Biases in Second Moments of Elliptic Curves

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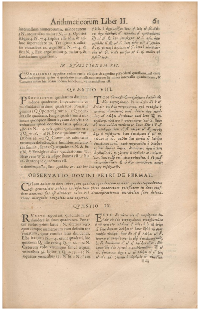
- Why Elliptic Curves?
- Definitions
- Bias conjecture
- Explicit formulae
- Counterexample

Why Elliptic Curves?

Modularity theorem for semistable elliptic curves (Andrew Wiles, 1995).

Andrew Wiles proved that elliptic curves over the field of rational numbers $\mathbb{Q}$ are related to modular forms.


Arithmeticorum Liber II. 61

IN QVAESTIONEM VII.

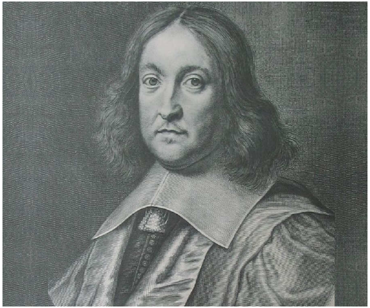
QVÆSTIO VIII.

Propositvm quadratum diuidere in duos quadratos. Imperatum sit vt 16. diuidatur in duos quadratos. Ponatur primus 1 Q. Oportet igitur 16 − 1 Q. æquales esse quadrato. Fingo quadratum a numeris quotquot libuerit, cum defectu tot vnitatum quod continet latus ipsius 16. esto 2 N. − 4. ipse igitur quadratus erit 4 Q. + 16. − 16 N. hæc æquabuntur vnitatibus 16 − 1 Q. Communis adiiciatur vtrimque defectus, & a similibus auferantur similia, fient 5 Q. æquales 16 N. & fit 1 N. $\frac{16}{5}$. Erit igitur alter quadratorum $\frac{256}{25}$. alter vero $\frac{144}{25}$ & vtriusque summa est $\frac{400}{25}$ seu 16. & vterque quadratus est.

OBSERVATIO DOMINI PETRI DE FERMAT.

Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos & generaliter nullam in infinitum vltra quadratum potestatem in duos eiusdem nominis fas est diuidere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.

QVÆSTIO IX.



Corollary. (Fermat's Last Theorem, 1637)

No three positive integers a, b, and c can satisfy the equation

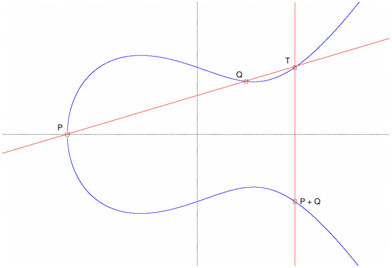
$$a^n + b^n = c^n, \qquad n \in \mathbb{N}_{\geq 3}.$$

An **elliptic curve** $\mathcal{E}$ is a non-singular curve of genus 1 of the form $y^2 = x^3 + A\,x + B$ where $A, B \in \mathbb{C}$. We may consider the set $\mathcal{E}(\mathbb{Q})$ of rational solutions of $\mathcal{E}$ plus the point at infinity $O_{\mathcal{E}}$.

Theorem. (Mordell-Weil, 1922)

Let P, Q, and $P * Q$ be points on $\mathcal{E}$ which lie on a line. Then the binary operation $P \cdot Q = \big(P * Q\big) * O_{\mathcal{E}}$ turns $\big(E(\mathbb{Q}), \cdot\big)$ into a finitely generated abelian group. In particular,

$$\mathcal{E}(\mathbb{Q}) \;\cong\; \mathcal{E}(\mathbb{Q})_{\text{torsion}} \oplus \mathbb{Z}^{\mathrm{rank}}$$



Motivating question

Can we find an elliptic curve of large rank?

Also known as twenty septendecillion sixty-seven sexdecillion seven hundred sixty-two quindecillion four hundred fifteen quattuordecillion five hundred seventy-five tredecillion five hundred twenty-six duodecillion five hundred eighty-five undecillion thirty-three decillion two hundred eight nonillion two hundred nine octillion three hundred thirty-eight septillion five hundred forty-two sextillion seven hundred fifty quintillion nine hundred thirty quadrillion two hundred thirty trillion three hundred twelve billion one hundred seventy-eight million nine hundred fifty-six thousand five hundred two

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In 2006, Noam Elkies set the record by finding an elliptic curve of rank at least 28:

$$\begin{aligned} y^2 + xy + y \;&=\; x^3 - x^2 \\ &- 20067762415575526585033208209338542750930230312178956502x \\ &+ 34481611795030556467032985690390720374855944359319180361266008296291939448732243429 \end{aligned}$$

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A *one-parameter family* of elliptic curves is given by

$$\mathcal{E} : y^2 = x^3 + A(T)x + B(T),$$

where $A(T)$, $B(T)$ are polynomials in $\mathbb{Z}[T]$.
Each specialization of T to an integer t gives an elliptic curve E_t over $\mathbb{Q}$.

