Sums and Differences of Correlated Random Sets

Thao Do - Stony Brook (thao.do@stonybrook.edu)

Archit Kulkarni - Carnegie-Mellon
David Moon - Williams College
Jake Wellens - Caltech
Advisor: Steven J Miller

GSUMC 2014, Rowan University
April 5, 2014

Introduction

Given $A \subset \mathbb{Z}$, let

$$A + A = \{a_1 + a_2 : a_1, a_2 \in A\},$$
$$A - A = \{a_1 - a_2 : a_1, a_2 \in A\}.$$

Introduction

Given $A \subset \mathbb{Z}$, let

$$A + A = \{a_1 + a_2 : a_1, a_2 \in A\},$$
$$A - A = \{a_1 - a_2 : a_1, a_2 \in A\}.$$

Example: For $A = \{1, 4, 5\}$:

$$A + A = \{2, 5, 6, 8, 9, 10\}$$
$$A - A = \{0, \pm 1, \pm 3, \pm 4\}.$$

Introduction

Given $A \subset \mathbb{Z}$, let

$$A + A = \{a_1 + a_2 : a_1, a_2 \in A\},$$
$$A - A = \{a_1 - a_2 : a_1, a_2 \in A\}.$$

Example: For $A = \{1, 4, 5\}$:

$$A + A = \{2, 5, 6, 8, 9, 10\}$$
$$A - A = \{0, \pm 1, \pm 3, \pm 4\}.$$

A finite set A is sum dominated, or more sum than difference (MSTD) if $|A + A| > |A - A|$.

The smallest example: $\{0; 2; 3; 4; 7; 11; 12; 14\}$.

Previous Results

Martin and O'Bryant (2006): There exists a positive constant c such that for any n large, the proportion of MSTD sets $A \subset \{0, \ldots, n\}$ is greater than c.

Previous Results

Martin and O'Bryant (2006): There exists a positive constant c such that for any n large, the proportion of MSTD sets $A \subset \{0, \dots, n\}$ is greater than c.

Zhao (2010): The proportion p_n of *MSTD* subset in $\{1, \cdots, n\}$ as $n \to \infty$ converges to a positive number which can be computed.

Previous Results

Martin and O'Bryant (2006): There exists a positive constant c such that for any n large, the proportion of MSTD sets $A \subset \{0, \dots, n\}$ is greater than c.

Zhao (2010): The proportion p_n of *MSTD* subset in $\{1, \cdots, n\}$ as $n \to \infty$ converges to a positive number which can be computed.

Hegarty-Miller (2009): If the probability to pick a number from 1 to n into set A decays with n, then probability A is *MSTD* converges to 0 as $n \to \infty$.

Previous Results

Martin and O'Bryant (2006): There exists a positive constant c such that for any n large, the proportion of MSTD sets $A \subset \{0, \ldots, n\}$ is greater than c.

Zhao (2010): The proportion p_n of *MSTD* subset in $\{1, \cdots, n\}$ as $n \to \infty$ converges to a positive number which can be computed.

Hegarty-Miller (2009): If the probability to pick a number from 1 to n into set A decays with n, then probability A is *MSTD* converges to 0 as $n \to \infty$.

Hegarty (2007): The smallest size of a sum-dominated set is 8.

Correlated Random Pairs

All of the literature to date has looked at sums and differences of a set *with itself*.

Correlated Random Pairs

All of the literature to date has looked at sums and differences of a set *with itself*.

We investigate sums and differences of *pairs* of subsets $(A, B) \subset \{0, \dots, n\}$. A pair is *sum dominated* or *MSTD* if

$$|A + B| > |\pm(A - B)| = |(A - B) \cup (B - A)|.$$

Correlated Random Pairs

All of the literature to date has looked at sums and differences of a set *with itself*.

We investigate sums and differences of *pairs* of subsets $(A, B) \subset \{0, \dots, n\}$. A pair is *sum dominated* or *MSTD* if

$$|A + B| > |\pm(A - B)| = |(A - B) \cup (B - A)|.$$

We select such pairs according to the dependent random process:

Correlated Random Pairs

All of the literature to date has looked at sums and differences of a set *with itself*.

