Low-lying zeros of GL(2) L-functions

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Introduction

Why study zeros of L-functions?

- Infinitude of primes, primes in arithmetic progression.
- Chebyshev's bias: $\pi_{3,4}(x) \geq \pi_{1,4}(x)$ 'most' of the time.
- Birch and Swinnerton-Dyer conjecture.
- Goldfeld, Gross-Zagier: bound for $h(D)$ from L-functions with many central point zeros.
- Even better estimates for $h(D)$ if a positive percentage of zeros of $\zeta(s)$ are at most $1/2 - \epsilon$ of the average spacing to the next zero.

Distribution of zeros

- $\zeta(s), L(s,\chi) \neq 0$ for $\mathfrak{Re}(s) = 1$: $\pi(x)$, $\pi_{a,q}(x)$.
- GRH: error terms.
- GSH: Chebyshev's bias.
- Analytic rank, adjacent spacings: $h(D)$.

Goals

- Determine correct scale and statistics to study zeros of L-functions.
- See similar behavior in different systems (random matrix theory).
- Discuss the tools and techniques needed to prove the results.
- State new GL(2) results, sketch proofs in simple families.

Fundamental Problem: Spacing Between Events

General Formulation: Studying system, observe values at $t_1, t_2, t_3, \ldots$.

Question: What rules govern the spacings between the t_i?

Examples:

- Spacings b/w Energy Levels of Nuclei.
- Spacings b/w Eigenvalues of Matrices.
- Spacings b/w Primes.
- Spacings b/w $n^k \alpha$ mod 1.
- Spacings b/w Zeros of L-functions.

Sketch of proofs

In studying many statistics, often three key steps:

1. Determine correct scale for events.
2. Develop an explicit formula relating what we want to study to something we understand.
3. Use an averaging formula to analyze the quantities above.

It is not always trivial to figure out what is the correct statistic to study!

Classical Random Matrix Theory

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Heavy nuclei (Uranium: 200+ protons / neutrons) worse!

Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation:

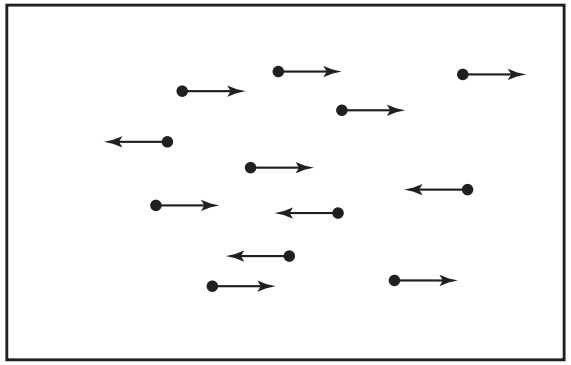
$$H\psi_n = E_n\psi_n$$

H : matrix, entries depend on system

E_n : energy levels

ψ_n : energy eigenfunctions

Origins of Random Matrix Theory



- Statistical Mechanics: for each configuration, calculate quantity (say pressure).
- Average over all configurations – most configurations close to system average.
- Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices (real Symmetric $A = A^T$, complex Hermitian $\overline{A}^T = A$).

Classical Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix p, define

$$\mathrm{Prob}(A) = \prod_{1 \le i \le j \le N} p(a_{ij}).$$

This means

$$\mathrm{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1 \le i \le j \le N} \int_{x_{ij} = \alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Want to understand eigenvalues of A.

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0:
$\int f(x)\delta(x - x_0)dx = f(x_0)$.

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$$\mu_{A,N}(x) = \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right)$$

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\int_a^b \mu_{A,N}(x)dx &= \frac{\#\left\{\lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a,b]\right\}}{N} \\
\text{k}^{\text{th}} \text{ moment} &= \frac{\sum_{i=1}^{N}\lambda_i(A)^k}{2^k N^{\frac{k}{2}+1}} = \frac{\mathrm{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}.
\end{aligned}$$

Wigner's Semi-Circle Law

Not most general case, gives flavor.

Wigner's Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$ with mean 0, variance 1, and other moments finite. Then for almost all A, as $N \to \infty$

$$\mu_{A,N}(x) \longrightarrow \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

SKETCH OF PROOF: Eigenvalue Trace Lemma

Want to understand the eigenvalues of A, but it is the matrix elements that are chosen randomly and independently.

Eigenvalue Trace Lemma

Let A be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\mathrm{Trace}(A^k) = \sum_{n=1}^{N} \lambda_i(A)^k,$$

where

$$\mathrm{Trace}(A^k) = \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_N i_1}.$$

SKETCH OF PROOF: Correct Scale

$$\mathrm{Trace}(A^2) = \sum_{i=1}^{N} \lambda_i(A)^2.$$

By the Central Limit Theorem:

$$\mathrm{Trace}(A^2) = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}a_{ji} = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^2 \sim N^2$$

$$\sum_{i=1}^{N} \lambda_i(A)^2 \sim N^2$$

Gives $N\,\mathrm{Ave}(\lambda_i(A)^2) \sim N^2$ or $\mathrm{Ave}(\lambda_i(A)) \sim \sqrt{N}$.

SKETCH OF PROOF: Averaging Formula

Recall k-th moment of $\mu_{A,N}(x)$ is $\mathrm{Trace}(A^k)/2^k N^{k/2+1}$.

Average k-th moment is

$$\int \cdots \int \frac{\mathrm{Trace}(A^k)}{2^k N^{k/2+1}} \prod_{i \le j} p(a_{ij}) da_{ij}.$$

Proof by method of moments: Two steps

- Show average of k-th moments converge to moments of semi-circle as $N \to \infty$;
- Control variance (show it tends to zero as $N \to \infty$).

SKETCH OF PROOF: Averaging Formula for Second Moment

Substituting into expansion gives

$$\frac{1}{2^2N^2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ji}^2\cdot p(a_{11})da_{11}\cdots p(a_{NN})da_{NN}$$

Integration factors as

$$\int_{a_{ij}=-\infty}^{\infty}a_{ij}^2p(a_{ij})da_{ij}\quad\cdot\prod_{\substack{(k,l)\neq(i,j)\\k<l}}\int_{a_{kl}=-\infty}^{\infty}p(a_{kl})da_{kl}\;=\;1.$$

Higher moments involve more advanced combinatorics (Catalan numbers).

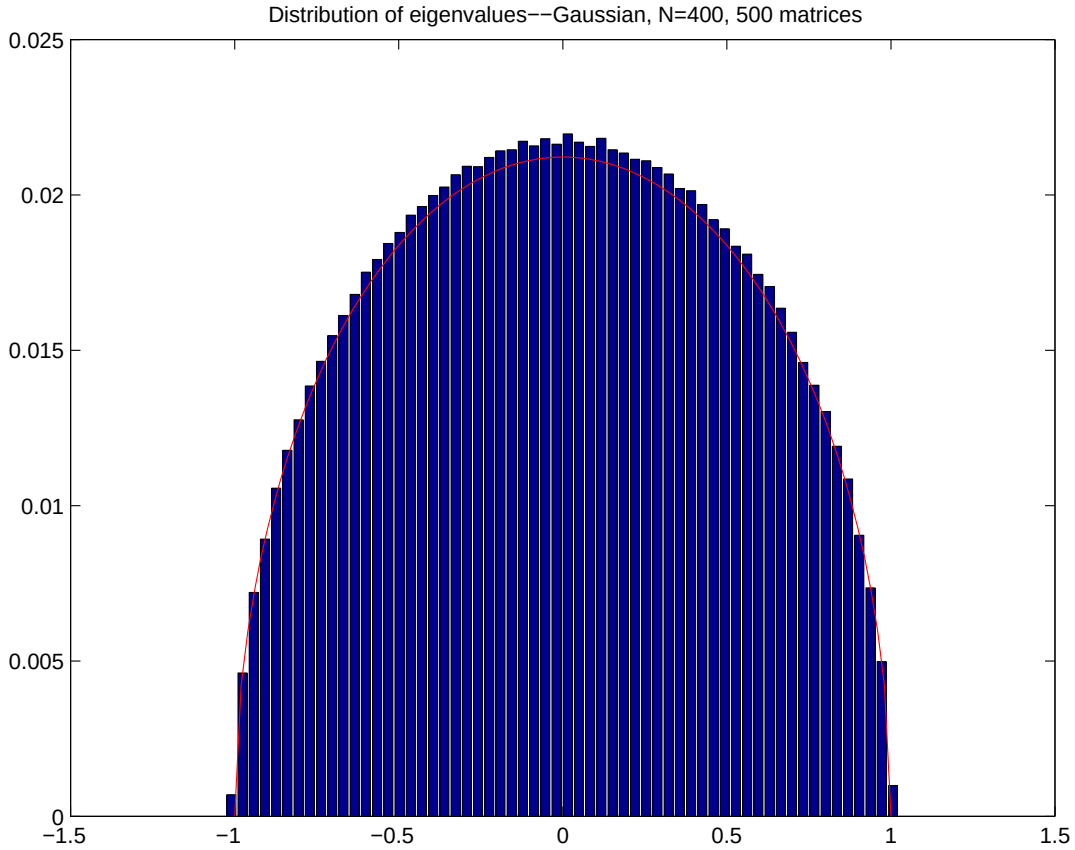
SKETCH OF PROOF: Averaging Formula for Higher Moments

Higher moments involve more advanced combinatorics (Catalan numbers).

$$\frac{1}{2^k N^{k/2+1}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} \cdots a_{i_k i_1} \cdot \prod_{i \leq j} p(a_{ij}) da_{ij}.$$

Main term $a_{i_\ell i_{\ell+1}}$'s matched in pairs, not all matchings contribute equally (if did have Gaussian, see in Real Symmetric Palindromic Toeplitz matrices; interesting results for circulant ensembles (joint with Gene Kopp, Murat Kologlu).

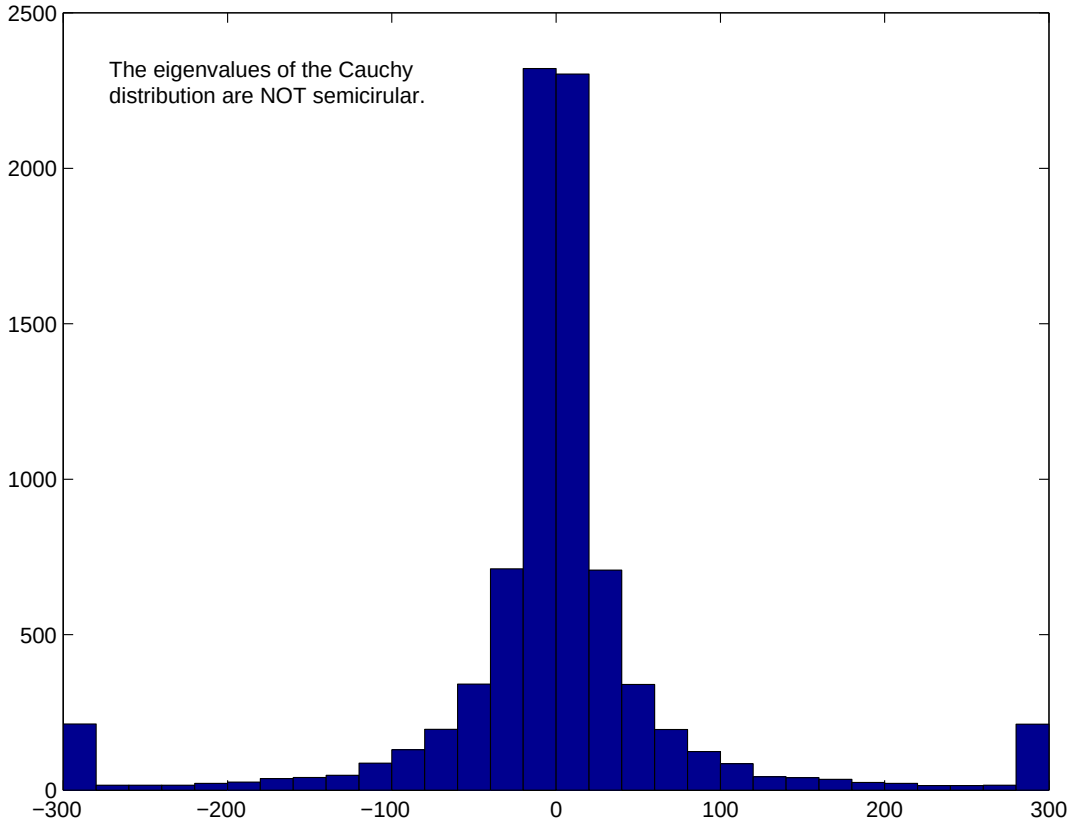
Numerical examples



500 Matrices: Gaussian 400×400

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Numerical examples



Cauchy Distribution: $p(x) = \frac{1}{\pi(1+x^2)}$

GOE Conjecture

GOE Conjecture:

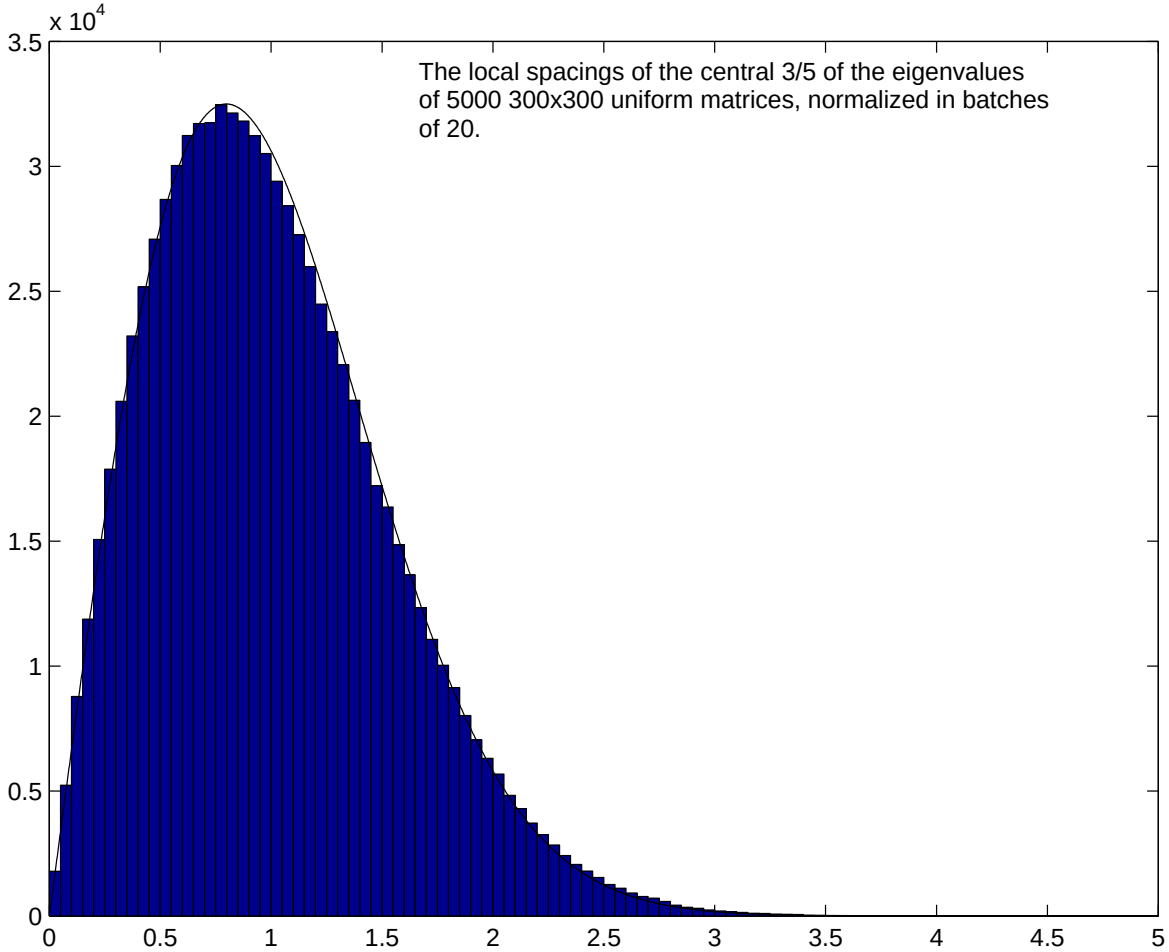
As $N \to \infty$, the probability density of the spacing b/w consecutive normalized eigenvalues approaches a limit independent of p.

