\log(S_n) &= \frac{x\sigma}{n} n \left(-\frac{\phi+1}{\phi+2}\frac{\phi+2}{\phi+1} - 1 + 2\frac{\phi}{\phi+2}\frac{\phi+2}{\phi} \right) \\
&\quad -\frac{1}{2}\left(\frac{x\sigma}{n}\right)^2 n(\phi+2)\left(-\frac{1}{\phi+1} + 1 + \frac{4}{\phi}\right) \\
&\quad +O\left(n\left(\frac{x\sigma}{n}\right)^3\right) \\
&= -\frac{1}{2}\frac{(x\sigma)^2}{n}(\phi+2)\left(\frac{3\phi+4}{\phi(\phi+1)} + 1\right) + O\left(n\left(\frac{x\sigma}{n}\right)^3\right) \\
&= -\frac{1}{2}\frac{(x\sigma)^2}{n}(\phi+2)\left(\frac{3\phi+4+2\phi+1}{\phi(\phi+1)}\right) + O\left(n\left(\frac{x\sigma}{n}\right)^3\right) \\
&= -\frac{1}{2}x^2\sigma^2\left(\frac{5(\phi+2)}{\phi n}\right) + O\left(n\,(x\sigma/n)^3\right).
\end{aligned}
$$

(Sketch of the) Proof of Gaussianity (cont)

But recall that

$$\sigma^2 = \frac{\phi n}{5(\phi+2)}.$$

Also, since $\sigma \sim n^{-1/2}$, $n\left(\frac{x\sigma}{n}\right)^3 \sim n^{-1/2}$. So for large n, the $O\left(n\left(\frac{x\sigma}{n}\right)^3\right)$ term vanishes. Thus we are left with

$$\begin{aligned}\log S_n &= -\frac{1}{2}x^2 \\ S_n &= e^{-\frac{1}{2}x^2}.\end{aligned}$$

Hence, as n gets large, the density converges to the normal distribution:

$$\begin{aligned}f_n(k)dk &= N_n S_n dk \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}x^2}\sigma dx \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx.\end{aligned}$$

□

Generalizations

Generalizations

Generalizing from Fibonacci numbers to linearly recursive sequences with arbitrary nonnegative coefficients.

$$H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1},\ n \geq L$$

with $H_1 = 1$, $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1,\ n < L$, coefficients $c_i \geq 0$; $c_1, c_L > 0$ if $L \geq 2$; $c_1 > 1$ if $L = 1$.

- Zeckendorf: Every positive integer can be written uniquely as $\sum a_i H_i$ with natural constraints on the a_i's (e.g. cannot use the recurrence relation to remove any summand).
- Lekkerkerker
- Central Limit Type Theorem

Generalizing Lekkerkerker

Generalized Lekkerkerker's Theorem

The average number of summands in the generalized Zeckendorf decomposition for integers in $[H_n, H_{n+1})$ tends to $Cn + d$ as $n \to \infty$, where $C > 0$ and d are computable constants determined by the c_i's.

$$C = -\frac{y'(1)}{y(1)} = \frac{\sum_{m=0}^{L-1}(s_m + s_{m+1} - 1)(s_{m+1} - s_m)y^m(1)}{2\sum_{m=0}^{L-1}(m+1)(s_{m+1} - s_m)y^m(1)}.$$

$$s_0 = 0, s_m = c_1 + c_2 + \cdots + c_m.$$

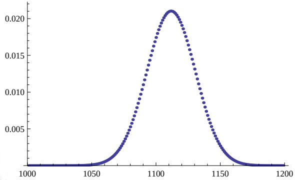
$y(x)$ is the root of $1 - \sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}$.

$y(1)$ is the root of $1 - c_1 y - c_2 y^2 - \cdots - c_L y^L$.

Central Limit Type Theorem

Central Limit Type Theorem

As $n \to \infty$, the distribution of the number of summands, i.e., $a_1 + a_2 + \cdots + a_m$ in the generalized Zeckendorf decomposition $\sum_{i=1}^{m} a_i H_i$ for integers in $[H_n, H_{n+1})$ is Gaussian.



Example: the Special Case of $L = 1$, $c_1 = 10$

$H_{n+1} = 10H_n$, $H_1 = 1$, $H_n = 10^{n-1}$.