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Moments of a family of elliptic curves

The r^{th} moment (note we do not normalize by $1/p$) is

$$\mathcal{A}_{r,\mathcal{E}}(p) \ = \ \sum_{t \in \mathbb{F}_p} a_{E_t}(p)^r,$$

where $a_{E_t}(p) = p + 1 - \#(\text{solutions to } E_t \bmod p)$ is the Frobenius trace of E_t.

The first moment is related to the rank of the elliptic curve family:

$\mathcal{A}_{1,\mathcal{E}}(p)$ and Family Rank (Nagao, Rosen-Silverman, 1998)

Given certain technical assumptions (Tate's Conjecture) hold for $\mathcal{E}$, then

$$\lim_{X\to\infty} \frac{1}{X} \sum_{p\le X} \mathcal{A}_{1,\mathcal{E}}(p) \frac{\log p}{p} \;=\; -\operatorname{rank} \mathcal{E}(\mathbb{Q}(T)).$$

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- By $\sum_{p\leq x} \log p \sim x$, if $\mathcal{A}_{1,\mathcal{E}(t)}(p) = -rp + O(1)$, then $\operatorname{rank} \mathcal{E}(\mathbb{Q}(T)) = r$.

Negative Bias in the First Moment

The first moment is related to the rank of the elliptic curve family:

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- By $\sum_{p\le x} \log p \sim x$, if $\mathcal{A}_{1,\mathcal{E}(t)}(p) = -rp + O(1)$, then $\operatorname{rank} \mathcal{E}(\mathbb{Q}(T)) = r$.
- The "rank" of the family means that except for finitely many t, the elliptic curve E_t has rank greater or equal to r.

Leitfaden

- Why Elliptic Curves?
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- Bias conjecture
- Explicit formulae
- Counterexample

The $j(T)$-invariant is $j(T) = 1728\frac{4A(T)^3}{4A(T)^3+27B(T)^2}$.

Second moment asymptotic (Michel, 1995)

For a one-parameter family $\mathcal{E}$ with $j(T)$-invariant non-constant, the second moment is

$$A_{2,\mathcal{E}} \;=\; p^2 + O(p^{3/2}),$$

with lower-order terms of size $p^{3/2}$, p, $p^{1/2}$, and 1.

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Strong and Weak Bias conjecture

- **Weak:** The largest lower term in the second moment expansion which does not average to 0 is on average **negative**.
- **Strong:** The largest lower term in the second moment expansion which does not average to 0 is **negative except for finitely many** p

Relation with Excess Rank

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- If we have lower order negative bias, then the bound for the average rank in families increases.
- However, lower order negative biases increases bound only by a small amount, which is not enough to explain observed excess rank.

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For a specialization $E_t : y^2 = x^3 + A(t)x + B(t)$, we may write

$$a_{E_t}(p) = -\sum_{x \in \mathbb{F}_p} \left(\frac{x^3 + A(t)x + B(t)}{p} \right)$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol mod p given by

$$\left(\frac{x}{p}\right) = \begin{cases} 1 & x \text{ a non-zero square modulo } p, \\ 0 & x \equiv 0 \mod p, \\ -1 & \text{otherwise.} \end{cases}$$

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Observe that $\left(\frac{x}{p}\right) + 1$ is precisely the number of solutions to $x = y^2 \pmod{p}$.

Lemmas on Legendre Symbols

Linear and quadratic Legendre sums

We have the following

$$\sum_{x(p)} \left(\frac{ax+b}{p}\right) = 0 \qquad p \nmid a,$$

$$\sum_{x(p)} \left(\frac{ax^2+bx+c}{p}\right) = \begin{cases} -\left(\frac{a}{p}\right) & p \nmid b^2-4ac, \\ (p-1)\left(\frac{a}{p}\right) & p \mid b^2-4ac. \end{cases}$$

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Average values of Legendre symbols

Taking the limit of the average of the Legendre symbol over all primes gives

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} \left(\frac{x}{p}\right) = \begin{cases} 1 & x \text{ a non-zero square,} \\ 0 & \text{otherwise.} \end{cases}$$

- The moments become intractible when $A(T)$ and $B(T)$ have high degree.
- For the following special families, the following is known:

Family	$A_{1,\mathcal{E}}(p)$	$A_{2,\mathcal{E}}(p)$
$y^2 = x^3 + 2^4(-3)^3(9T+1)^2$	0	$\begin{cases} 2p^2-2p & p\equiv 2 \bmod 3 \\ 0 & p\equiv 1 \bmod 3 \end{cases}$
$y^2 = x^3 \pm 4(4T+2)x$	0	$\begin{cases} 2p^2-2p & p\equiv 1 \bmod 4 \\ 0 & p\equiv 3 \bmod 4 \end{cases}$
$y^2 = x^3 + (T+1)x^2 + Tx$	0	$p^2 - 2p - 1$
$y^2 = x^3 + x^2 + 2T + 1$	0	$p^2 - 2p - -3$
$y^2 = x^3 + Tx^2 + 1$	$-p$	$p^2 - n_{3,2,p}p - 1 + c_{3/2}(p)$
$y^2 = x^3 - T^2x + T^2$	$-2p$	$p^2 - p - c_1(p) - c_0(p)$
$y^2 = x^3 - T^2x + T^4$	$-2p$	$p^2 - p - c_1(p) - c_0(p)$
$y^2 = x^3 + Tx^2 - (T+3)x + 1$	$-2c_{p,1;4}p$	$p^2 - 4c_{p,1;6}p - 1$

where $c_{p,a;m} = 1$ if $p \equiv a \bmod m$ and 0 otherwise; $n_{3,2,p}$ is the number of cubes roots of 2 mod p; $c_\alpha(p)$ are certain legendre sums multiplied by p.

Example

Consider $\mathcal{F} : y^2 = x^3 - T^2x + T^4$. Then the first moment is

$$\begin{aligned} \mathcal{A}_{1,\mathcal{F}}(p) &= \sum_{T \in \mathbb{F}_p} a_{p,E_t} \\ &= -\sum_{t \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_p} \left(\frac{x^3 - t^2x + t^4}{p} \right). \end{aligned}$$

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In general, quartic Legendre sums are intractible.
But we may apply the clever substitution $x \mapsto tx$ which gives

$$\begin{aligned} &= -\sum_{x\in\mathbb{F}_p}\left(\frac{x^3}{p}\right) - \sum_{t\not\equiv 0(p)}\sum_{x\in\mathbb{F}_p}\left(\frac{t^3x^3 - t^3x + t^4}{p}\right) \\ &= -\sum_{t\not\equiv 0\in\mathbb{F}_p}\sum_{x\in\mathbb{F}_p}\left(\frac{tx^3 - tx + t^2}{p}\right).\end{aligned}$$

So, we obtained a closed-form expression.

- We computationally evaluated second moments of various families of elliptic curves.
- By Michel's theorem, we assume that

$$\mathcal{A}_{2,\mathcal{E}}(p) = p^2 + \alpha(p)p^{3/2} + \beta(p)p + O(p^{1/2})$$

where $\alpha(p)$ and $\beta(p)$ are $O(1)$. To investigate the $\alpha(p)$ coefficient, we graphed the *bias* of the second moment

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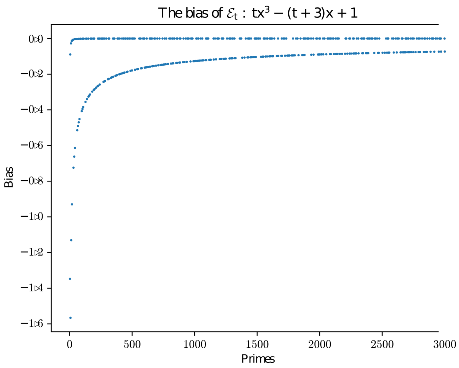
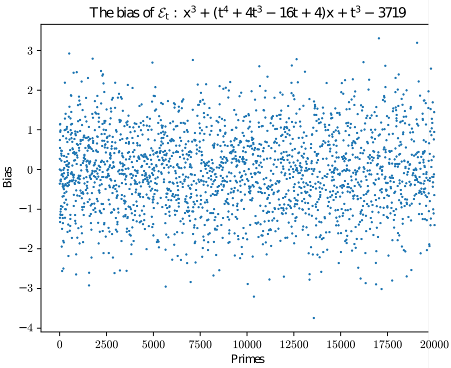
Bias

We compute the *bias* of $\mathcal{A}_{2,\mathcal{E}}$ defined by

$$\mathcal{B}_{\mathcal{E}}(p) \;=\; \frac{\mathcal{A}_{2,\mathcal{E}} - p^2}{p^{3/2}}.$$