Until recently only known if p is a Gaussian.

$$\mathrm{GOE}(x) \approx \frac{\pi}{2} x e^{-\pi x^2/4}.$$

Numerical Experiment: Uniform Distribution

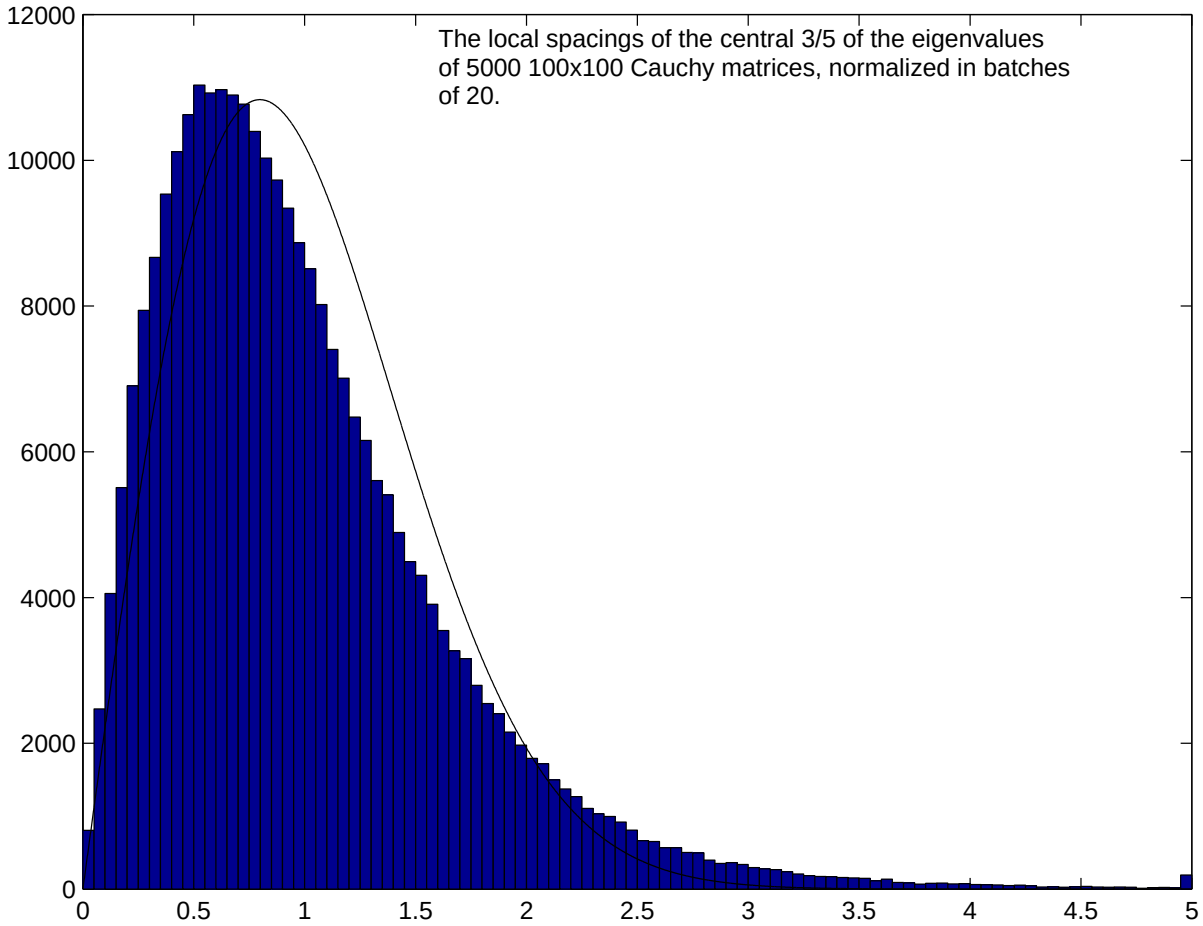
Let $p(x) = \frac{1}{2}$ for $|x| \leq 1$.



5000: 300×300 uniform on $[-1, 1]$

Cauchy Distribution

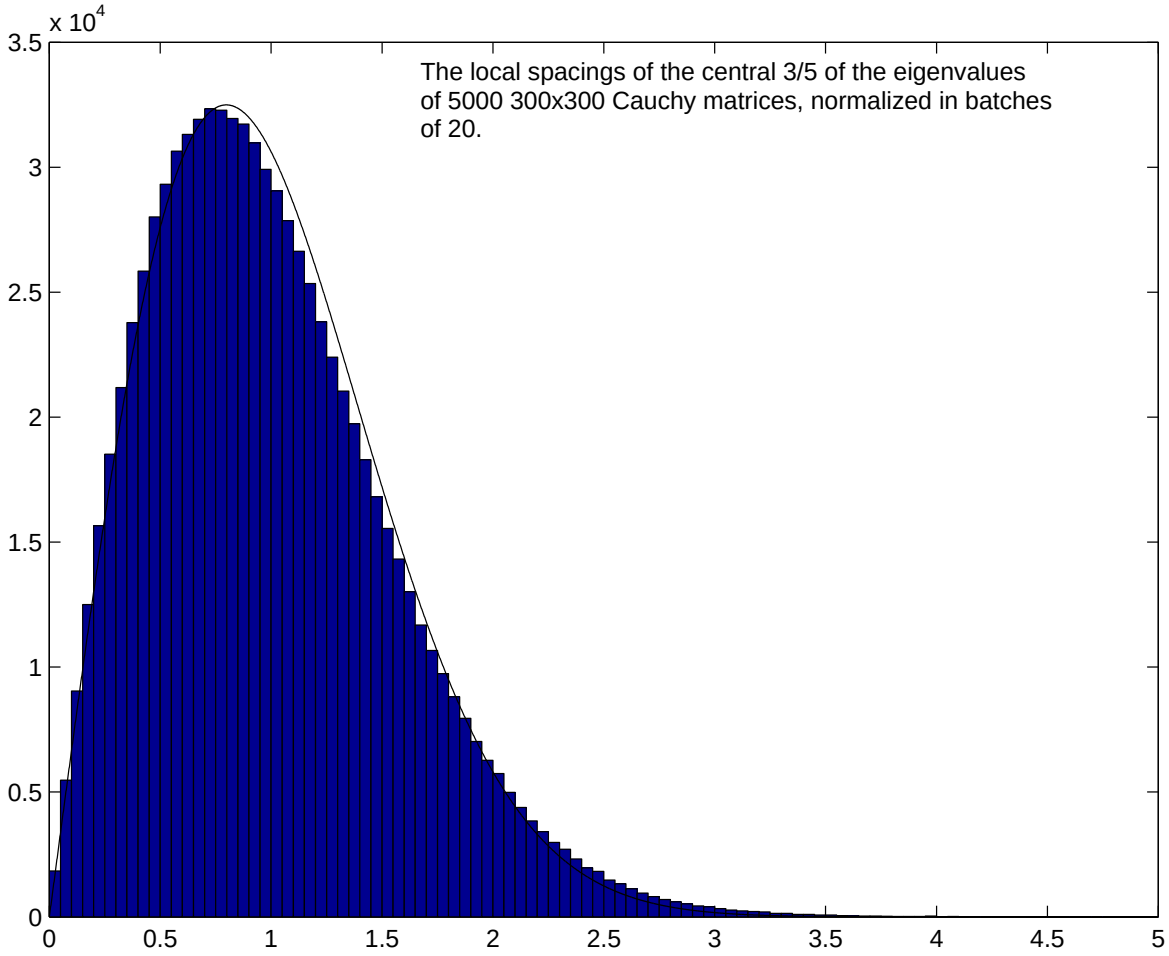
Let $p(x) = \frac{1}{\pi(1+x^2)}$.



5000: 100×100 Cauchy

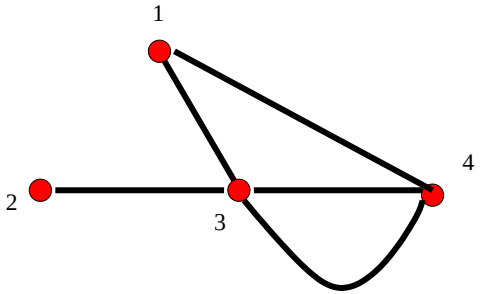
Cauchy Distribution

Let $p(x) = \frac{1}{\pi(1+x^2)}$.



5000: 300×300 Cauchy

Random Graphs



Degree of a vertex $=$ number of edges leaving the vertex.
Adjacency matrix: a_{ij} $=$ number edges b/w Vertex i and Vertex j.

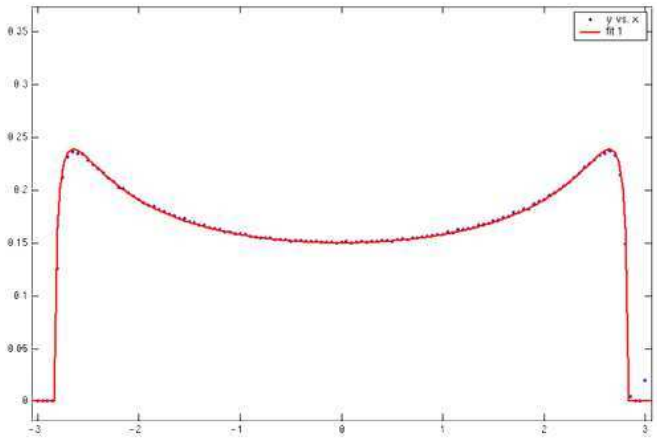
$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix}$$

These are Real Symmetric Matrices.

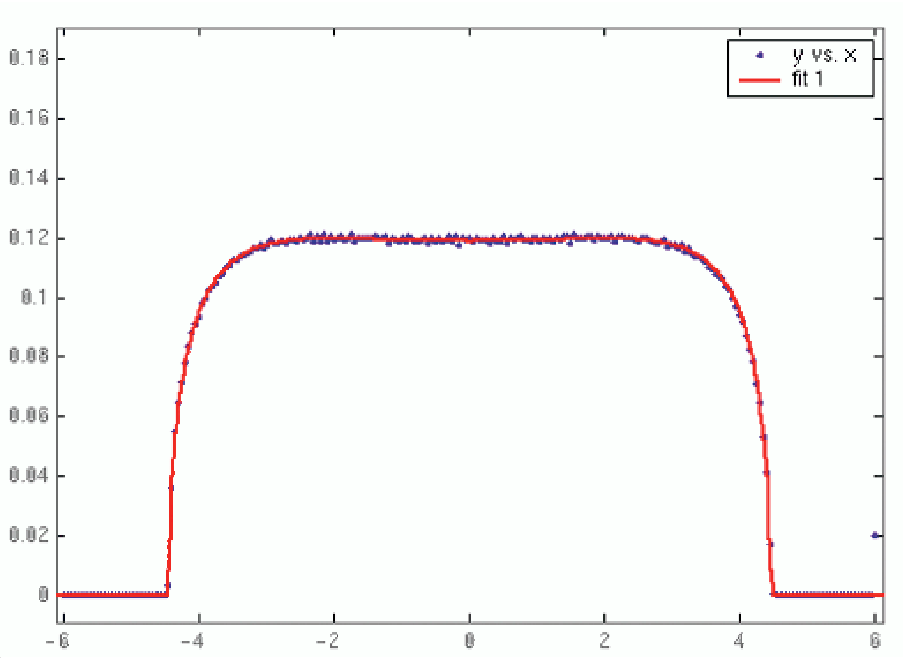
McKay's Law (Kesten Measure) with $d = 3$

Density of Eigenvalues for d-regular graphs

$$f(x) = \begin{cases} \frac{d}{2\pi(d^2-x^2)}\sqrt{4(d-1)-x^2} & |x| \leq 2\sqrt{d-1} \\ 0 & \text{otherwise.} \end{cases}$$



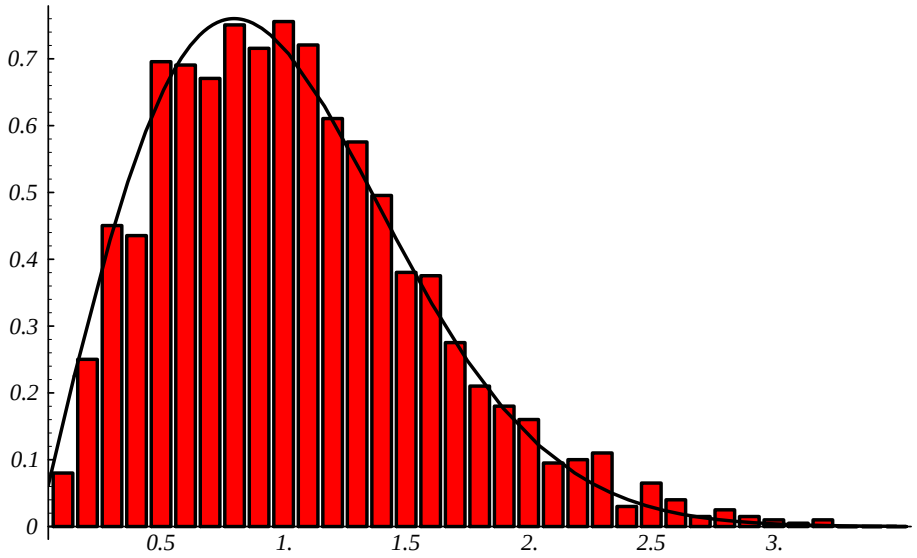
McKay's Law (Kesten Measure) with $d = 6$



Fat Thin: fat enough to average, thin enough to get something different than semi-circle (though as $d \to \infty$ recover semi-circle).

3-Regular Graph with 2000 Vertices: Comparison with the GOE

Spacings between eigenvalues of 3-regular graphs and the GOE:



Block Circulant Ensemble
(with Murat Koloğlu, Gene Kopp, Fred Strauch and Wentao Xiong)

The Ensemble of m-Block Circulant Matrices

Symmetric matrices periodic with period m on wrapped diagonals, i.e., symmetric block circulant matrices.

8-by-8 real symmetric 2-block circulant matrix:

$$\left(\begin{array}{cc|cc|cc|cc}
c_0 & c_1 & c_2 & c_3 & c_4 & d_3 & c_2 & d_1 \\
c_1 & d_0 & d_1 & d_2 & d_3 & d_4 & c_3 & d_2 \\
\hline
c_2 & d_1 & c_0 & c_1 & c_2 & c_3 & c_4 & d_3 \\
c_3 & d_2 & c_1 & d_0 & d_1 & d_2 & d_3 & d_4 \\
\hline
c_4 & d_3 & c_2 & d_1 & c_0 & c_1 & c_2 & c_3 \\
d_3 & d_4 & c_3 & d_2 & c_1 & d_0 & d_1 & d_2 \\
\hline
c_2 & c_3 & c_4 & d_3 & c_2 & d_1 & c_0 & c_1 \\
d_1 & d_2 & d_3 & d_4 & c_3 & d_2 & c_1 & d_0
\end{array}\right).$$

Choose distinct entries i.i.d.r.v.

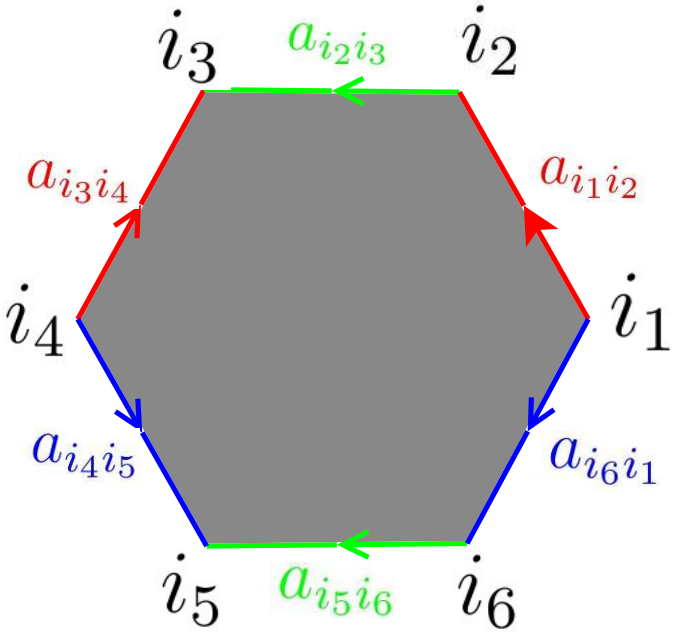
Oriented Matchings and Dualization

Compute moments of eigenvalue distribution (as m stays fixed and $N \to \infty$) using the combinatorics of pairings. Rewrite:

$$\begin{aligned} M_n(N) &= \frac{1}{N^{\frac{n}{2}+1}} \sum_{1 \le i_1, \ldots, i_n \le N} \mathbb{E}(a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_n i_1}) \\ &= \frac{1}{N^{\frac{n}{2}+1}} \sum_{\sim} \eta(\sim) m_{d_1(\sim)} \cdots m_{d_l(\sim)}. \end{aligned}$$

where the sum is over oriented matchings on the edges $\{(1,2),(2,3),\ldots,(n,1)\}$ of a regular n-gon.