- Legal decomposition is decimal expansion: $\sum_{i=1}^{m} a_i H_i$: $a_i \in \{0, 1, \dots, 9\}$ $(1 \leq i < m)$, $a_m \in \{1, \dots, 9\}$.
- For $N \in [H_n, H_{n+1})$, $m = n$, i.e., first term is $a_n H_n = a_n 10^{n-1}$.
- A_i: the corresponding random variable of a_i. The A_i's are independent.
- For large n, the contribution of A_n is immaterial. A_i $(1 \leq i < n)$ are identically distributed random variables with mean 4.5 and variance 8.25.
- Central Limit Theorem: $A_2 + A_3 + \cdots + A_n \to$ Gaussian with mean $4.5n + O(1)$ and variance $8.25n + O(1)$.

Far-difference Representation

Theorem (Alpert, 2009) (Analogue to Zeckendorf)

Every integer can be written uniquely as a sum of the $\pm F_n$'s, such that every two terms of the same (opposite) sign differ in index by at least 4 (3).

Example: $1900 = F_{17} - F_{14} - F_{10} + F_6 + F_2$.

K: # of positive terms, L: # of negative terms.

Generalized Lekkerkerker's Theorem

As $n \to \infty$, $E[K]$ and $E[L] \to n/10$. $E[K] - E[L] = \varphi/2 \approx .809$.

Central Limit Type Theorem

As $n \to \infty$, K and L converges to a bivariate Gaussian.

- $\text{corr}(K, L) = -(21 - 2\varphi)/(29 + 2\varphi) \approx -.551$, $\varphi = \frac{\sqrt{5}+1}{2}$.
- $K + L$ and $K - L$ are independent.

Gaps Between Summands

Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \dots, i_2 - i_1$.

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Can ask similar questions about binary or other expansions: $2011 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^4 + 2^3 + 2^1 + 2^0$.

Main Results (Beckwith-Miller 2011)

Theorem (Base B Gap Distribution)

For base B decompositions, $P(0) = \frac{(B-1)(B-2)}{B^2}$, and for $k \geq 1$, $P(k) = c_B B^{-k}$, with $c_B = \frac{(B-1)(3B-2)}{B^2}$.

Theorem (Zeckendorf Gap Distribution)

For Zeckendorf decompositions, $P(k) = \frac{\phi(\phi-1)}{\phi^k}$ for $k \geq 2$, with $\phi = \frac{1+\sqrt{5}}{2}$ the golden mean.

Proof of Fibonacci Result

Lekkerkerker $\Rightarrow$ total number of gaps $\sim F_{n-1}\frac{n}{\phi^2+1}$.

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$$P(k) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n-k} X_{i,i+k}}{F_{n-1}\frac{n}{\phi^2+1}}.$$

Calculating $X_{i,i+k}$

How many decompositions contain a gap from F_i to F_{i+k}?

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For the indices less than i: F_{i-1} choices.

For the indices greater than $i+k$: $F_{n-k-2-i}$ choices.

Calculating $X_{i,i+k}$

How many decompositions contain a gap from F_i to F_{i+k}?

$1 \leq i \leq n - k - 2$:

For the indices less than i: F_{i-1} choices.

For the indices greater than $i + k$: $F_{n-k-2-i}$ choices.

So total choices number of choices is $F_{n-k-2-i}F_{i-1}$.

Determining $P(k)$

$$\sum_{i=1}^{n-k} X_{i,i+k} = F_{n-k-1} + \sum_{i=1}^{n-k-2} F_{i-1}F_{n-k-i-2}$$

$\sum_{i=0}^{n-k-3} F_i F_{n-k-i-3}$ is the x^{n-k-3} coefficient of $(g(x))^2$, where $g(x)$ is the generating function of the Fibonaccis.

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$P(k) = \frac{C}{\phi^k}$ for some constant C, so $P(k) = \frac{\phi(\phi-1)}{\phi^k}$.

Tribonacci Gaps

Tribonacci Numbers: $T_{n+1} = T_n + T_{n-1} + T_{n-2}$;
$F_1 = 1, F_2 = 2, F_3 = 4, F_4 = 7, \dots$.