We investigate sums and differences of *pairs* of subsets $(A, B) \subset \{0, \dots, n\}$. A pair is *sum dominated* or *MSTD* if

$$|A + B| > |\pm(A - B)| = |(A - B) \cup (B - A)|.$$

We select such pairs according to the dependent random process:

$$P(k \in A) = p; \quad P(k \in B | k \in A) = \rho_1; \quad P(k \in B | k \notin A) = \rho_2.$$

Correlated Random Pairs

- $(\rho_1, \rho_2) = (1, 0) \implies (A, A)$.

Correlated Random Pairs

- $(\rho_1, \rho_2) = (1, 0) \implies (A, A)$.
- $(\rho_1, \rho_2) = (0, 1) \implies (A, A^c)$.

Correlated Random Pairs

- $(\rho_1, \rho_2) = (1, 0) \implies (A, A)$.
- $(\rho_1, \rho_2) = (0, 1) \implies (A, A^c)$.
- $\rho_1 = \rho_2, \implies (A, B)$ independent.

Probability function

Let $P(\vec{\rho}, n)$ be the probability that a $\vec{\rho}$-correlated pair $(A, B) \subset \{0, \ldots, n\}$ is MSTD.

Probability function

Let $P(\vec{\rho}, n)$ be the probability that a $\vec{\rho}$-correlated pair $(A, B) \subset \{0, \ldots, n\}$ is MSTD.

Theorem

For any $\vec{\rho} \in [0,1]^3$, the limit

$$\lim_{n\to\infty} P(\vec{\rho}, n) =: P(\vec{\rho})$$

exists. Moreover, as long as $p \notin \{0, 1\}$ and $(\rho_1, \rho_2) \neq (0,0), (1,1)$, then $P(\vec{\rho})$ is strictly positive.

Probability function

Let $P(\vec{\rho}, n)$ be the probability that a $\vec{\rho}$-correlated pair $(A, B) \subset \{0, \ldots, n\}$ is MSTD.

Theorem

For any $\vec{\rho} \in [0,1]^3$, the limit

$$\lim_{n\to\infty} P(\vec{\rho}, n) =: P(\vec{\rho})$$

exists. Moreover, as long as $p \notin \{0,1\}$ and $(\rho_1, \rho_2) \neq (0,0), (1,1)$, then $P(\vec{\rho})$ is strictly positive.

Main idea of proof: same approach with Martin O'Bryant (2007) and Zhao (2010), construct an appropriate fringe (edge elements), then fill in the middle.

The function $P(\vec{\rho})$

Theorem

The function $P(\vec{\rho})$ is continuous on $[0, 1]^3$.

The function $P(\vec{\rho})$

Theorem

The function $P(\vec{\rho})$ is continuous on $[0,1]^3$.

Main idea of proof: write this function as an infinite sum of polynomial-type term, show the sum converges uniformly.

The function $P(\vec{\rho})$

Theorem

The function $P(\vec{\rho})$ is continuous on $[0,1]^3$.

Main idea of proof: write this function as an infinite sum of polynomial-type term, show the sum converges uniformly.

Corollary: P must attain a maximum in $[0,1]^3$.

The function $P(\vec{\rho})$

Theorem

The function $P(\vec{\rho})$ is continuous on $[0,1]^3$.

Main idea of proof: write this function as an infinite sum of polynomial-type term, show the sum converges uniformly.

Corollary: P must attain a maximum in $[0,1]^3$.

Conjecture 1: Function $P(p, \rho_1, \rho_2)$ is differentiable.

The function $P(\vec{\rho})$

Theorem

The function $P(\vec{\rho})$ is continuous on $[0,1]^3$.

Main idea of proof: write this function as an infinite sum of polynomial-type term, show the sum converges uniformly.

Corollary: P must attain a maximum in $[0,1]^3$.

Conjecture 1: Function $P(p, \rho_1, \rho_2)$ is differentiable.

Conjecture 2: max $P = P(0, 1, 1/2) \approx 0.03$.