Oriented Matchings and Dualization

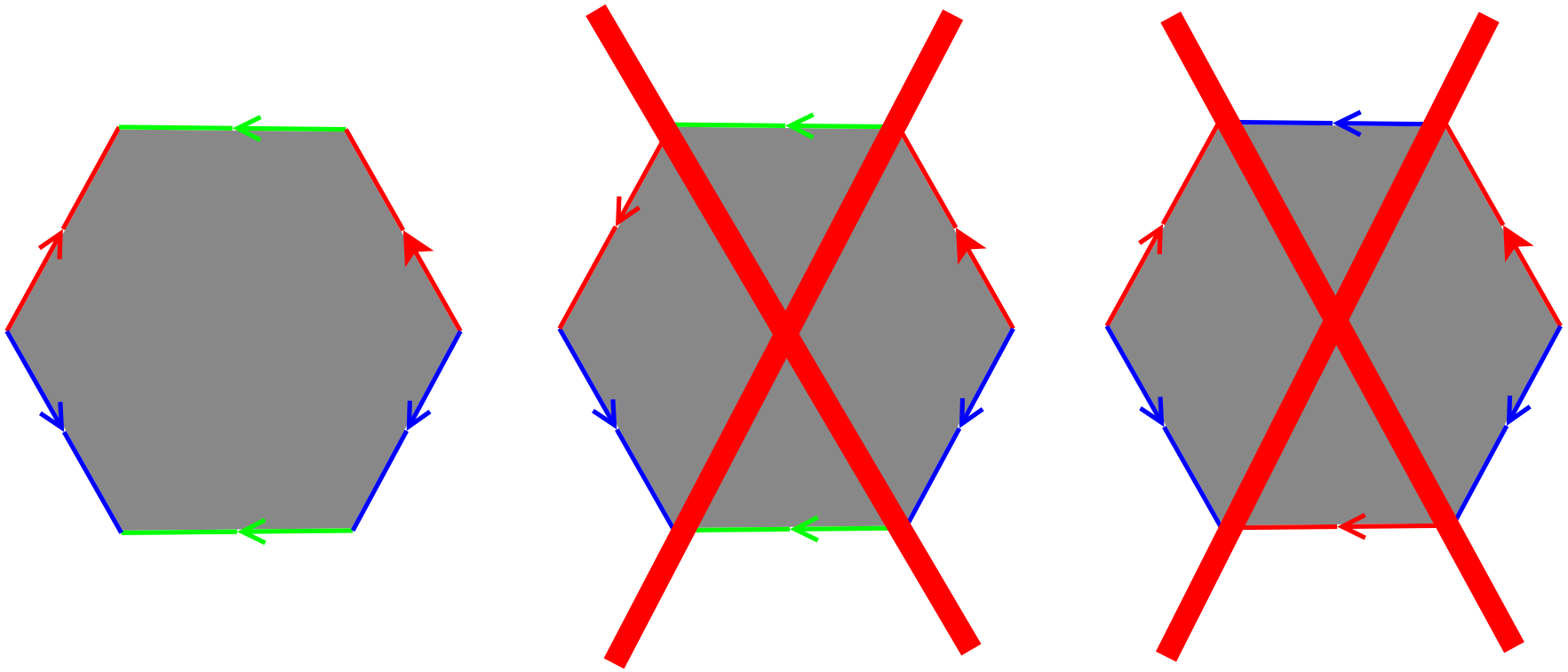


$$\left(\begin{array}{cc|cc|cc|cc} c_0 & c_1 & c_2 & c_3 & c_4 & d_3 & c_2 & d_1 \\ c_1 & d_0 & d_1 & d_2 & d_3 & d_4 & c_3 & d_2 \\ \hline c_2 & d_1 & c_0 & c_1 & c_2 & c_3 & c_4 & d_3 \\ c_3 & d_2 & c_1 & d_0 & d_1 & d_2 & d_3 & d_4 \\ \hline c_4 & d_3 & c_2 & d_1 & c_0 & c_1 & c_2 & c_3 \\ d_3 & d_4 & c_3 & d_2 & c_1 & d_0 & d_1 & d_2 \\ \hline c_2 & c_3 & c_4 & d_3 & c_2 & d_1 & c_0 & c_1 \\ d_1 & d_2 & d_3 & d_4 & c_3 & d_2 & c_1 & d_0 \end{array}\right)$$

Figure: An oriented matching in the expansion for $M_n(N) = M_6(8)$.

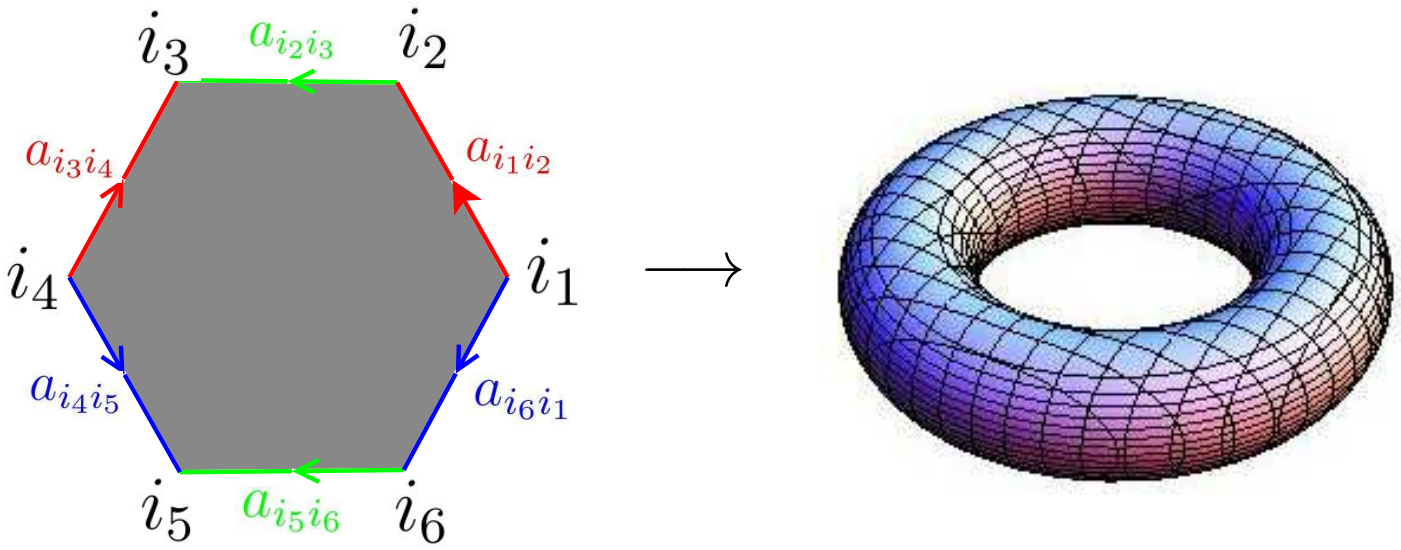
Contributing Terms

As $N \to \infty$, the only terms that contribute to this sum are those in which the entries are matched in pairs and with opposite orientation.



Only Topology Matters

Think of pairings as topological identifications; the contributing ones give rise to orientable surfaces.



Contribution from such a pairing is m^{-2g}, where g is the genus (number of holes) of the surface. Proof: combinatorial argument involving Euler characteristic.

Computing the Even Moments

Theorem: Even Moment Formula

$$M_{2k} = \sum_{g=0}^{\lfloor k/2 \rfloor} \varepsilon_g(k) m^{-2g} + O_k\left(\frac{1}{N}\right),$$

with $\varepsilon_g(k)$ the number of pairings of the edges of a $(2k)$-gon giving rise to a genus g surface.

J. Harer and D. Zagier (1986) gave generating functions for the $\varepsilon_g(k)$.

Harer and Zagier

$$\sum_{g=0}^{\lfloor k/2 \rfloor} \varepsilon_g(k) r^{k+1-2g} = (2k-1)!!\, c(k,r)$$

where

$$1 + 2\sum_{k=0}^{\infty} c(k,r) x^{k+1} = \left(\frac{1+x}{1-x}\right)^r.$$

Thus, we write

$$M_{2k} = m^{-(k+1)}(2k-1)!!\, c(k,m).$$

A multiplicative convolution and Cauchy's residue formula yield the characteristic function of the distribution.

$$\begin{aligned}
\phi(t) &= \sum_{k=0}^{\infty} \frac{(it)^{2k} M_{2k}}{(2k)!} = \frac{1}{m} \sum_{k=0}^{\infty} \frac{(-t^2/2m)^k}{k!} c(k,m) \\
&= \frac{1}{2\pi i m} \oint_{|z|=2} \frac{1}{2z^{-1}} \left(\left(\frac{1+z^{-1}}{1-z^{-1}} \right)^m - 1 \right) e^{-t^2 z/2m} \frac{dz}{z} \\
&= \frac{1}{m} e^{\frac{-t^2}{2m}} \sum_{\ell=1}^{m} \binom{m}{\ell} \frac{1}{(\ell-1)!} \left(\frac{-t^2}{m} \right)^{\ell-1}
\end{aligned}$$

Results

Fourier transform and algebra yields

Theorem: Koloğlu, Kopp and Miller

The limiting spectral density function $f_m(x)$ of the real symmetric m-block circulant ensemble is given by the formula

$$f_m(x) = \frac{e^{-\frac{mx^2}{2}}}{\sqrt{2\pi m}} \sum_{r=0}^{m} \frac{1}{(2r)!} \sum_{s=0}^{m-r} \binom{m}{r+s+1} \frac{(2r+2s)!}{(r+s)!s!} \left(-\frac{1}{2}\right)^s (mx^2)^r.$$

As $m \to \infty$, the limiting spectral densities approach the semicircle distribution.

Results

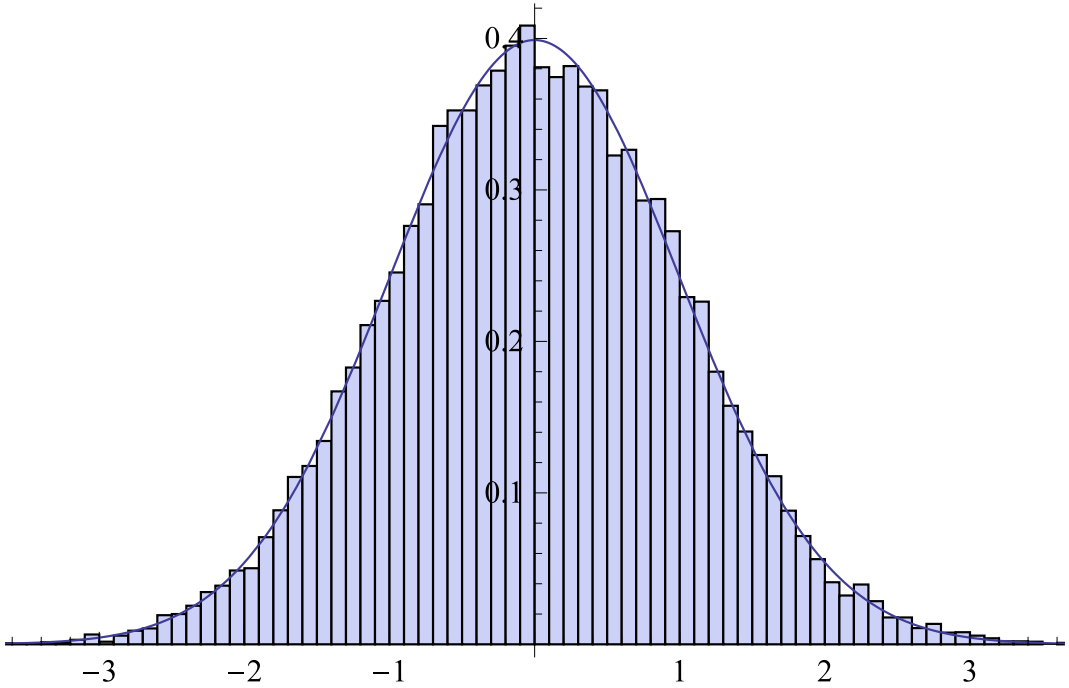


Figure: Plot for f_1 and histogram of eigenvalues of 100 circulant matrices of size 400×400.

Results

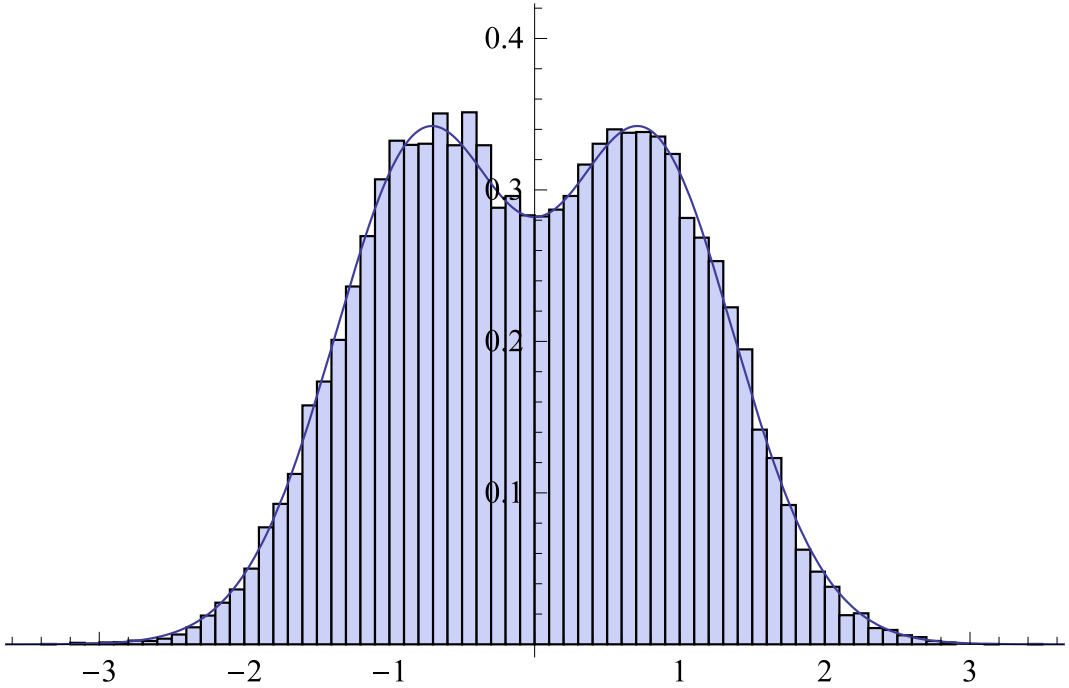


Figure: Plot for f_2 and histogram of eigenvalues of 100 2-block circulant matrices of size 400×400.

Results

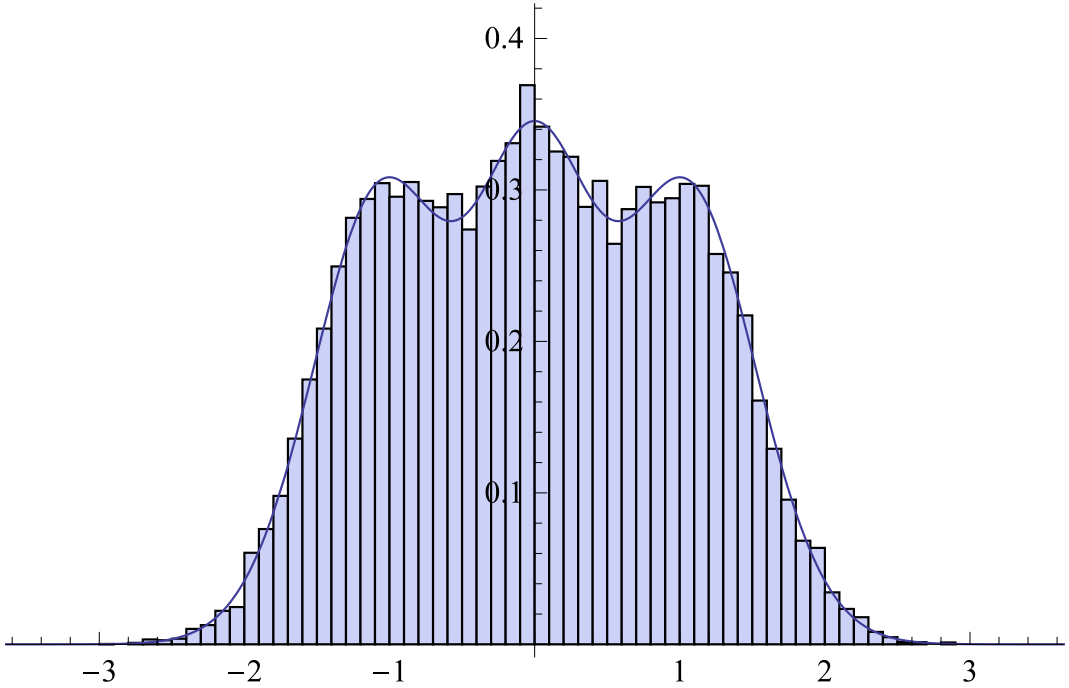


Figure: Plot for f_3 and histogram of eigenvalues of 100 3-block circulant matrices of size 402×402.