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Counting:

$$X_{i,i+k}(n) = \begin{cases} T_{i-1}(T_{n-i-3} + T_{n-i-4}) & \text{if } k = 1 \\ (T_{i-1} + T_{i-2})(T_{n-k-i-1} + T_{n-k-i-3}) & \text{if } k \geq 2 \end{cases}$$

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Constants st $P(1) = \frac{c_1}{C\lambda_1^3}$, $P(k) = \frac{2c_1}{C(1+\lambda_1)}\lambda_1^{-k}$ (for $k \geq 2$).

Other gaps?

- Gaps longer than recurrence – geometric decay.
- Interesting behavior with "short" gaps.
- "Skiponaccis": $S_{n+1} = S_n + S_{n-2}$.
- "Doublanaccis": $H_{n+1} = 2H_n + H_{n-1}$.
- Hope: Generalize to all positive linear recurrences.

Method of General Proof

Generating Function (Example: Binet's Formula)

Binet's Formula

$$F_1 = F_2 = 1; \; F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right].$$

- Recurrence relation: $F_{n+1} = F_n + F_{n-1}$ (1)
- Generating function: $g(x) = \sum_{n>0} F_n x^n$.

$$\begin{aligned}
(1) \quad &\Rightarrow \sum_{n\geq 2} F_{n+1}x^{n+1} = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 2} F_{n-1}x^{n+1} \\
&\Rightarrow \sum_{n\geq 3} F_n x^n = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 1} F_n x^{n+2} \\
&\Rightarrow \sum_{n\geq 3} F_n x^n = x\sum_{n\geq 2} F_n x^n + x^2\sum_{n\geq 1} F_n x^n \\
&\Rightarrow g(x) - F_1 x - F_2 x^2 = x(g(x) - F_1 x) + x^2 g(x) \\
&\Rightarrow g(x) = x/(1 - x - x^2).
\end{aligned}$$

Partial Fraction Expansion (Example: Binet's Formula)

- Generating function: $g(x) = \sum_{n>0} F_n x^n = \frac{x}{1-x-x^2}$.
- Partial fraction expansion:

$$\Rightarrow \; g(x) \; = \; \frac{x}{1-x-x^2} \; = \; \frac{1}{\sqrt{5}}\left(\frac{\frac{1+\sqrt{5}}{2}x}{1-\frac{1+\sqrt{5}}{2}x} - \frac{\frac{-1+\sqrt{5}}{2}x}{1-\frac{-1+\sqrt{5}}{2}x}\right).$$

Coefficient of x^n (power series expansion):

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right]$$ - Binet's Formula!

(using geometric series: $\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots$).

Differentiating Identities and Method of Moments

- Differentiating identities
  Example: Given a random variable X such that
  $\text{Prob}(X=1)=\frac{1}{2}$, $\text{Prob}(X=2)=\frac{1}{4}$, $\text{Prob}(X=3)=\frac{1}{8}$, ... ,
  then what's the mean of X (i.e., $E[X]$)?
  Solution: Let $f(x)=\frac{1}{2}x+\frac{1}{4}x^2+\frac{1}{8}x^3+\cdots=\frac{1}{1-x/2}-1$.
  $$f'(x)=1\cdot\frac{1}{2}+2\cdot\frac{1}{4}x+3\cdot\frac{1}{8}x^2+\cdots.$$
  $$f'(1)=1\cdot\frac{1}{2}+2\cdot\frac{1}{4}+3\cdot\frac{1}{8}+\cdots=E[X].$$

- Method of moments: Random variables $X_1, X_2, \dots$.
  If the ℓ^{th} moment $E[X_n^\ell]$ converges to that of the standard normal distribution $(\forall\ell)$, then X_n converges to a Gaussian.

  Standard normal distribution:
  $2m^{\text{th}}$ moment: $(2m-1)!!=(2m-1)(2m-3)\cdots1$,
  $(2m-1)^{\text{th}}$ moment: 0.

New Approach: Case of Fibonacci Numbers

$p_{n,k} = \#\left\{N \in [F_n, F_{n+1})\text{: the Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands}\right\}$.