Here are two examples for the graph of the biases, one for a tractable family, and one for not





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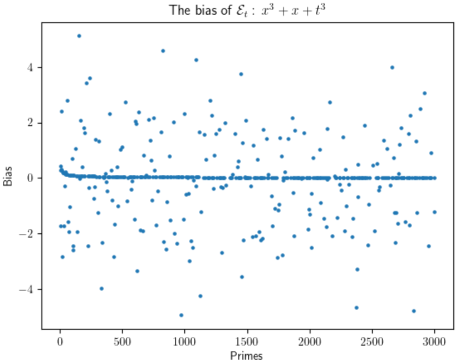

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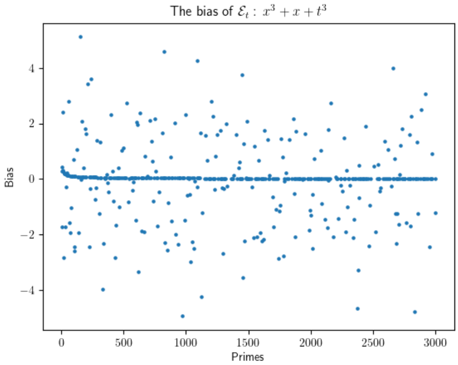

The graph indicates a clear line where the bias is positive, compared to the graphs in the previous slides.

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$$\begin{aligned}
\mathcal{A}_{2,\mathcal{F}}(p) &= \sum_{t \in \mathbb{F}_p} \sum_{x,y \in \mathbb{F}_p} \left(\frac{x^3 + x + t^3}{p}\right)\left(\frac{y^3 + y + t^3}{p}\right) \\
&= \sum_{t \in \mathbb{F}_p} \sum_{x,y \in \mathbb{F}_p} \left(\frac{x^3 + x + t}{p}\right)\left(\frac{y^3 + y + t}{p}\right) \\
&= \mathcal{A}_{2,\tilde{\mathcal{F}}}(p) \;=\; p^2 - \left(\frac{-3}{p}\right)p \;=\; p^2 + p.
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\end{aligned}$$

Bias Revisited

We graph the *bias* of $\mathcal{A}_{2,\mathcal{E}}$, for calculated values, defined by

$$\mathcal{B}_{\mathcal{E}}(p) \;=\; \frac{\mathcal{A}_{2,\mathcal{E}} - p^2}{p^{3/2}}.$$

Recall by Michel's theorem, we have

$$\mathcal{A}_{2,\mathcal{E}}(p) = p^2 + \alpha(p)p^{3/2} + \beta(p)p + O(p^{1/2})$$

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$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leq x} \alpha(p) = 0.$$

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- Show that $\beta(p)$ averages to a positive number.

By the prime number theorem, one shows

$$\frac{1}{\pi(x)}\sum_{p\leq x}\alpha(p) \;=\; \frac{1}{\pi(x)}\sum_{p\leq x}\mathcal{B}_{\mathcal{E}}(p)+O(x^{-1/2}\log x)$$

Problem: The constant in the big O term might dominate.

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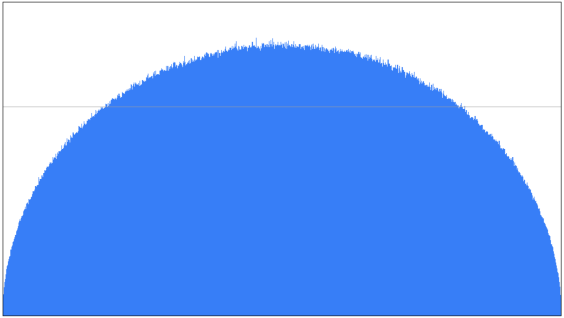
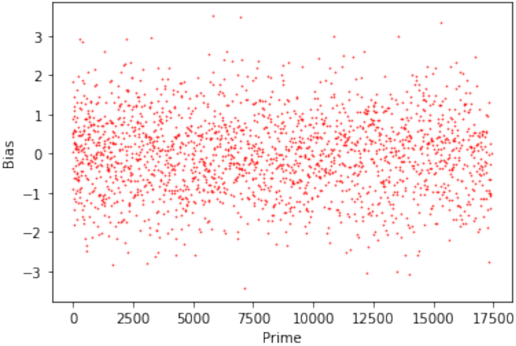
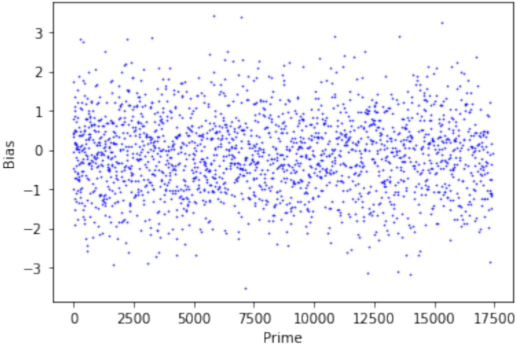
Problem: The constant in the big O term might dominate.
Solution: Randomly simulate elliptic moments using the *Sato-Tate distribution*.