Results

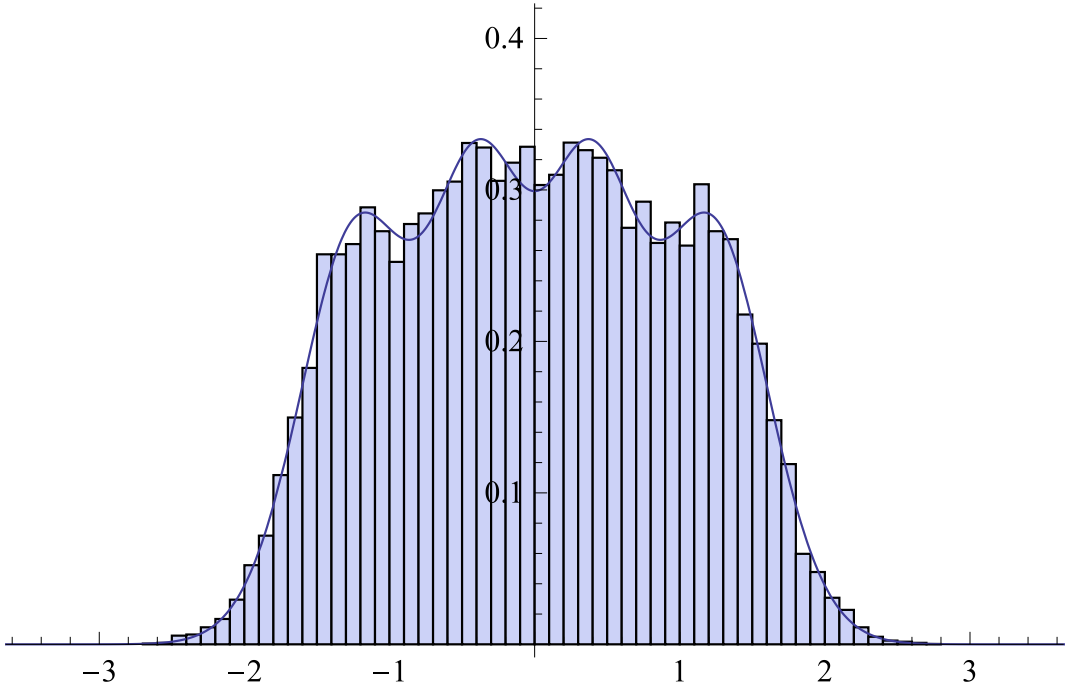


Figure: Plot for f_4 and histogram of eigenvalues of 100 4-block circulant matrices of size 400×400.

Results

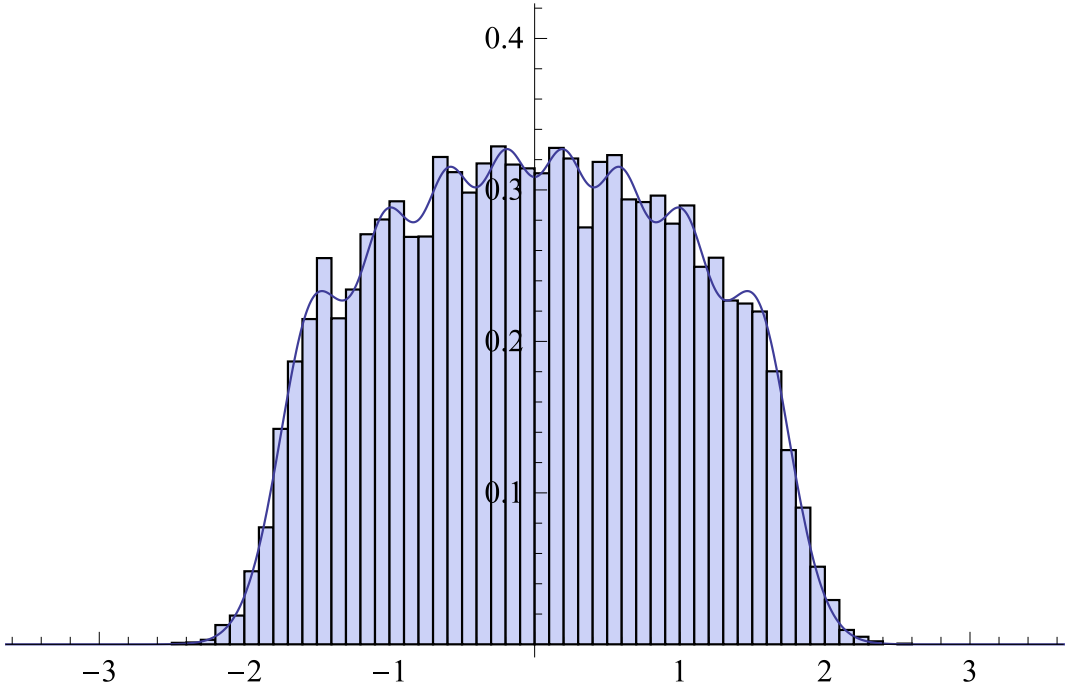


Figure: Plot for f_8 and histogram of eigenvalues of 100 8-block circulant matrices of size 400×400.

Results

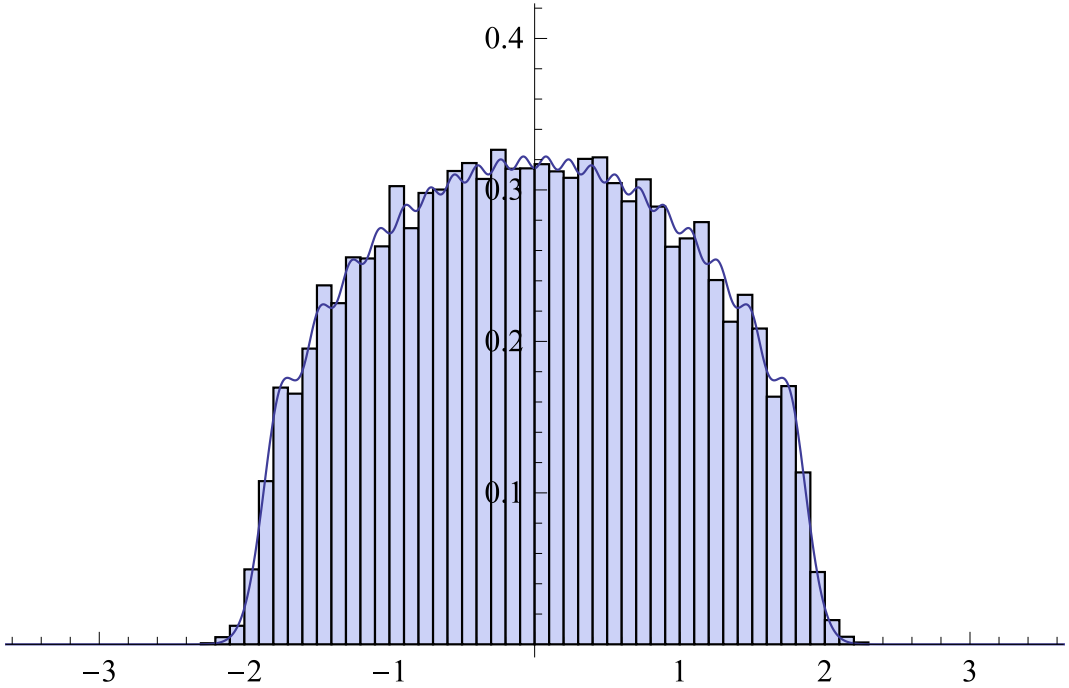


Figure: Plot for f_{20} and histogram of eigenvalues of 100 20-block circulant matrices of size 400×400.

Results

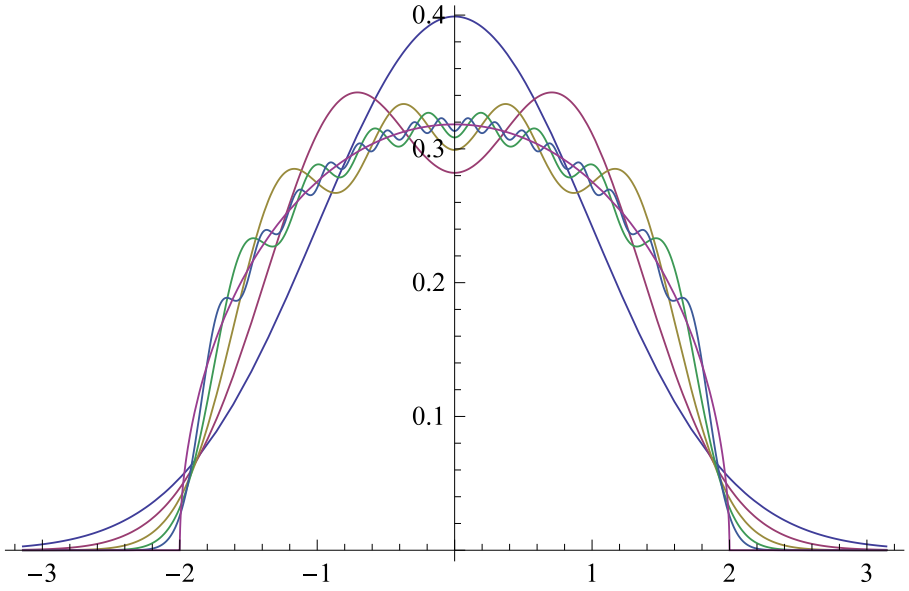


Figure: Plot of convergence to the semi-circle.

The Limiting Spectral Measure for Ensembles of Symmetric Block Circulant Matrices (with Murat Koloğlu, Gene S. Kopp, Frederick W. Strauch and Wentao Xiong), to appear in the Journal of Theoretical Probability. http://arxiv.org/abs/1008.4812

Introduction to L-Functions

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

General L-functions

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

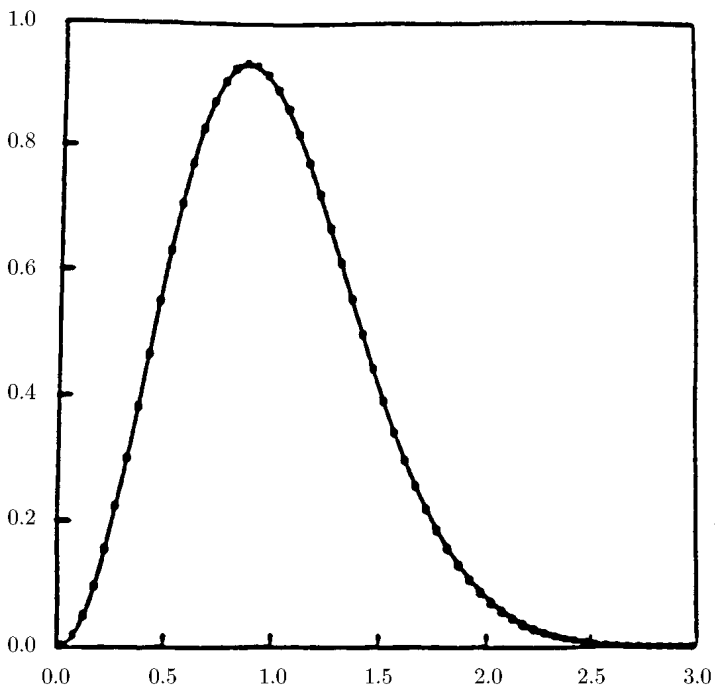
$$\Lambda(s,f) = \Lambda_\infty(s,f)L(s,f) = \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

Zeros of $\zeta(s)$ vs GUE



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko).

Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

Explicit Formula (Contour Integration)

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1} \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\sum_p\log\left(1-p^{-s}\right) \\
&= \sum_p\frac{\log p\ \cdot\ p^{-s}}{1-p^{-s}} = \sum_p\frac{\log p}{p^s} + \mathrm{Good}(s).
\end{aligned}
$$

Explicit Formula (Contour Integration)

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1} \\
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&= \sum_p\frac{\log p\ \cdot\ p^{-s}}{1-p^{-s}} = \sum_p\frac{\log p}{p^s} + \mathrm{Good}(s).
\end{aligned}
$$

Contour Integration:

$$
\int -\frac{\zeta'(s)}{\zeta(s)}\,\frac{x^s}{s}\,ds \quad \text{vs} \quad \sum_p\log p\int\left(\frac{x}{p}\right)^s\frac{ds}{s}.
$$

Explicit Formula (Contour Integration)

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1} \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\sum_p\log\left(1-p^{-s}\right) \\
&= \sum_p\frac{\log p\ \cdot\ p^{-s}}{1-p^{-s}} = \sum_p\frac{\log p}{p^s} + \mathrm{Good}(s).
\end{aligned}
$$

Contour Integration:

$$
\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p\log p\int\phi(s)p^{-s}ds.
$$

Explicit Formula (Contour Integration)

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1} \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\sum_p \log\left(1-p^{-s}\right) \\
&= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \mathrm{Good}(s).
\end{aligned}
$$

Contour Integration (see Fourier Transform arising):

$$\int -\frac{\zeta'(s)}{\zeta(s)}\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s)e^{-\sigma\log p}e^{-it\log p}ds.$$

Knowledge of zeros gives info on coefficients.

Explicit Formula: Examples

Riemann Zeta Function: Let $\sum_\rho$ denote the sum over the zeros of $\zeta(s)$ in the critical strip, g an even Schwartz function of compact support and $\phi(r) = \int_{-\infty}^{\infty} g(u)e^{iru}du$. Then

$$\sum_\rho \phi(\gamma_\rho) = 2\phi\left(\frac{i}{2}\right) - \sum_p \sum_{k=1}^{\infty} \frac{2\log p}{p^{k/2}} g\left(k\log p\right)$$
$$+ \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{iy-\frac{1}{2}} + \frac{\Gamma'(\frac{iy}{2}+\frac{5}{4})}{\Gamma(\frac{iy}{2}+\frac{5}{4}))} - \frac{1}{2}\log\pi\right)\phi(y)\,dy.$$

Explicit Formula: Examples

Dirichlet L-functions: Let h be an even Schwartz function and $L(s,\chi)=\sum_n \chi(n)/n^s$ a Dirichlet L-function from a non-trivial character χ with conductor m and zeros $\rho=\frac{1}{2}+i\gamma_\chi$; if the Generalized Riemann Hypothesis is true then $\gamma\in\mathbb{R}$. Then

$$\sum_\rho h\left(\gamma_\rho\frac{\log(m/\pi)}{2\pi}\right) = \int_{-\infty}^{\infty} h(y)dy$$
$$-2\sum_p \frac{\log p}{\log(m/\pi)}\widehat{h}\left(\frac{\log p}{\log(m/\pi)}\right)\frac{\chi(p)}{p^{1/2}}$$
$$-2\sum_p \frac{\log p}{\log(m/\pi)}\widehat{h}\left(2\frac{\log p}{\log(m/\pi)}\right)\frac{\chi^2(p)}{p}+O\left(\frac{1}{\log m}\right).$$

Explicit Formula: Examples

Cuspidal Newforms: Let $\mathcal{F}$ be a family of cupsidal newforms (say weight k, prime level N and possibly split by sign) $L(s,f) = \sum_n \lambda_f(n)/n^s$. Then

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{\gamma_f}\phi\left(\frac{\log R}{2\pi}\gamma_f\right) = \widehat{\phi}(0) + \frac{1}{2}\phi(0) - \frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}P(f;\phi) + O\left(\frac{\log\log R}{\log R}\right)$$

$$P(f;\phi) = \sum_{p\nmid N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}.$$

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbf{R}^{n-1}$.

n-level correlation

$$\lim_{N\to\infty} \frac{\#\left\{\left(\alpha_{j_1}-\alpha_{j_2},\ldots,\alpha_{j_{n-1}}-\alpha_{j_n}\right)\in B, j_i\neq j_k\right\}}{N}$$

(Instead of using a box, can use a smooth test function.)

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbf{R}^{n-1}$.

1. Normalized spacings of $\zeta(s)$ starting at 10^{20} (Odlyzko).
2. 2 and 3-correlations of $\zeta(s)$ (Montgomery, Hejhal).
3. n-level correlations for all automorphic cupsidal L-functions (Rudnick-Sarnak).
4. n-level correlations for the classical compact groups (Katz-Sarnak).
5. Insensitive to any finite set of zeros.

Measures of Spacings: n-Level Density and Families

$\phi(x) := \prod_i \phi_i(x_i)$, ϕ_i even Schwartz functions whose Fourier Transforms are compactly supported.

n-level density

$$D_{n,f}(\phi) = \sum_{\substack{j_1,\dots,j_n \\ \text{distinct}}} \phi_1\left(L_f\gamma_f^{(j_1)}\right)\cdots\phi_n\left(L_f\gamma_f^{(j_n)}\right)$$

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1. Individual zeros contribute in limit.
2. Most of contribution is from low zeros.
3. Average over similar curves (family).

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1. Individual zeros contribute in limit.
2. Most of contribution is from low zeros.
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Katz-Sarnak Conjecture

For a 'nice' family of L-functions, the n-level density depends only on a symmetry group attached to the family.

