On a Pair of Diophantine Equations

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Motivation

For relatively prime $a, b \in \mathbb{N}$, consider

$$ax + by = \frac{(a-1)(b-1)}{2}, \tag{1}$$

$$ax + by + 1 = \frac{(a-1)(b-1)}{2}. \tag{2}$$

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Theorem (Beiter (1964), extended by Chu (2020))

Exactly one of the equations has a nonnegative solution (x, y). The solution is unique.

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Exactly one of the equations has a nonnegative solution (x, y). The solution is unique.

Goal

Investigate which equation is used by two consecutive terms of a given sequence.

Notations

$\Gamma(a, b)$ tells which Diophantine equation is used by (a/d) and (b/d), for $d = \gcd(a, b)$.

$\Gamma(5, 15) = \Gamma(1, 3) = 1.$

$\Gamma(12, 18) = \Gamma(2, 3) = 2.$

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$\Gamma(12,18) = \Gamma(2,3) = 2.$

Given a sequence $(a_n)_n$,

$$\Delta((a_n)_n) = (\Gamma(a_n, a_{n+1}))_n.$$

$\Delta(\mathbb{N}) = 1,2,1,2,1,2,\ldots.$

Definition of Θ

Definition

Let $a, b \in \mathbb{N}$ and $d = \gcd(a, b)$. Then $\Theta(a, b)$ is defined as the multiplicative inverse of a/d under modulo b/d.

Pairwise theorem

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If a divides b or b divides a, then $\Gamma(a, b) = 1$.

Compute $\Gamma(25, 110)$:

$d = \gcd(25, 110) = 5,$

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Let $a, b \in \mathbb{N}$ and $d = \gcd(a, b)$.
If a divides b or b divides a, then $\Gamma(a, b) = 1$. Otherwise,

$$\text{if } a/d \text{ is odd, then } \Gamma(a, b) = 1 \iff \Theta(b, a) \text{ is odd;}$$

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$$\text{if } a/d \text{ is even, then } \Gamma(a, b) = 1 \iff \Theta(a, b) \text{ is odd.}$$

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if a/d is even, then $\Gamma(a, b) = 1 \iff \Theta(a, b)$ is odd.

Compute $\Gamma(25, 110)$:

$d = \gcd(25, 110) = 5$,
$25/d$ is odd,
$\theta(110, 25) = 3$ is odd.

Therefore, $\Gamma(25, 110) = 1$.

The sequence $(n^k)_n$

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When $k = 4$, $\Delta((n^4)_n)$ is

$$1, 2, 1, 1, 1, 2, 2, 1, 2, 2, 1, 2, 1, 2, 1, 2, \ldots.$$

General theorem

Theorem

Fix $k \in \mathbb{N}$. Then $\Delta((n^k)_n)$ is eventually $1, 2, 1, 2, 1, 2, \ldots$.

Bound for when the pattern starts

Define

$$g(x) = \left(\sum_{i=1}^{k} x^{i-1}\right)^k \mod x^k.$$

Find M_k

If k is odd, let M_k be the smallest positive, even integer such that for all $n \geq M_k$,

$$0 \ < \ n^k - g(n) \ < \ n^k \quad \text{and} \quad 0 \ < \ g(-n) \ < \ n^k;$$

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if k is even, let M_k be the smallest positive, even integer such that for all $n \geq M_k$,

$$0 < n^k + g(-n) < n^k \quad \text{and} \quad 0 < g(n) < n^k.$$

A bound for when the pattern starts

Then the sequence $(\Gamma(n^k, (n+1)^k))_{n\geq 1}$ starts to be $1, 2, 1, 2, 1, 2, \ldots$ at $n \leq M_k + 1$.