Taking $p \to 0$

In previous section, we know that for any fixed (p, ρ_1, ρ_2) there is a positive percentage of MSTD pairs.

Taking $p \to 0$

In previous section, we know that for any fixed (p, ρ_1, ρ_2) there is a positive percentage of MSTD pairs.

Here we let some of p, ρ_1, ρ_2 vary and depend on n.

Taking $p \to 0$

In previous section, we know that for any fixed (p, ρ_1, ρ_2) there is a positive percentage of MSTD pairs.

Here we let some of p, ρ_1, ρ_2 vary and depend on n.

We get similar results to Hegarty-Miller (2009): if $\vec{\rho}$ decays with n (either $p \to 0$ or $\rho_1 + \rho_2 \to 0$) then the probability a correlated pair (A, B) in $\{1, \cdots, n\}$ is *MSTD* converges to 0 as $n \to \infty$.

The minimal MSTD pair

Hegarty (2007) proved the smallest MSTD set has size 8.

The minimal MSTD pair

Hegarty (2007) proved the smallest MSTD set has size 8. We prove

Theorem

The smallest MSTD pair has size $(3, 5)$ or $(4, 4)$.

The minimal MSTD pair

Hegarty (2007) proved the smallest MSTD set has size 8. We prove

Theorem

The smallest MSTD pair has size $(3, 5)$ or $(4, 4)$.

Examples of minimal size MSTD pair:

$$A = \{1, 2, 5, 7\}, \quad B = \{1, 3, 6, 7\}$$

$$A = \{3, 4, 6\}, \quad B = \{1, 2, 5, 7, 8\}$$

$$A = \{3, 5, 6\}, \quad B = \{1, 2, 4, 7, 8\}.$$

The minimal MSTD pair

Hegarty (2007) proved the smallest MSTD set has size 8. We prove

Theorem

The smallest MSTD pair has size $(3,5)$ or $(4,4)$.

Examples of minimal size MSTD pair:

$$A = \{1,2,5,7\}, \quad B = \{1,3,6,7\}$$

$$A = \{3,4,6\}, \quad B = \{1,2,5,7,8\}$$

$$A = \{3,5,6\}, \quad B = \{1,2,4,7,8\}.$$

Idea of proof: different from Hegarty (2007), we use combinatorial approach. Show that if (A,B) is MSTD then there must exist $a_1, a_2, a_3 \in A$ and $b_1, b_2, b_3 \in B$ such that $a_1 + b_1 = a_2 + b_2 = a_3 + b_3$.

Summary of Results and Future Research

- We prove results similar to previous research in a more general setting.

Summary of Results and Future Research

- We prove results similar to previous research in a more general setting.
- We show that the limit $P(\vec{\rho})$ exists for each chosen p, ρ_1, ρ_2, and prove P is continuous.

Summary of Results and Future Research

- We prove results similar to previous research in a more general setting.
- We show that the limit $P(\vec{\rho})$ exists for each chosen p, ρ_1, ρ_2, and prove P is continuous.
- In the future, we would like to prove our two conjectures, and find more analytic properties of P.

Acknowledgements

We would like to thank our advisor, Steven J. Miller, our co-authors David Moon and Archit Kulkarni, the rest of the team at the Williams College SMALL REU 2013, and the National Science Foundation.

This research was funded by NSF grant DMS0850577.

Link to our paper:
http://arxiv.org/abs/1401.2588

References

- P. Hegarty, Some explicit constructions of sets with more sums than differences. Acta Arithmetica **130** (2007), no. 1, 61–77.
- P. Hegarty and S. Miller, When almost all sets are difference dominated, *Random Structures and Algorithms* 35 (2009), no. 1, 118-136.
- G. Martin and K. O'Bryant, Many sets have more sums than differences, Additive Combinatorics, CRM Proc. Lecture Notes, vol. 43, Amer. Math. Soc., Providence, RI, 2007, pp. 287-305.
- Y. Zhao, Sets Characterized by Missing Sums and Differences, *Journal of Number Theory*, 131 (2010), pp. 2107-2134.