Normalization of Zeros

Local (hard, use C_f) vs Global (easier, use $\log C = |\mathcal{F}_N|^{-1}\sum_{f\in\mathcal{F}_N}\log C_f$). Hope: ϕ a good even test function with compact support, as $|\mathcal{F}| \to \infty$,

$$\frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N} D_{n,f}(\phi) = \frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N}\sum_{\substack{j_1,\dots,j_n\\ j_i\neq\pm j_k}}\prod_i \phi_i\left(\frac{\log C_f}{2\pi}\gamma_E^{(j_i)}\right)$$

$$\to \int\cdots\int \phi(x)W_{n,\mathcal{G}(\mathcal{F})}(x)dx.$$

Katz-Sarnak Conjecture

As $C_f \to \infty$ the behavior of zeros near $1/2$ agrees with $N \to \infty$ limit of eigenvalues of a classical compact group.

1-Level Densities

The Fourier Transforms for the 1-level densities are

$$\begin{aligned}
\widehat{W_{1,\mathrm{SO(even)}}}(u) &= \delta_0(u) + \frac{1}{2}\eta(u) \\
\widehat{W_{1,\mathrm{SO}}}(u) &= \delta_0(u) + \frac{1}{2} \\
\widehat{W_{1,\mathrm{SO(odd)}}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) + 1 \\
\widehat{W_{1,Sp}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) \\
\widehat{W_{1,U}}(u) &= \delta_0(u)
\end{aligned}$$

where $\delta_0(u)$ is the Dirac Delta functional and

$$\eta(u) = \begin{cases} 1 & \text{if } |u| < 1 \\ \frac{1}{2} & \text{if } |u| = 1 \\ 0 & \text{if } |u| > 1 \end{cases}$$

Correspondences

Similarities between L-Functions and Nuclei:

Zeros	$\longleftrightarrow$	Energy Levels
Schwartz test function	$\longrightarrow$	Neutron
Support of test function	$\longleftrightarrow$	Neutron Energy.

Results

Some Number Theory Results

- Orthogonal: Iwaniec-Luo-Sarnak, Ricotta-Royer: 1-level density for holomorphic even weight k cuspidal newforms of square-free level N (SO(even) and SO(odd) if split by sign).
- Symplectic: Rubinstein, Gao, Levinson-Miller, and Entin, Roddity-Gershon and Rudnick: n-level densities for twists $L(s, \chi_d)$ of the zeta-function.
- Unitary: Fiorilli-Miller, Hughes-Rudnick: Families of Primitive Dirichlet Characters.
- Orthogonal: Miller, Young: One and two-parameter families of elliptic curves.

Main Tools

1. Control of conductors: Usually monotone, gives scale to study low-lying zeros.
2. Explicit Formula: Relates sums over zeros to sums over primes.
3. Averaging Formulas: Petersson formula in Iwaniec-Luo-Sarnak, Orthogonality of characters in Fiorilli-Miller, Gao, Hughes-Rudnick, Levinson-Miller, Rubinstein.

Applications of n-level density

One application: bounding the order of vanishing at the central point.

$\text{Average rank} \cdot \phi(0) \ \leq \ \int \phi(x) W_{G(\mathcal{F})}(x) dx$ if ϕ non-negative.

Applications of n-level density

One application: bounding the order of vanishing at the central point.

$\text{Average rank} \cdot \phi(0) \leq \int \phi(x) W_{G(\mathcal{F})}(x) dx$ if ϕ non-negative.

Can also use to bound the percentage that vanish to order r for any r.

Theorem (Miller, Hughes-Miller)

Using n-level arguments, for the family of cuspidal newforms of prime level $N \to \infty$ (split or not split by sign), for any r there is a c_r such that probability of at least r zeros at the central point is at most $c_n r^{-n}$.

Better results using 2-level than Iwaniec-Luo-Sarnak using the 1-level for $r \geq 5$.

Identifying the Symmetry Groups

- Often an analysis of the monodromy group in the function field case suggests the answer.
- Tools: Explicit Formula, Orthogonality of Characters / Petersson Formula.
- How to identify symmetry group in general? One possibility is by the signs of the functional equation:
- **Folklore Conjecture:** If all signs are even and no corresponding family with odd signs, Symplectic symmetry; otherwise SO(even). (False!)

Explicit Formula

- π: cuspidal automorphic representation on GL_n.
- $Q_\pi > 0$: analytic conductor of $L(s,\pi) = \sum \lambda_\pi(n)/n^s$.
- By GRH the non-trivial zeros are $\frac{1}{2} + i\gamma_{\pi,j}$.
- Satake parameters $\{\alpha_{\pi,i}(p)\}_{i=1}^n$; $\lambda_\pi(p^\nu) = \sum_{i=1}^n \alpha_{\pi,i}(p)^\nu$.
- $L(s,\pi) = \sum_n \frac{\lambda_\pi(n)}{n^s} = \prod_p \prod_{i=1}^n (1 - \alpha_{\pi,i}(p)p^{-s})^{-1}$.

$$\sum_j g\left(\gamma_{\pi,j}\frac{\log Q_\pi}{2\pi}\right) = \widehat{g}(0) - 2\sum_{p,\nu}\widehat{g}\left(\frac{\nu\log p}{\log Q_\pi}\right)\frac{\lambda_\pi(p^\nu)\log p}{p^{\nu/2}\log Q_\pi}$$

1-Level Density

Assuming conductors constant in family $\mathcal{F}$, have to study

$$\lambda_f(p^\nu) = \alpha_{f,1}(p)^\nu + \cdots + \alpha_{f,n}(p)^\nu$$

$$S_1(\mathcal{F}) = -2\sum_p \hat{g}\left(\frac{\log p}{\log R}\right)\frac{\log p}{\sqrt{p}\log R}\left[\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p)\right]$$

$$S_2(\mathcal{F}) = -2\sum_p \hat{g}\left(2\frac{\log p}{\log R}\right)\frac{\log p}{p\log R}\left[\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p^2)\right]$$

The corresponding classical compact group is determined by

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p^2) = c_{\mathcal{F}} = \begin{cases} 0 & \text{Unitary} \\ 1 & \text{Symplectic} \\ -1 & \text{Orthogonal.} \end{cases}$$

Some Results: Rankin-Selberg Convolution of Families

Symmetry constant: $c_{\mathcal{L}} = 0$ (resp, 1 or -1) if family $\mathcal{L}$ has unitary (resp, symplectic or orthogonal) symmetry.

Rankin-Selberg convolution: Satake parameters for $\pi_{1,p} \times \pi_{2,p}$ are

$$\{\alpha_{\pi_1 \times \pi_2}(k)\}_{k=1}^{nm} = \{\alpha_{\pi_1}(i) \cdot \alpha_{\pi_2}(j)\}_{\substack{1 \le i \le n \\ 1 \le j \le m}}.$$

Theorem (Dueñez-Miller)

If $\mathcal{F}$ and $\mathcal{G}$ are *nice* families of L-functions, then $c_{\mathcal{F} \times \mathcal{G}} = c_{\mathcal{F}} \cdot c_{\mathcal{G}}$.

Breaks analysis of compound families into simple ones.

Takeaways

Very similar to Central Limit Theorem.

- Universal behavior: main term controlled by first two moments of Satake parameters, agrees with RMT.
- First moment zero save for families of elliptic curves.
- Higher moments control convergence and can depend on arithmetic of family.

Elliptic Curves: First Zero Above Central Point
(With E. Dueñez, D. K. Huynh, J. P. Keating, N. Snaith)

Theoretical results

Theorem: M– '04

For small support, one-param family of rank r over $\mathbb{Q}(T)$:

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{E_t \in \mathcal{F}_N} \sum_j \varphi\left(\frac{\log C_{E_t}}{2\pi}\gamma_{E_t,j}\right) = \int \varphi(x)\rho_{\mathcal{G}}(x)dx + r\varphi(0)$$

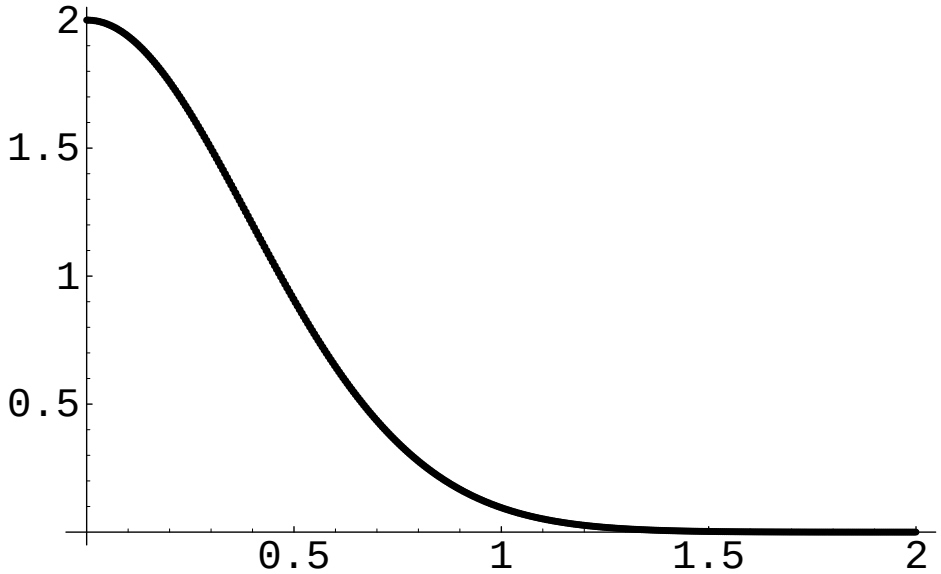
$$\text{where} \quad \mathcal{G} = \begin{cases} \mathrm{SO(odd)} & \text{if half odd} \\ \mathrm{SO(even)} & \text{if all even} \\ \text{weighted average} & \text{otherwise.} \end{cases}$$

Supports Katz-Sarnak, B-SD, and Independent model in limit.

Independent Model:

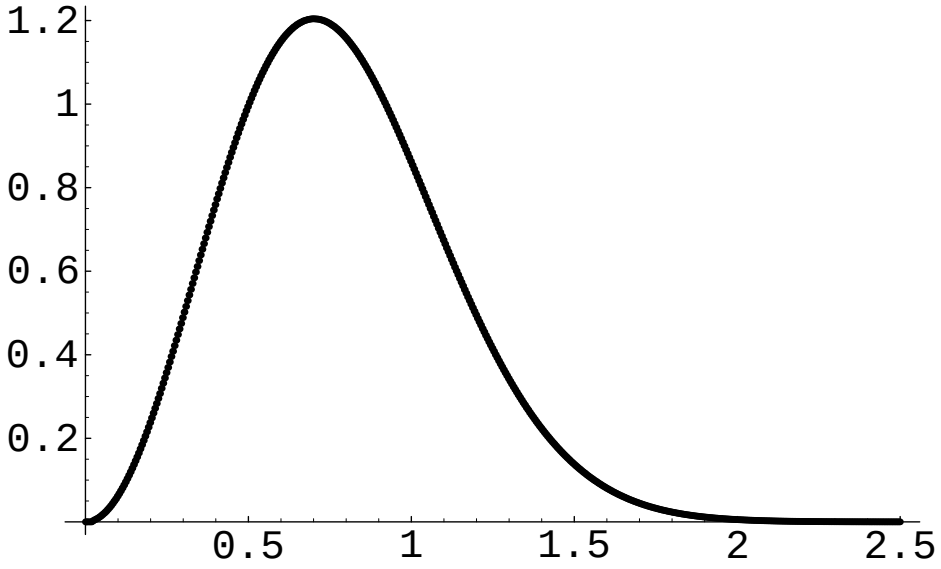
$$\mathcal{A}_{2N,2r} = \left\{ \begin{pmatrix} I_{2r\times 2r} & \\ & g \end{pmatrix} : g \in SO(2N-2r) \right\}.$$

RMT: Theoretical Results ($N \to \infty$)



1st normalized evalue above 1: SO(even)

RMT: Theoretical Results ($N \to \infty$)



1st normalized evalue above 1: SO(odd)

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

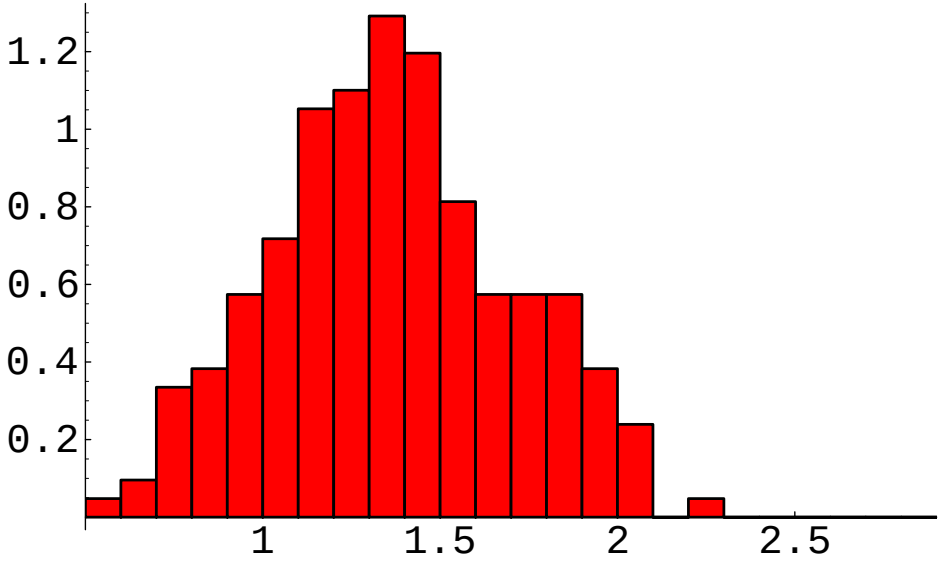


Figure 4a: 209 rank 0 curves from 14 rank 0 families, $\log(\mathrm{cond}) \in [3.26, 9.98]$, median $= 1.35$, mean $= 1.36$

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

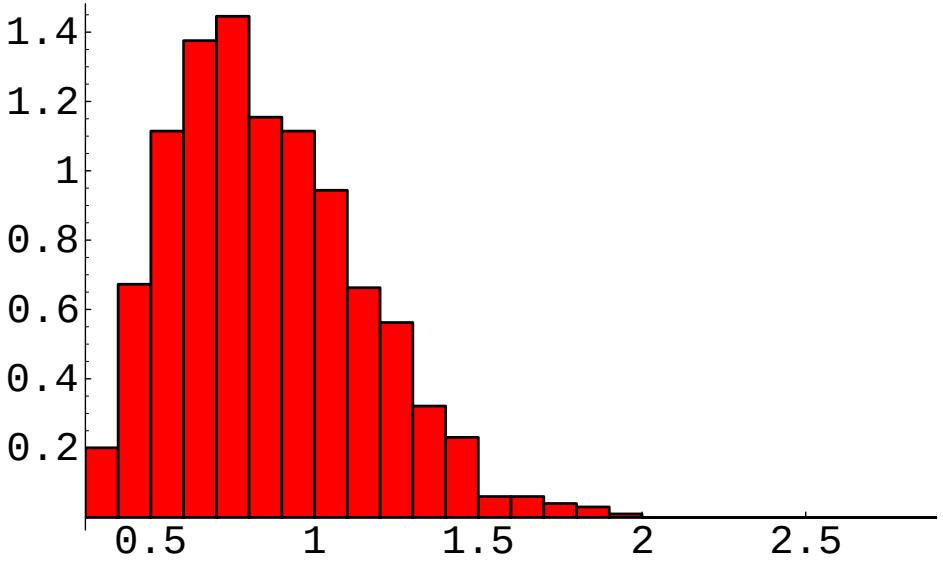


Figure 4b: 996 rank 0 curves from 14 rank 0 families, $\log(\mathrm{cond}) \in [15.00, 16.00]$, median $= .81$, mean $= .86$.

Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$) 1st Normalized Zero above Central Point

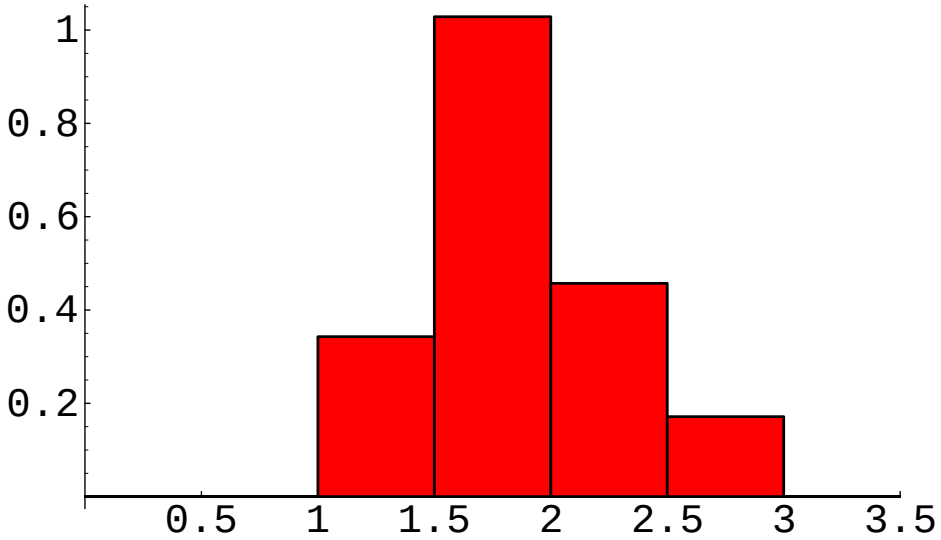


Figure 5a: 35 curves, $\log(\mathrm{cond}) \in [7.8, 16.1]$, $\widetilde{\mu} = 1.85$, $\mu = 1.92$, $\sigma_\mu = .41$

Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$) 1st Normalized Zero above Central Point

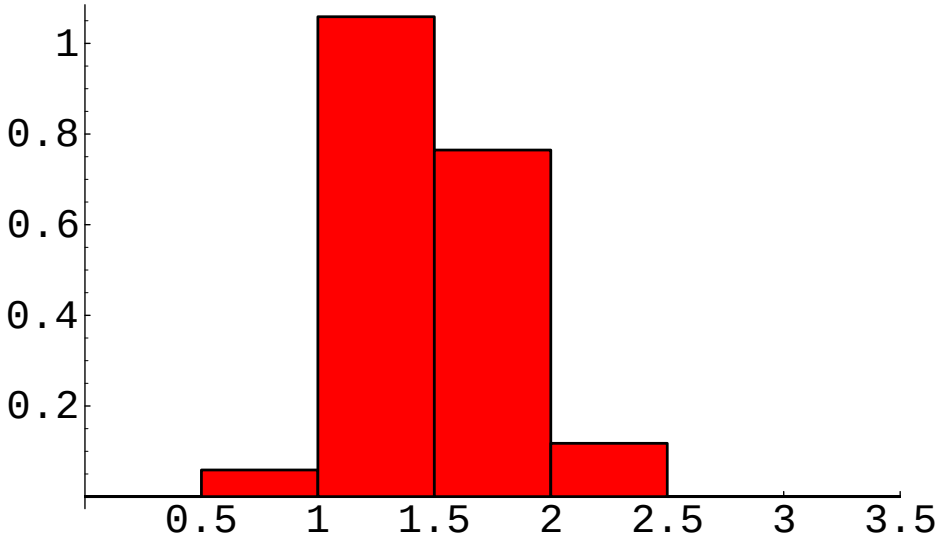


Figure 5b: 34 curves, $\log(\mathrm{cond}) \in [16.2, 23.3]$, $\widetilde{\mu} = 1.37$, $\mu = 1.47$, $\sigma_\mu = .34$

Summary of Data

- The repulsion of the low-lying zeros increased with increasing rank, and was present even for rank 0 curves.
- As the conductors increased, the repulsion decreased.
- Statistical tests failed to reject the hypothesis that, on average, the first three zeros were all repelled equally (i.e., shifted by the same amount).

Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- All curves have $\log(\mathrm{cond}) \in [15, 16]$;
- z_j = imaginary part of jth normalized zero above the central point;
- 863 rank 0 curves from the 14 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$.

	863 Rank 0 Curves	**701 Rank 2 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.28	1.30	
Mean $z_2 - z_1$	1.30	1.34	-1.60
StDev $z_2 - z_1$	0.49	0.51	
Median $z_3 - z_2$	1.22	1.19	
Mean $z_3 - z_2$	1.24	1.22	0.80
StDev $z_3 - z_2$	0.52	0.47	
Median $z_3 - z_1$	2.54	2.56	
Mean $z_3 - z_1$	2.55	2.56	-0.38
StDev $z_3 - z_1$	0.52	0.52	

Spacings b/w Norm Zeros: Rank 2 one-param families over $\mathbb{Q}(T)$

- All curves have $\log(\mathrm{cond}) \in [15, 16]$;
- z_j = imaginary part of the jth norm zero above the central point;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$;
- 23 rank 4 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	64 Rank 2 Curves	**23 Rank 4 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.26	1.27	
Mean $z_2 - z_1$	1.36	1.29	0.59
StDev $z_2 - z_1$	0.50	0.42	
Median $z_3 - z_2$	1.22	1.08	
Mean $z_3 - z_2$	1.29	1.14	1.35
StDev $z_3 - z_2$	0.49	0.35	
Median $z_3 - z_1$	2.66	2.46	
Mean $z_3 - z_1$	2.65	2.43	2.05
StDev $z_3 - z_1$	0.44	0.42	

Rank 2 Curves from Rank 0 & Rank 2 Families over $\mathbb{Q}(T)$

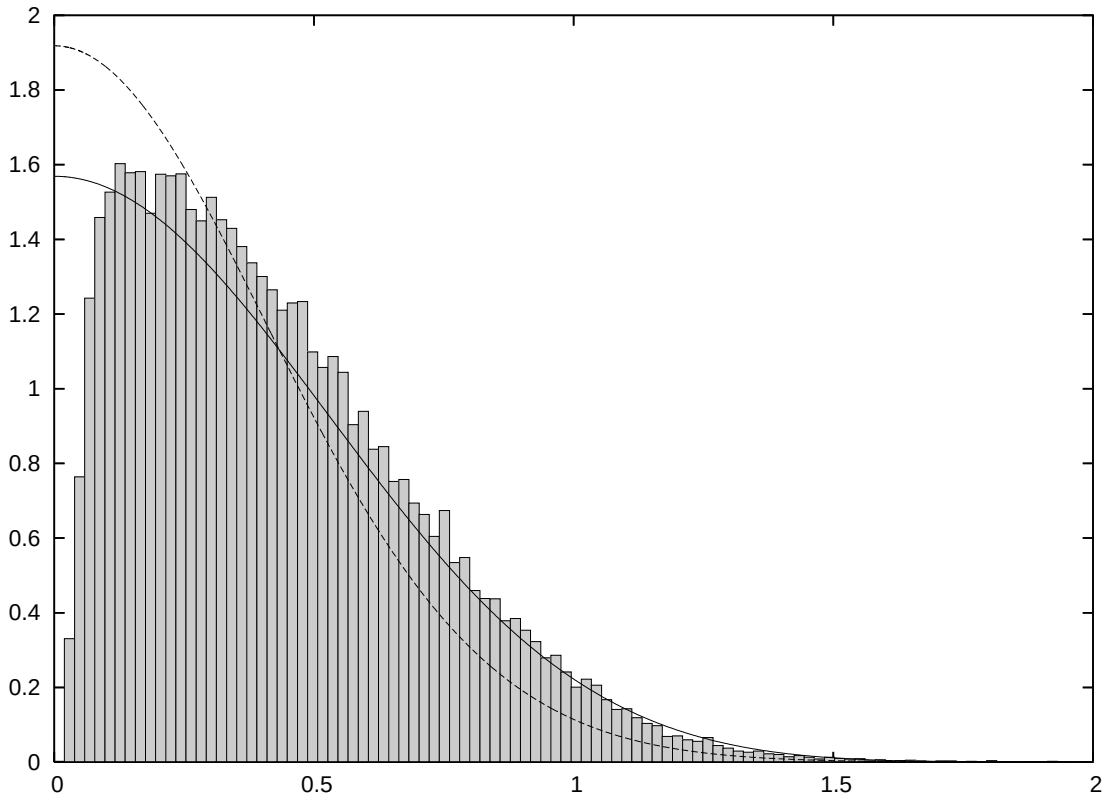
- All curves have $\log(\mathrm{cond}) \in [15, 16]$;
- z_j = imaginary part of the jth norm zero above the central point;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	701 Rank 2 Curves	**64 Rank 2 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.30	1.26	
Mean $z_2 - z_1$	1.34	1.36	0.69
StDev $z_2 - z_1$	0.51	0.50	
Median $z_3 - z_2$	1.19	1.22	
Mean $z_3 - z_2$	1.22	1.29	1.39
StDev $z_3 - z_2$	0.47	0.49	
Median $z_3 - z_1$	2.56	2.66	
Mean $z_3 - z_1$	2.56	2.65	1.93
StDev $z_3 - z_1$	0.52	0.44	

New Model for Finite Conductors

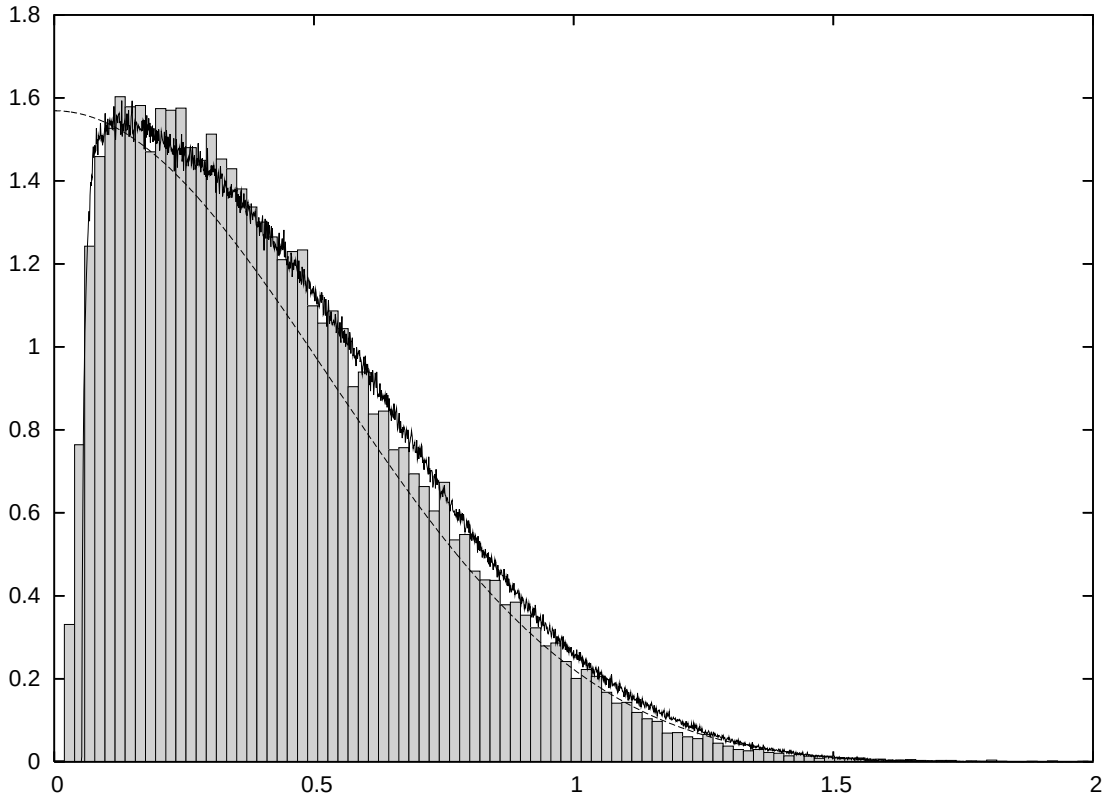
- Replace conductor N with $N_{\text{effective}}$.
  - ◇ Arithmetic info, predict with L-function Ratios Conj.
  - ◇ Do the number theory computation.
- Excised Orthogonal Ensembles.
  - ◇ $L(1/2, E)$ discretized.
  - ◇ Study matrices in $\mathrm{SO}(2N_{eff})$ with $|\Lambda_A(1)| \geq ce^N$.
- Painlevé VI differential equation solver.
  - ◇ Use explicit formulas for densities of Jacobi ensembles.
  - ◇ Key input: Selberg-Aomoto integral for initial conditions.

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of SO(2N) with N_{eff} (solid), standard N_0 (dashed).

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart); lowest eigenvalue of SO(2N): N_{eff} = 2 (solid) with discretisation, and $N_{\text{eff}} = 2.32$ (dashed) without discretisation.

History of the Ratios Conjecture

- Farmer (1993): Considered

$$\int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)}\,dt,$$

conjectured (for appropriate values)

$$T\frac{(\alpha+\delta)(\beta+\gamma)}{(\alpha+\beta)(\gamma+\delta)} - T^{1-\alpha-\beta}\frac{(\delta-\beta)(\gamma-\alpha)}{(\alpha+\beta)(\gamma+\delta)}.$$

- Conrey-Farmer-Zirnbauer (2007): conjecture formulas for averages of products of L-functions over families:

$$R_{\mathcal{F}} = \sum_{f\in\mathcal{F}} \omega_f \frac{L\left(\frac{1}{2}+\alpha, f\right)}{L\left(\frac{1}{2}+\gamma, f\right)}.$$

Uses of the Ratios Conjecture

- Applications:
  - ◇ n-level correlations and densities;
  - ◇ mollifiers;
  - ◇ moments;
  - ◇ vanishing at the central point;
- Advantages:
  - ◇ RMT models often add arithmetic ad hoc;
  - ◇ predicts lower order terms, often to square-root level.

Inputs for 1-level density

- Approximate Functional Equation:

$$L(s,f) = \sum_{m\le x}\frac{a_m}{m^s} + \epsilon\mathbb{X}_L(s)\sum_{n\le y}\frac{a_n}{n^{1-s}};$$

◇ ϵ sign of the functional equation,
◇ $\mathbb{X}_L(s)$ ratio of Γ-factors from functional equation.

- Explicit Formula: g Schwartz test function,

$$\sum_{f\in\mathcal{F}}\omega_f\sum_{\gamma}g\left(\gamma\frac{\log N_f}{2\pi}\right) = \frac{1}{2\pi i}\int_{(c)} - \int_{(1-c)} R'_{\mathcal{F}}(\cdots)g(\cdots)$$

◇ $R'_{\mathcal{F}}(r) = \frac{\partial}{\partial\alpha}R_{\mathcal{F}}(\alpha,\gamma)\Big|_{\alpha=\gamma=r}$.

Procedure (Recipe)

- Use approximate functional equation to expand numerator.
- Expand denominator by generalized Mobius function: cusp form
$$\frac{1}{L(s,f)} = \sum_h \frac{\mu_f(h)}{h^s},$$
where $\mu_f(h)$ is the multiplicative function equaling 1 for $h = 1$, $-\lambda_f(p)$ if $n = p$, $\chi_0(p)$ if $h = p^2$ and 0 otherwise.
- Execute the sum over $\mathcal{F}$, keeping only main (diagonal) terms.
- Extend the m and n sums to infinity (complete the products).
- Differentiate with respect to the parameters.

Procedure ('Illegal Steps')

- Use approximate functional equation to expand numerator.
- Expand denominator by generalized Mobius function: cusp form

$$\frac{1}{L(s,f)} = \sum_h \frac{\mu_f(h)}{h^s},$$

where $\mu_f(h)$ is the multiplicative function equaling 1 for $h = 1$, $-\lambda_f(p)$ if $n = p$, $\chi_0(p)$ if $h = p^2$ and 0 otherwise.
- Execute the sum over $\mathcal{F}$, keeping only main (diagonal) terms.
- Extend the m and n sums to infinity (complete the products).
- Differentiate with respect to the parameters.