- Recurrence relation:
  $N \in [F_{n+1}, F_{n+2})$: $N = F_{n+1} + F_t + \cdots$, $t \leq n-1$.

$$p_{n+1,k+1} \;=\; p_{n-1,k} + p_{n-2,k} + \cdots$$

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$$\begin{aligned} p_{n+1,k+1} &= p_{n-1,k} + p_{n-2,k} + \cdots \\ p_{n,k+1} &= p_{n-2,k} + p_{n-3,k} + \cdots \end{aligned}$$

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  $$\begin{aligned} p_{n+1,k+1} &= p_{n-1,k} + p_{n-2,k} + \cdots \\ p_{n,k+1} &= p_{n-2,k} + p_{n-3,k} + \cdots \\ \Rightarrow p_{n+1,k+1} &= p_{n,k+1} + p_{n-1,k}. \end{aligned}$$
- Generating function: $\sum_{n,k>0} p_{n,k} x^k y^n = \frac{y}{1-y-xy^2}$.
- Partial fraction expansion:
  $$\frac{y}{1-y-xy^2} = -\frac{y}{y_1(x)-y_2(x)}\left(\frac{1}{y-y_1(x)} - \frac{1}{y-y_2(x)}\right)$$
  where $y_1(x)$ and $y_2(x)$ are the roots of $1-y-xy^2=0$.

Coefficient of y^n: $g(x) = \sum_{k>0} p_{n,k} x^k$.

New Approach: Case of Fibonacci Numbers (Continued)

K_n: the corresponding random variable associated with k.

$g(x) = \sum_{k>0} p_{n,k} x^k$.

- Differentiating identities:

  $g(1) = \sum_{k>0} p_{n,k} = F_{n+1} - F_n$,

  $g'(x) = \sum_{k>0} k p_{n,k} x^{k-1}$, $g'(1) = g(1)E[K_n]$,

  $(xg'(x))' = \sum_{k>0} k^2 p_{n,k} x^{k-1}$,

  $(xg'(x))'|_{x=1} = g(1)E[K_n^2]$, $\left(x\,(xg'(x))'\right)'|_{x=1} = g(1)E[K_n^3]$, ...

  Similar results hold for the centralized K_n: $K_n' = K_n - E[K_n]$.

- Method of moments (for normalized K_n'):

  $E[(K_n')^{2m}]/(SD(K_n'))^{2m} \to (2m-1)!!$,

  $E[(K_n')^{2m-1}]/(SD(K_n'))^{2m-1} \to 0$. $\Rightarrow K_n \to$ Gaussian.

New Approach: General Case

Let $p_{n,k} = \#\{N \in [H_n, H_{n+1})$: the generalized Zeckendorf decomposition of N has exactly k summands$\}$.

- Recurrence relation:

  Fibonacci: $p_{n+1,k+1} = p_{n,k+1} + p_{n,k}$.

  General: $p_{n+1,k} = \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} p_{n-m,k-j}$.

  where $s_0 = 0, s_m = c_1 + c_2 + \cdots + c_m$.

- Generating function:

  Fibonacci: $\frac{y}{1-y-xy^2}$.

  General:

$$\frac{\sum_{n\le L} p_{n,k} x^k y^n - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1} \sum_{n<L-m} p_{n,k} x^k y^n}{1 - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}}.$$

New Approach: General Case (Continued)

- Partial fraction expansion:

  Fibonacci: $-\frac{y}{y_1(x)-y_2(x)}\left(\frac{1}{y-y_1(x)}-\frac{1}{y-y_2(x)}\right)$.

  General:

$$-\frac{1}{\sum_{j=s_{L-1}}^{s_L-1} x^j}\sum_{i=1}^{L}\frac{B(x,y)}{(y-y_i(x))\prod_{j\neq i}(y_j(x)-y_i(x))}.$$

$$B(x,y)=\sum_{n\leq L} p_{n,k}x^k y^n-\sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}\sum_{n<L-m} p_{n,k}x^k y^n,$$

$$y_i(x)\text{: root of } 1-\sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}=0.$$

  Coefficient of y^n: $g(x)=\sum_{n,k>0} p_{n,k}x^k$.