Period k

Chu (2020) observed that $\Delta((F_n)_n)$ is eventually

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Period k

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$$1, 1, 1, 2, 2, 2, 1, 1, 1, 2, 2, 2, \ldots.$$

Given $k \in \mathbb{N}$, can we find a sequence $(a_n)_{n\geq 1}$ with

$$\Delta((a_n)_n) = \underbrace{1, \ldots, 1}_{k}, \underbrace{2, \ldots, 2}_{k}, \underbrace{1, \ldots, 1}_{k}, \underbrace{2, \ldots, 2}_{k}, \ldots ?$$

Period k

Theorem

Fix $k \in \mathbb{N}$. Then $\Delta\left(\left(\left\lceil \frac{2^{n+k-1}}{2^k+1} \right\rceil\right)_n\right)$ is

$$\underbrace{1, \ldots, 1}_{k}, \underbrace{2, \ldots, 2}_{k}, \underbrace{1, \ldots, 1}_{k}, \underbrace{2, \ldots, 2}_{k}, \ldots.$$

Step 1 of proof: construct a desired sequence

Lemma

Let $a \in \mathbb{N}$. Then the pair $(a, 2a)$ uses the first equation. If $a \geq 2$, the pair $(a, 2a-1)$ uses the second equation.

Fix $k \in \mathbb{N}$. Construct $(a_n)_{n\geq 1}$ as follows: let $a_1 := 1$, and for all integer $n \geq 1$,

$$a_{n+1} := \begin{cases} 2a_n & \text{if } n \equiv 1, \ldots, \text{or } k \pmod{2k} \\ 2a_n - 1 & \text{if } n \equiv k+1, \ldots, \text{or } 2k \pmod{2k}. \end{cases}$$

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$$\Delta((a_n)_n) = \underbrace{1, \ldots, 1}_{k}, \underbrace{2, \ldots, 2}_{k}, \underbrace{1, \ldots, 1}_{k}, \underbrace{2, \ldots, 2}_{k}, \ldots.$$

Step 2 of proof: finding the closed formula

It remains to show that for all $n \geq 1$, $a_n = \left\lceil \frac{2^{n+k-1}}{2^k+1} \right\rceil$.

Lemma

Let $k, n \in \mathbb{N}$ and suppose $n \equiv m \pmod{2k}$ $(1 \leq m \leq 2k)$. Then

$$2^{n+k-1} \equiv \begin{cases} 2^k - 2^{m-1} + 1 \pmod{2^k+1} & \text{if } 1 \leq m \leq k, \\ 2^{m-k-1} \pmod{2^k+1} & \text{if } k+1 \leq m \leq 2k. \end{cases}$$

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$$2^{n+k-1} = (2^k + 1)\left(\left\lceil \frac{2^{n+k-1}}{2^k+1} \right\rceil - 1\right) + 2^k - 2^{m-1} + 1$$

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Case 2: $k+1 \leq m \leq 2k$

$$2^{n+k-1} = (2^k+1)\left\lfloor \frac{2^{n+k-1}}{2^k+1} \right\rfloor + 2^{m-k-1}$$

$$2^{n+k} = (2^k+1)\left(2\left\lceil \frac{2^{n+k-1}}{2^k+1} \right\rceil - 1\right) + 2^{m-k} - 2^k$$

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Arithmetic sequences

Given $a, d \in \mathbb{N}$, consider the sequence $x_n = a + (n-1)d$.

Theorem

Consecutive terms of x_n use the 2 equations alternatively.

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Eg: For $x_n = 3 + 6(n-1)$,
$\Delta((x_n)_n) = 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, \ldots.$

Geometric sequences

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What about the shifted geometric sequence $x_n = ar^{n-1} + 1$?

Theorem

If $\gcd(a+1,\ r-1)$ is odd, then $\Delta((x_n)_{n\geq 2})$ is constant.
If $\gcd(a+1,\ r-1)$ is even, $\Delta((x_n)_{n\geq 2})$ alternates between 1 and 2.

Shifted geometric sequences

Lemma

$$\gcd(ar^{n-1}+1,\ ar^n+1) \ = \ \gcd(a+1,\ r-1)$$

Shifted geometric sequences

Lemma

$$\gcd(ar^{n-1}+1,\ ar^n+1) = \gcd(a+1,\ r-1)$$

Examples

- $x_n = 4.11^{n-1} + 1$; $\gcd(a+1,\ r-1) = 5$.
  $\Delta((x_n)_n) = 1, 2, 2, 2, 2, 2, \ldots$
- $x_n = 3.9^{n-1} + 1$; $\gcd(a+1,\ r-1) = 4$.
  $\Delta((x_n)_n) = 1, 1, 2, 1, 2, 1, 2, \ldots$

Thank you for listening