1-Level Prediction from Ratio's Conjecture

$$
\begin{aligned}
& A_E(\alpha,\gamma) \\
= \quad & Y_E^{-1}(\alpha,\gamma) \times \prod_{p|M} \left(\sum_{m=0}^{\infty} \left(\frac{\lambda(p^m)\omega_E^m}{p^{m(1/2+\alpha)}} - \frac{\lambda(p)}{p^{1/2+\gamma}} \frac{\lambda(p^m)\omega_E^{m+1}}{p^{m(1/2+\alpha)}} \right) \right) \times \\
& \prod_{p\nmid M} \left(1 + \frac{p}{p+1} \left(\sum_{m=1}^{\infty} \frac{\lambda(p^{2m})}{p^{m(1+2\alpha)}} - \frac{\lambda(p)}{p^{1+\alpha+\gamma}} \sum_{m=0}^{\infty} \frac{\lambda(p^{2m+1})}{p^{m(1+2\alpha)}} \right.\right. \\
& \left.\left. + \frac{1}{p^{1+2\gamma}} \sum_{m=0}^{\infty} \frac{\lambda(p^{2m})}{p^{m(1+2\alpha)}} \right) \right)
\end{aligned}
$$

where

$$Y_E(\alpha,\gamma) = \frac{\zeta(1+2\gamma)L_E(\mathrm{sym}^2, 1+2\alpha)}{\zeta(1+\alpha+\gamma)L_E(\mathrm{sym}^2, 1+\alpha+\gamma)}.$$

Huynh, Morrison and Miller confirmed Ratios' prediction, which is

1-Level Prediction from Ratio's Conjecture

$$\frac{1}{X^*}\sum_{d\in\mathcal{F}(X)}\sum_{\gamma_d} g\Big(\frac{\gamma_d L}{\pi}\Big)$$

$$= \frac{1}{2LX^*}\int_{-\infty}^{\infty} g(\tau)\sum_{d\in\mathcal{F}(X)}\left[2\log\left(\frac{\sqrt{M}|d|}{2\pi}\right)+\frac{\Gamma'}{\Gamma}\Big(1+\frac{i\pi\tau}{L}\Big)+\frac{\Gamma'}{\Gamma}\Big(1-\frac{i\pi\tau}{L}\Big)\right]d\tau$$

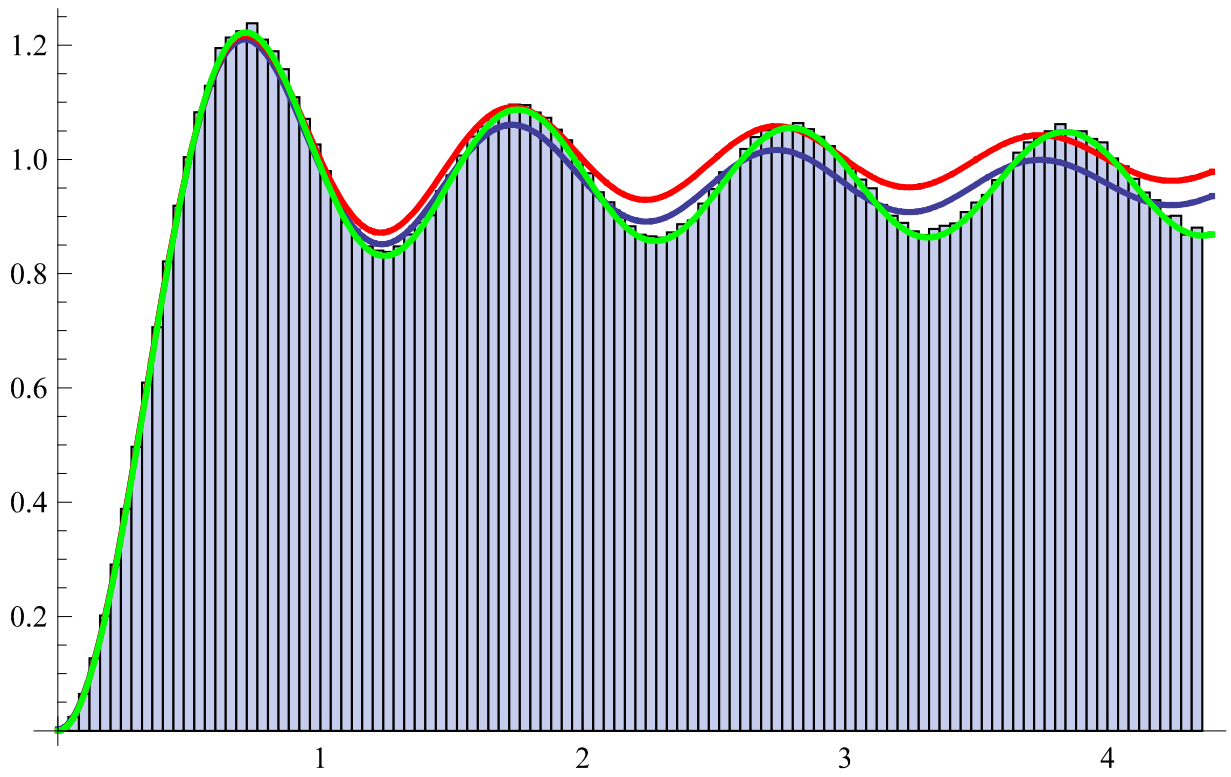
$$+\frac{1}{L}\int_{-\infty}^{\infty} g(\tau)\left(-\frac{\zeta'}{\zeta}\Big(1+\frac{2\pi i\tau}{L}\Big)+\frac{L_E'}{L_E}\Big(\mathrm{sym}^2,1+\frac{2\pi i\tau}{L}\Big)-\sum_{\ell=1}^{\infty}\frac{(M^\ell-1)\log M}{M^{\left(2+\frac{2i\pi\tau}{L}\right)\ell}}\right)d\tau$$

$$-\frac{1}{L}\sum_{k=0}^{\infty}\int_{-\infty}^{\infty} g(\tau)\frac{\log M}{M^{(k+1)(1+\frac{\pi i\tau}{L})}}d\tau+\frac{1}{L}\int_{-\infty}^{\infty} g(\tau)\sum_{p\nmid M}\frac{\log p}{(p+1)}\sum_{k=0}^{\infty}\frac{\lambda(p^{2k+2})-\lambda(p^{2k})}{p^{(k+1)(1+\frac{2\pi i\tau}{L})}}d\tau$$

$$-\frac{1}{LX^*}\int_{-\infty}^{\infty} g(\tau)\sum_{d\in\mathcal{F}(X)}\left[\left(\frac{\sqrt{M}|d|}{2\pi}\right)^{-2i\pi\tau/L}\frac{\Gamma(1-\frac{i\pi\tau}{L})}{\Gamma(1+\frac{i\pi\tau}{L})}\frac{\zeta(1+\frac{2i\pi\tau}{L})L_E(\mathrm{sym}^2,1-\frac{2i\pi\tau}{L})}{L_E(\mathrm{sym}^2,1)}\right.$$

$$\left.\times A_E\Big(-\frac{i\pi\tau}{L},\frac{i\pi\tau}{L}\Big)\right]d\tau+O(X^{-1/2+\varepsilon});$$

Numerics (J. Stopple): 1,003,083 negative fundamental discriminants $-d \in [10^{12}, 10^{12} + 3.3 \cdot 10^6]$



Histogram of normalized zeros ($\gamma \leq 1$, about 4 million).

◇ Red: main term. ◇ Blue: includes $O(1/\log X)$ terms.

◇ Green: all lower order terms.

Cuspidal Maass Forms
(Joint with Levent Alpoge)

Maass Forms

Definition: Maass Forms

A Maass form on a group $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ is a function $f : \mathcal{H} \to \mathbb{R}$ which satisfies:

1. $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$,
2. f vanishes at the cusps of Γ, and
3. $\Delta f = \lambda f$ for some $\lambda = s(1-s) > 0$, where

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

is the Laplace-Beltrami operator on $\mathcal{H}$.

- Coefficients contain information about partitions.
- For full modular group, $s = 1/2 + it_j$ with $t_j \in \mathbb{R}$.

L-function associated to Maass forms

Write Fourier expansion of Maass form u_j as

$$u_j(z) = \cosh(t_j) \sum_{n \neq 0} \sqrt{y} \lambda_j(n) K_{it_j}(2\pi |n| y) e^{2\pi i n x}.$$

Define L-function attached to u_j as

$$L(s, u_j) = \sum_{n \geq 1} \frac{\lambda_j(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_j(p)}{p^s}\right)^{-1}$$

where $\alpha_j(p) + \beta_j(p) = \lambda_j(p), \quad \alpha_j(p)\beta_j(p) = 1, \quad \lambda_j(1) = 1.$
Also,

$$\lambda_j(p) \ll p^{7/64}.$$

Kuznetsov Trace Formula

$$\sum_j \frac{h(t_j)}{\|u_j\|^2}\lambda_j(m)\overline{\lambda_j(n)} + \frac{1}{4\pi}\int_{\mathbb{R}} \overline{\tau(m,r)}\tau(n,r)\frac{h(r)}{\cosh(\pi r)}dr =$$

$$\frac{\delta_{n,m}}{\pi^2}\int_{\mathbb{R}} r\tanh(r)h(r)dr + \frac{2i}{\pi}\sum_{c\geq 1}\frac{S(n,m;c)}{c}\int_{\mathbb{R}} J_{ir}\left(\frac{4\pi\sqrt{mn}}{c}\right)\frac{h(r)r}{\cosh(\pi r)}dr$$

where
$\tau(m,r) = \pi^{\frac{1}{2}+ir}\Gamma(1/2+ir)^{-1}\zeta(1+2ir)^{-1}n^{-\frac{1}{2}}\sum_{ab=|m|}\left(\frac{a}{b}\right)^{ir}$.

$$S(n,m;c) = \sum_{0\leq x\leq c-1,\gcd(x,c)=1} e^{2\pi i(nx+mx^*)/c}$$

$$J_{ir}(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+ir+1)}\left(\frac{1}{2}x\right)^{2m+ir}.$$

Main Result

Theorem (Alpoge-Miller 2013)

For $h_T(r) = \frac{r}{T} h(ir/T)/\sinh(\pi r/T)$ the Katz-Sarnak conjecture holds for level 1 cuspidal Maass forms for test functions whose Fourier transform is supported in $(-4/3, 4/3)$.

- Write $\int_{-\infty}^{\infty} J_{2ir}(X) \frac{r h_T(r)}{\cosh(\pi r)} dr$ as sum of $J_{2k+1}(X)$ and h_T at imaginary arguments and $J_{2kT}(X)$ and h at k.
- Bound contributions from sums. Apply Poisson summation, analyze, and Poisson summation again.
- Key steps: Taylor expanding, Fourier transform identities relating differentiation and multiplication.

Cuspidal Newforms
(Joint with C. Hughes, G. Iyer, N. Triantafillou)

Modular Form Preliminaries

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{c} ad - bc = 1 \\ c \equiv 0(N) \end{array} \right\}$$

f is a weight k holomorphic cuspform of level N if

$$\forall \gamma \in \Gamma_0(N), \;\; f(\gamma z) = (cz + d)^k f(z).$$

- Fourier Expansion: $f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i z}$, $L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}$.
- Petersson Norm: $\langle f, g \rangle = \int_{\Gamma_0(N)\backslash \mathbf{H}} f(z)\overline{g(z)} y^{k-2} dxdy$.
- Normalized coefficients:

$$\psi_f(n) = \sqrt{\frac{\Gamma(k-1)}{(4\pi n)^{k-1}}} \frac{1}{||f||} a_f(n).$$

Modular Form Preliminaries: Petersson Formula

$B_k(N)$ an orthonormal basis for weight k level N. Define

$$\Delta_{k,N}(m,n) = \sum_{f \in B_k(N)} \psi_f(m)\overline{\psi}_f(n).$$

Petersson Formula

$$\Delta_{k,N}(m,n) = 2\pi i^k \sum_{c \equiv 0(N)} \frac{S(m,n,c)}{c} J_{k-1}\left(4\pi \frac{\sqrt{mn}}{c}\right) + \delta(m,n).$$

2-Level Density

$$\int_{x_1=2}^{R^\sigma}\int_{x_2=2}^{R^\sigma}\widehat{\phi}\left(\frac{\log x_1}{\log R}\right)\widehat{\phi}\left(\frac{\log x_2}{\log R}\right)J_{k-1}\left(4\pi\frac{\sqrt{m^2x_1x_2N}}{c}\right)\frac{dx_1dx_2}{\sqrt{x_1x_2}}$$

Change of variables and Jacobian:

$$\begin{aligned} u_2 &= x_1x_2 & x_2 &= \tfrac{u_2}{u_1} \\ u_1 &= x_1 & x_1 &= u_1 \end{aligned}$$

$$\left|\frac{\partial x}{\partial u}\right| = \begin{vmatrix} 1 & 0 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1} \end{vmatrix} = \frac{1}{u_1}.$$

Left with

$$\int\int\widehat{\phi}\left(\frac{\log u_1}{\log R}\right)\widehat{\phi}\left(\frac{\log\left(\frac{u_2}{u_1}\right)}{\log R}\right)\frac{1}{\sqrt{u_2}}J_{k-1}\left(4\pi\frac{\sqrt{m^2u_2N}}{c}\right)\frac{du_1du_2}{u_1}$$

2-Level Density

Changing variables, u_1-integral is

$$\int_{w_1=\frac{\log u_2}{\log R}-\sigma}^{\sigma} \widehat{\phi}(w_1)\,\widehat{\phi}\left(\frac{\log u_2}{\log R}-w_1\right) dw_1.$$

Support conditions imply

$$\Psi_2\left(\frac{\log u_2}{\log R}\right) = \int_{w_1=-\infty}^{\infty} \widehat{\phi}(w_1)\,\widehat{\phi}\left(\frac{\log u_2}{\log R}-w_1\right) dw_1.$$

Substituting gives

$$\int_{u_2=0}^{\infty} J_{k-1}\left(4\pi\frac{\sqrt{m^2u_2N}}{c}\right)\Psi_2\left(\frac{\log u_2}{\log R}\right)\frac{du_2}{\sqrt{u_2}}$$

n-Level Density: Sketch of proof

Expand Bessel-Kloosterman piece, use GRH to drop non-principal characters, change variables, main term is

$$\frac{b\sqrt{N}}{2\pi m}\int_0^\infty J_{k-1}(x)\widehat{\Phi_n}\left(\frac{2\log(bx\sqrt{N}/4\pi m)}{\log R}\right)\frac{\mathrm{d}x}{\log R}$$

with $\Phi_n(x) = \phi(x)^n$.

Main Idea

Difficulty in comparison with classical RMT is that instead of having an n-dimensional integral of $\phi_1(x_1)\cdots\phi_n(x_n)$ we have a 1-dimensional integral of a new test function. This leads to harder combinatorics but allows us to appeal to the result from ILS.

Results

Theorem (Iyer-Miller-Triantafillou):
The n-level densities agree for $\operatorname{supp}(\widehat{\phi}) \subset \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$.

Philosophy:
Number theory harder - adapt tools to get an answer.
Random matrix theory easier - manipulate known answer.

Theorem (ILS)

Let Ψ be an even Schwartz function with $\operatorname{supp}(\widehat{\Psi}) \subset (-2, 2)$. Then

$$\sum_{m \le N^{\epsilon}} \frac{1}{m^2} \sum_{(b,N)=1} \frac{R(m^2,b)R(1,b)}{\varphi(b)} \int_{y=0}^{\infty} J_{k-1}(y) \widehat{\Psi}\left(\frac{2\log(by\sqrt{N}/4\pi m)}{\log R}\right) \frac{\mathrm{d}y}{\log R}$$
$$= -\frac{1}{2}\left[\int_{-\infty}^{\infty} \Psi(x) \frac{\sin 2\pi x}{2\pi x} \mathrm{d}x - \frac{1}{2}\Psi(0)\right] + O\left(\frac{k \log\log kN}{\log kN}\right),$$

where $R = k^2 N$, φ is Euler's totient function, and $R(n,q)$ is a Ramanujan sum.

Number Theory Side: Iyer-Miller-Triantafillou: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-2}, \frac{1}{n-2})$

Sequence of Lemmas - New Contributions Arise

1. Apply Petersson Formula
2. Restrict Certain Sums
3. Convert Kloosterman Sums to Gauss Sums
4. Remove Non-Principal Characters
5. Convert Sums to Integrals

New Results: Number Theory Side: Iyer-Miller-Triantafillou: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-2}, \frac{1}{n-2})$

Theorem

Fix $n \geq 4$ and let ϕ be an even Schwartz function with $\text{supp}(\widehat{\phi}) \subset \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$. Then, the nth centered moment of the 1-level density for holomorphic cusp forms is

$$\frac{1+(-1)^n}{2}(-1)^n(n-1)!!\left(2\int_{-\infty}^{\infty}\hat{\phi}(y)^2|y|dy\right)^{n/2}$$
$$\pm(-2)^{n-1}\left(\int_{-\infty}^{\infty}\phi(x)^n\frac{\sin 2\pi x}{2\pi x}dx-\frac{1}{2}\phi(0)^n\right)$$
$$\mp(-2)^{n-1}n\left(\int_{-\infty}^{\infty}\hat{\phi}(x_2)\int_{-\infty}^{\infty}\phi^{n-1}(x_1)\frac{\sin(2\pi x_1(1+|x_2|))}{2\pi x_1}dx_1dx_2-\frac{1}{2}\phi^n(0)\right).$$

Agrees with RMT.

n-Level Density: Katz-Sarnak Determinant Expansions

Example: $\mathrm{SO(even)}$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \widehat{\phi}(x_1) \cdots \widehat{\phi}(x_n) \det \Big(K_1(x_j, x_k) \Big)_{1 \le j,k \le n} dx_1 \cdots dx_n,$$

where

$$K_1(x,y) = \frac{\sin\Big(\pi(x-y)\Big)}{\pi(x-y)} + \frac{\sin\Big(\pi(x+y)\Big)}{\pi(x+y)}.$$

Problem: n-dimensional integral - looks very different.

Preliminaries

Easier to work with cumulants.

$$\sum_{n=1}^{\infty} C_n \frac{(it)^n}{n!} = \log \widehat{P}(t),$$

where P is the probability density function.

$$\mu'_n = C_n + \sum_{m=1}^{n-1} \binom{n-1}{m-1} C_m \mu'_{n-m},$$

where μ'_n is uncentered moment.