- Differentiating identities
- Method of moments $\Rightarrow K_n \to$ Gaussian

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Appendix:
Combinatorial Identities and Lekkerkerker's Theorem

Needed Binomial Identity

Binomial identity involving Fibonacci Numbers

Let F_m denote the m^{th} Fibonacci number, with $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5$ and so on. Then

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} = F_{n-1}.$$

Proof by induction: The base case is trivially verified. Assume our claim holds for n and show that it holds for $n+1$. We may extend the sum to $n-1$, as $\binom{n-1-k}{k} = 0$ whenever $k > \lfloor \frac{n-1}{2} \rfloor$. Using the standard identity that

$$\binom{m}{\ell} + \binom{m}{\ell+1} = \binom{m+1}{\ell+1},$$

and the convention that $\binom{m}{\ell} = 0$ if ℓ is a negative integer, we find

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n-k}{k} &= \sum_{k=0}^{n} \left[\binom{n-1-k}{k-1} + \binom{n-1-k}{k} \right] \\
&= \sum_{k=1}^{n} \binom{n-1-k}{k-1} + \sum_{k=0}^{n} \binom{n-1-k}{k} \\
&= \sum_{k=1}^{n} \binom{n-2-(k-1)}{k-1} + \sum_{k=0}^{n} \binom{n-1-k}{k} = F_{n-2} + F_{n-1}
\end{aligned}$$

by the inductive assumption; noting $F_{n-2} + F_{n-1} = F_n$ completes the proof. □

Preliminaries for Lekkerkerker's Theorem

$$\mathcal{E}(n) \ := \ \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} k \binom{n-1-k}{k}.$$

Average number of summands in $[F_n, F_{n+1})$ is

$$\frac{\mathcal{E}(n)}{F_{n-1}} + 1.$$

Recurrence Relation for $\mathcal{E}(n)$

$$\mathcal{E}(n) + \mathcal{E}(n-2) \ = \ (n-2)F_{n-3}.$$

Recurrence Relation

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$$\mathcal{E}(n) + \mathcal{E}(n-2) \ = \ (n-2)F_{n-3}.$$

Proof by algebra (details later):

$$\begin{aligned} \mathcal{E}(n) &= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} k \binom{n-1-k}{k} \\ &= (n-2)\sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} \binom{n-3-\ell}{\ell} - \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} \ell \binom{n-3-\ell}{\ell} \\ &= (n-2)F_{n-3} - \mathcal{E}(n-2). \end{aligned}$$

Solving Recurrence Relation

Formula for $\mathcal{E}(n)$ (i.e., Lekkerkerker's Theorem)

$$\mathcal{E}(n) \;=\; \frac{nF_{n-1}}{\varphi^2+1} + O(F_{n-2}).$$

$$\sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (-1)^{\ell} \left(\mathcal{E}(n-2\ell) + \mathcal{E}(n-2(\ell+1))\right)$$

$$= \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (-1)^{\ell}(n-2-2\ell)F_{n-3-2\ell}.$$

Result follows from Binet's formula, the geometric series formula, and differentiating identities: $\sum_{j=0}^{m} jx^j = x\frac{(m+1)x^m(x-1)-(x^{m+1}-1)}{(x-1)^2}$. Details later in the appendix.

Derivation of Recurrence Relation for $\mathcal{E}(n)$

$$\begin{aligned}
\mathcal{E}(n) &= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} k \binom{n-1-k}{k} \\
&= \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} k \frac{(n-1-k)!}{k!(n-1-2k)!} \\
&= \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (n-1-k) \frac{(n-2-k)!}{(k-1)!(n-1-2k)!} \\
&= \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (n-2-(k-1)) \frac{(n-3-(k-1)!}{(k-1)!(n-3-2(k-1))!} \\
&= \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (n-2-\ell) \binom{n-3-\ell}{\ell} \\
&= (n-2) \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} \binom{n-3-\ell}{\ell} - \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} \ell \binom{n-3-\ell}{\ell} \\
&= (n-2)F_{n-3} - \mathcal{E}(n-2),
\end{aligned}$$

which proves the claim (note we used the binomial identity to replace the sum of binomial coefficients with a Fibonacci number).