The following are two graphs which randomly simulate the bias.
One graph has coefficient $\alpha(p) = -.1$ and the other has $\alpha(p) = 0$.
Can you guess which is which?





Taking the running average of the biases, it is clear there is a bias:

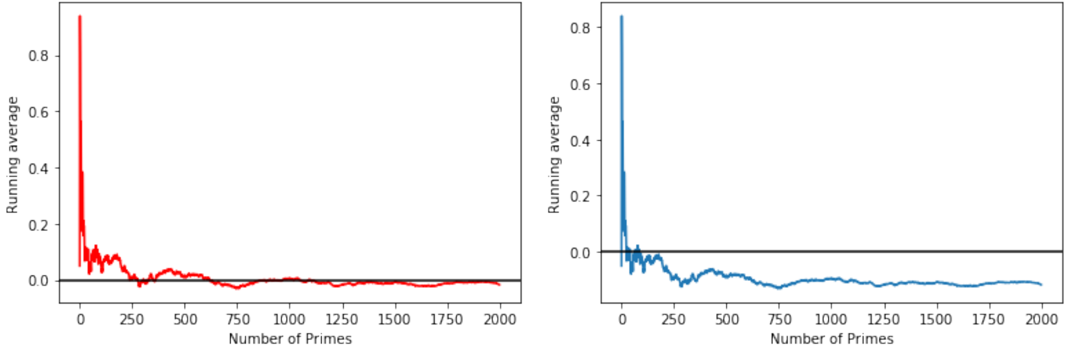
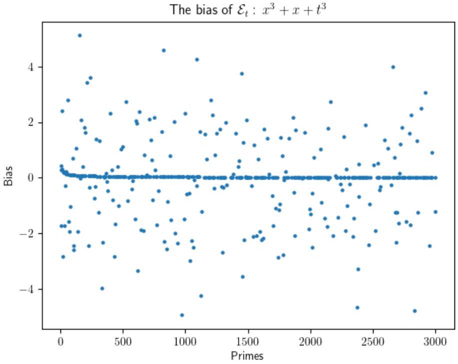
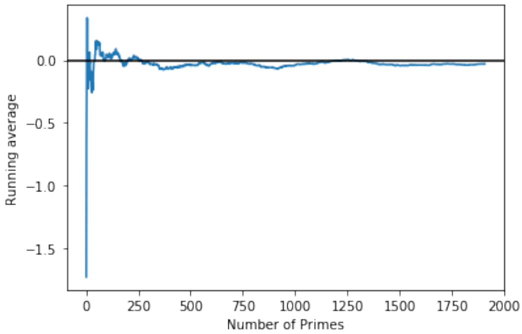


Figure: Unbiased Running Averages (Red) versus Biased Running Averages (Blue) for a random simulation

Doing the same with our family of interest, that is, $y^2 = x^3 + x + t^3$, we get





So we have strong computational evidence the largest term averages to 0.

Acknowledgements

Special thanks to Professor Steven J. Miller and the Churchill Foundation.

Thanks to our SMALL 2023 faculty, research assistants, and peers for their support.

Thanks to the National Science Foundation for making SMALL 2023 possible.

A Curiosity

The conjectured first moment of $y^2 = x^3 + x + t^3$

The first moment $\mathcal{A}_{1,p}$ satisfies

$$|\mathcal{A}_{1,p}| = \begin{cases} 4p & p \text{ is of the form } a^2 + 36b^2, \\ 0 & \text{otherwise.} \end{cases}$$

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For a prime $p \not\equiv 1(12)$, the Chinese remainder theorem in conjunction with the changes of variable

$$t \mapsto tx, \quad \text{and} \quad t \mapsto t^3 \implies \mathcal{A}_{1,p} = 0.$$

The conjectured first moment of $y^2 = x^3 + x + t^3$

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Using binary quadratic forms, $\mathcal{A}_{1,p} \neq 0$ forces p to be of the form

$$p = a^2 + 36b^2 \quad \text{or} \quad p = 4a^2 + 9b^2.$$

We computationally found $|\mathcal{A}_{1,p}| = 4p$ in the former and $\mathcal{A}_{1,p} = 0$ in the latter.