Formula for $\mathcal{E}(n)$

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$$\mathcal{E}(n) \;=\; \frac{nF_{n-1}}{\varphi^2+1} + O(F_{n-2}).$$

Proof: The proof follows from using telescoping sums to get an expression for $\mathcal{E}(n)$, which is then evaluated by inputting Binet's formula and differentiating identities. Explicitly, consider

$$\sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (-1)^{\ell}\left(\mathcal{E}(n-2\ell) + \mathcal{E}(n-2(\ell+1))\right) \;=\; \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (-1)^{\ell}(n-2-2\ell)F_{n-3-2\ell}$$

$$= \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (-1)^{\ell}(n-3-2\ell)F_{n-3-2\ell} + \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (-1)^{\ell}(2\ell)F_{n-3-2\ell}$$

$$= \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (-1)^{\ell}(n-3-2\ell)F_{n-3-2\ell} + O(F_{n-2});$$

while we could evaluate the last sum exactly, trivially estimating it suffices to obtain the main term (as we have a sum of every other Fibonacci number, the sum is at most the next Fibonacci number after the largest one in our sum).

Formula for $\mathcal{E}(n)$ (continued)

We now use Binet's formula to convert the sum into a geometric series. Letting $\varphi = \frac{1+\sqrt{5}}{2}$ be the golden mean, we have

$$F_n = \frac{\varphi}{\sqrt{5}} \cdot \varphi^n - \frac{1-\varphi}{\sqrt{5}} \cdot (1-\varphi)^n$$

(our constants are because our counting has $F_1 = 1$, $F_2 = 2$ and so on). As $|1-\varphi| < 1$, the error from dropping the $(1-\varphi)^n$ term is $O(\sum_{\ell \le n} n) = O(n^2) = o(F_{n-2})$, and may thus safely be absorbed in our error term. We thus find

$$\begin{aligned} \mathcal{E}(n) &= \frac{\varphi}{\sqrt{5}} \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (n-3-2\ell)(-1)^\ell \varphi^{n-3-2\ell} + O(F_{n-2}) \\ &= \frac{\varphi^{n-2}}{\sqrt{5}} \left[(n-3) \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} (-\varphi^{-2})^\ell - 2 \sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} \ell(-\varphi^{-2})^\ell \right] + O(F_{n-2}). \end{aligned}$$

Formula for $\mathcal{E}(n)$ (continued)

We use the geometric series formula to evaluate the first term. We drop the upper boundary term of $(-\varphi^{-1})^{\lfloor \frac{n-3}{2} \rfloor}$, as this term is negligible since $\varphi > 1$. We may also move the 3 from the $n-3$ into the error term, and are left with

$$\mathcal{E}(n) = \frac{\varphi^{n-2}}{\sqrt{5}}\left[\frac{n}{1+\varphi^{-2}} - 2\sum_{\ell=0}^{\lfloor \frac{n-3}{2} \rfloor} \ell(-\varphi^{-2})^{\ell}\right] + O(F_{n-2})$$
$$= \frac{\varphi^{n-2}}{\sqrt{5}}\left[\frac{n}{1+\varphi^{-2}} - 2S\left(\left\lfloor \frac{n-3}{2} \right\rfloor, -\varphi^{-2}\right)\right] + O(F_{n-2}),$$

where

$$\mathcal{S}(m,x) = \sum_{j=0}^{m} jx^{j}.$$

There is a simple formula for $\mathcal{S}(m,x)$. As

$$\sum_{j=0}^{m} x^{j} = \frac{x^{m+1}-1}{x-1},$$

applying the operator $x\frac{d}{dx}$ gives

$$\mathcal{S}(m,x) = \sum_{j=0}^{m} jx^{j} = x\frac{(m+1)x^{m}(x-1)-(x^{m+1}-1)}{(x-1)^2} = \frac{mx^{m+2}-(m+1)x^{m+1}+x}{(x-1)^2}.$$

Formula for $\mathcal{E}(n)$ (continued)

Taking $x = -\varphi^{-2}$, we see that the contribution from this piece may safely be absorbed into the error term $O(F_{n-2})$, leaving us with

$$\mathcal{E}(n) \ = \ \frac{n\varphi^{n-2}}{\sqrt{5}(1+\varphi^{-2})} + O(F_{n-2}) \ = \ \frac{n\varphi^{n}}{\sqrt{5}(\varphi^{2}+1)} + O(F_{n-2}).$$

Noting that for large n we have $F_{n-1} = \frac{\varphi^n}{\sqrt{5}} + O(1)$, we finally obtain

$$\mathcal{E}(n) \ = \ \frac{nF_{n-1}}{\varphi^2+1} + O(F_{n-2}). \qquad \Box$$