Preliminaries

Manipulating determinant expansions leads to analysis of

$$K(y_1,\ldots,y_n) = \sum_{m=1}^{n} \sum_{\substack{\lambda_1+\ldots+\lambda_m=n \\ \lambda_j\geq 1}} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1!\cdots\lambda_m!}$$

$$\sum_{\epsilon_1,\ldots,\epsilon_n=\pm 1} \prod_{\ell=1}^{m} \chi_{\left\{\left|\sum_{j=1}^{n}\eta(\ell,j)\epsilon_j y_j\right|\leq 1\right\}},$$

where

$$\eta(\ell,j) = \begin{cases} +1 & \text{if } j \leq \sum_{k=1}^{\ell}\lambda_k \\ -1 & \text{if } j > \sum_{k=1}^{\ell}\lambda_k. \end{cases}$$

New Result: Iyer-Miller-Triantafillou: Large Support: $\mathrm{supp}(\widehat{\phi}) \subseteq \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$

Hughes-Miller solved for $\mathrm{supp}(\widehat{\phi}) \subseteq (-\frac{1}{n-1}, \frac{1}{n-1})$.

New Complications: If $\mathrm{supp}(\widehat{\phi}) \subseteq (-\frac{1}{n-2}, \frac{1}{n-2})$,

1. $\eta(\ell, j)\epsilon_j y_j$ need not have same sign (at most one can differ);
2. more than one term in product can be zero (for fixed m, λ_j, ϵ_j).

Solution: Double count terms and subtract a correcting term ρ_j.

New Result: Iyer-Miller-Triantafillou: Large Support: $\mathrm{supp}(\widehat{\phi}) \subseteq \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$

After Fourier transform identities:

$$\begin{aligned} C_n^{\mathrm{SO}}(\phi) &= \frac{(-1)^{n-1}}{2}\left(\int_{-\infty}^{\infty} \phi(x)^n \frac{\sin 2\pi x}{2\pi x} dx - \frac{1}{2}\phi(0)^n\right) \\ &\quad + \frac{n(-1)^n}{2}\left(\int_{-\infty}^{\infty} \hat{\phi}(x_2) \int_{-\infty}^{\infty} \phi^{n-1}(x_1) \right. \\ &\qquad \left. \frac{\sin(2\pi x_1(1+|x_2|))}{2\pi x_1} dx_1 dx_2 - \frac{1}{2}\phi^n(0)\right). \end{aligned}$$

Agrees with number theory!

Conclusion and Bibliography

Recap

- Understand compound families in terms of simple ones.
- Choose combinatorics to simplify calculations.
- Extending support often related to deep arithmetic questions.

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Excised Orthogonal Ensembles

Excised Orthogonal Ensemble: Preliminaries

Characteristic polynomial of $A \in \mathrm{SO}(2N)$ is

$$\Lambda_A(e^{i\theta}, N) := \det(I - Ae^{-i\theta}) = \prod_{k=1}^{N} (1 - e^{i(\theta_k - \theta)})(1 - e^{i(-\theta_k - \theta)}),$$

with $e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_N}$ the eigenvalues of A.

Motivated by the arithmetical size constraint on the central values of the L-functions, consider Excised Orthogonal Ensemble $T_{\mathcal{X}}$: $A \in \mathrm{SO}(2N)$ with $|\Lambda_A(1, N)| \geq \exp(\mathcal{X})$.

One-Level Densities

One-level density $R_1^{G(N)}$ for a (circular) ensemble $G(N)$:

$$R_1^{G(N)}(\theta) = N \int \dots \int P(\theta, \theta_2, \dots, \theta_N) d\theta_2 \dots d\theta_N,$$

where $P(\theta, \theta_2, \dots, \theta_N)$ is the joint probability density function of eigenphases.

One-Level Densities

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where $P(\theta, \theta_2, \dots, \theta_N)$ is the joint probability density function of eigenphases. The one-level density excised orthogonal ensemble:

$$R_1^{T\mathcal{X}}(\theta_1) := C_{\mathcal{X}} \cdot N \int_0^\pi \cdots \int_0^\pi H(\log|\Lambda_A(1,N)| - \mathcal{X}) \times \\ \times \prod_{j<k} (\cos\theta_j - \cos\theta_k)^2 d\theta_2 \cdots d\theta_N,$$

with $H(x)$ the Heaviside function

$$H(x) = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x < 0 \end{cases}$$

and $C_{\mathcal{X}}$ is a normalization constant.

One-Level Densities

One-level density $R_1^{G(N)}$ for a (circular) ensemble $G(N)$:

$$R_1^{G(N)}(\theta) = N \int \dots \int P(\theta, \theta_2, \dots, \theta_N) d\theta_2 \dots d\theta_N,$$

where $P(\theta, \theta_2, \dots, \theta_N)$ is the joint probability density function of eigenphases. The one-level density excised of the orthogonal ensemble is

$$R_1^{T_{\mathcal{X}}}(\theta_1) = \frac{C_{\mathcal{X}}}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{Nr} \frac{\exp(-r\mathcal{X})}{r} R_1^{J_N}(\theta_1; r - 1/2, -1/2) dr,$$

where $C_{\mathcal{X}}$ is a normalization constant and

$$R_1^{J_N}(\theta_1; r - 1/2, -1/2) = N \int_0^{\pi} \cdots \int_0^{\pi} \prod_{j=1}^{N} w^{(r-1/2,-1/2)}(\cos\theta_j) \times \prod_{j<k} (\cos\theta_j - \cos\theta_k)^2 d\theta_2 \cdots d\theta_N$$

is the one-level density for the Jacobi ensemble J_N with weight function

$$w^{(\alpha,\beta)}(\cos\theta) = (1-\cos\theta)^{\alpha+1/2}(1+\cos\theta)^{\beta+1/2}, \qquad \alpha = r - 1/2 \text{ and } \beta = -1/2.$$

Results

- With $C_{\mathcal{X}}$ normalization constant and $P(N,r,\theta)$ defined in terms of Jacobi polynomials,

$$R_1^{T_{\mathcal{X}}}(\theta) = \frac{C_{\mathcal{X}}}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\exp(-r\mathcal{X})}{r} 2^{N^2+2Nr-N} \times$$
$$\times \prod_{j=0}^{N-1} \frac{\Gamma(2+j)\Gamma(1/2+j)\Gamma(r+1/2+j)}{\Gamma(r+N+j)} \times$$
$$\times (1-\cos\theta)^r \frac{2^{1-r}}{2N+r-1} \frac{\Gamma(N+1)\Gamma(N+r)}{\Gamma(N+r-1/2)\Gamma(N-1/2)} P(N,r,\theta)\, dr.$$

- Residue calculus implies $R_1^{T_{\mathcal{X}}}(\theta) = 0$ for $d(\theta,\mathcal{X}) < 0$ and

$$R_1^{T_{\mathcal{X}}}(\theta) = R_1^{\mathrm{SO}(2N)}(\theta) + C_{\mathcal{X}} \sum_{k=0}^{\infty} b_k \exp((k+1/2)\mathcal{X}) \quad \text{for } d(\theta,\mathcal{X}) \geq 0,$$

where $d(\theta,\mathcal{X}) := (2N-1)\log 2 + \log(1-\cos\theta) - \mathcal{X}$ and b_k are coefficients arising from the residues. As $\mathcal{X} \to -\infty$, θ fixed, $R_1^{T_{\mathcal{X}}}(\theta) \to R_1^{\mathrm{SO}(2N)}(\theta)$.

Numerical check

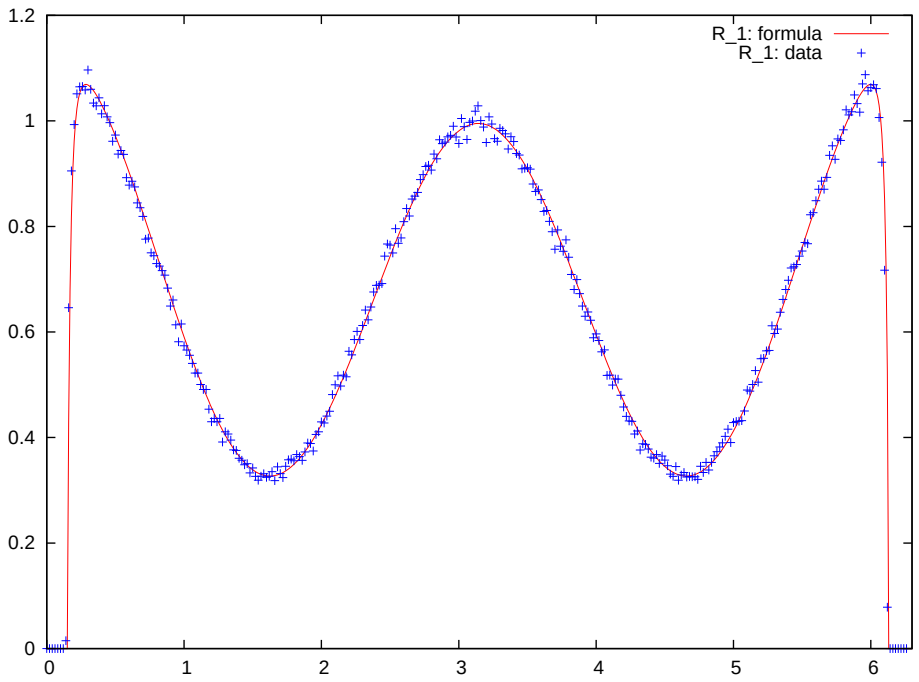

Figure: One-level density of excized $SO(2N), N = 2$ with cut-off $|\Lambda_A(1, N)| \geq 0.1$. The red curve uses our formula. The blue crosses give the empirical one-level density of 200,000 numerically generated matrices.

Theory vs Experiment

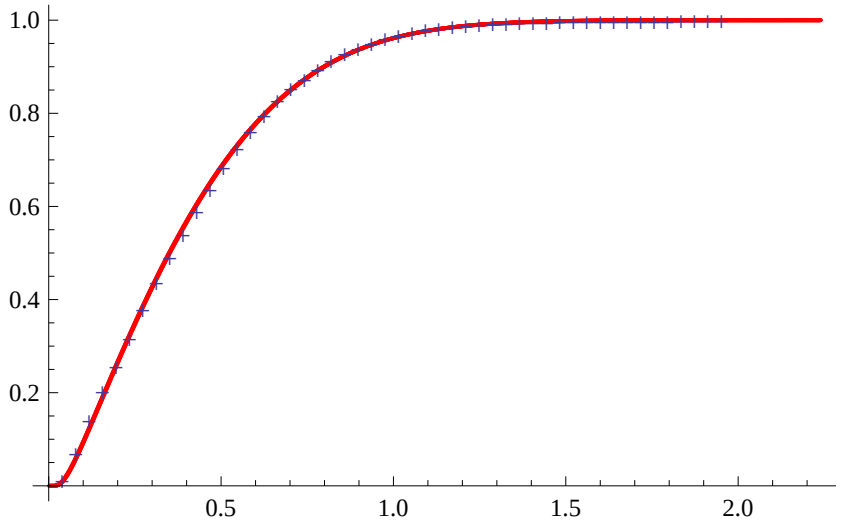

Figure: Cumulative probability density of the first eigenvalue from 3×10^6 numerically generated matrices $A \in SO(2N_{\text{std}})$ with $|\Lambda_A(1, N_{\text{std}})| \geq 2.188 \times \exp(-N_{\text{std}}/2)$ and $N_{\text{std}} = 12$ red dots compared with the first zero of even quadratic twists $L_{E_{11}}(s, \chi_d)$ with prime fundamental discriminants $0 < d \leq 400{,}000$ blue crosses. The random matrix data is scaled so that the means of the two distributions agree.

Example:
Dirichlet L-functions

Dirichlet Characters and L-Functions (m prime)

$(\mathbb{Z}/m\mathbb{Z})^*$ is cyclic of order $m-1$ with generator g. Let $\zeta_{m-1} = e^{2\pi i/(m-1)}$. The principal character χ_0 is given by

$$\chi_0(k) = \begin{cases} 1 & \text{if } (k,m) = 1 \\ 0 & \text{if } (k,m) > 1. \end{cases}$$

The $m-2$ primitive characters are determined (by multiplicativity) by action on g.

As each $\chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*$, for each χ there exists an l such that $\chi(g) = \zeta_{m-1}^l$. Hence for each $l, 1 \le l \le m-2$ we have

$$\chi_l(k) = \begin{cases} \zeta_{m-1}^{la} & k \equiv g^a(m) \\ 0 & (k,m) > 0 \end{cases}$$

Dirichlet Characters and L-Functions (m prime)

Let χ be a primitive character mod m. Let

$$c(m,\chi) = \sum_{k=0}^{m-1} \chi(k) e^{2\pi i k/m}.$$

$c(m,\chi)$ is a Gauss sum of modulus $\sqrt{m}$.

$$\begin{aligned} L(s,\chi) &= \prod_p (1-\chi(p)p^{-s})^{-1} \\ \Lambda(s,\chi) &= \pi^{-\frac{1}{2}(s+\epsilon)} \Gamma\left(\frac{s+\epsilon}{2}\right) m^{\frac{1}{2}(s+\epsilon)} L(s,\chi), \end{aligned}$$

where

$$\epsilon = 0 \text{ if } \chi(-1) = 1 \text{ and } \epsilon = 1 \text{ if } \chi(-1) = -1.$$

Explicit Formula and Expansion

Let ϕ be an even Schwartz function with compact support $(-\sigma, \sigma)$, let χ be a non-trivial primitive Dirichlet character of conductor m.

$$\begin{aligned}
& \sum \phi\left(\gamma \frac{\log(\frac{m}{\pi})}{2\pi}\right) \\
= & \int_{-\infty}^{\infty} \phi(y) dy \\
& - \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)[\chi(p) + \bar{\chi}(p)]p^{-\frac{1}{2}} \\
& - \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\Big(2\frac{\log p}{\log(m/\pi)}\Big)[\chi^2(p) + \bar{\chi}^2(p)]p^{-1} \\
& + O\Big(\frac{1}{\log m}\Big).
\end{aligned}$$

Explicit Formula and Expansion

$\{\chi_0\} \cup \{\chi_l\}_{1 \le l \le m-2}$ are all the characters mod m.
Consider the family of primitive characters mod a prime m ($m-2$ characters):

$$\begin{aligned} & \int_{-\infty}^{\infty} \phi(y)dy \\ - & \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)[\chi(p) + \bar{\chi}(p)]p^{-\frac{1}{2}} \\ - & \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\Big(2\frac{\log p}{\log(m/\pi)}\Big)[\chi^2(p) + \bar{\chi}^2(p)]p^{-1} \\ + & O\Big(\frac{1}{\log m}\Big). \end{aligned}$$

Note can pass Character Sum through Test Function.

Character Sums

$$\sum_{\chi} \chi(k) = \begin{cases} m-1 & k \equiv 1(m) \\ 0 & \text{otherwise.} \end{cases}$$

For any prime $p \neq m$

$$\sum_{\chi \neq \chi_0} \chi(p) \;=\; \begin{cases} -1+m-1 & p \equiv 1(m) \\ -1 & \text{otherwise.} \end{cases}$$

Substitute into

$$\frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_{p} \frac{\log p}{\log(m/\pi)} \widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)[\chi(p) + \bar{\chi}(p)]p^{-\frac{1}{2}}$$

First Sum: no contribution if $\sigma < 2$

$$\begin{aligned}
& \frac{-2}{m-2}\sum_{p}^{m^\sigma}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)p^{-\frac{1}{2}} \\
+ \quad & 2\frac{m-1}{m-2}\sum_{p\equiv 1(m)}^{m^\sigma}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)p^{-\frac{1}{2}} \\
\ll \quad & \frac{1}{m}\sum_{p}^{m^\sigma}p^{-\frac{1}{2}} + \sum_{p\equiv 1(m)}^{m^\sigma}p^{-\frac{1}{2}} \ll \frac{1}{m}\sum_{k}^{m^\sigma}k^{-\frac{1}{2}} + \sum_{\substack{k\equiv 1(m)\\ k\geq m+1}}^{m^\sigma}k^{-\frac{1}{2}} \\
\ll \quad & \frac{1}{m}\sum_{k}^{m^\sigma}k^{-\frac{1}{2}} + \frac{1}{m}\sum_{k}^{m^\sigma}k^{-\frac{1}{2}} \ll \frac{1}{m}m^{\sigma/2}.
\end{aligned}$$

Second Sum

$$\frac{1}{m-2}\sum_{\chi\neq\chi_0}\sum_{p}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\Big(2\frac{\log p}{\log(m/\pi)}\Big)\frac{\chi^2(p)+\bar{\chi}^2(p)}{p}.$$

$$\sum_{\chi\neq\chi_0}[\chi^2(p)+\bar{\chi}^2(p)] = \begin{cases} 2(m-2) & p\equiv\pm1(m) \\ -2 & p\not\equiv\pm1(m) \end{cases}$$

Second Sum

Up to $O\left(\frac{1}{\log m}\right)$ we find that

$$\begin{aligned}
&\ll \frac{1}{m-2}\sum_{p}^{m^{\sigma/2}} p^{-1} + \frac{2m-2}{m-2}\sum_{p\equiv\pm 1(m)}^{m^{\sigma/2}} p^{-1} \\
&\ll \frac{1}{m-2}\sum_{k}^{m^{\sigma/2}} k^{-1} + \sum_{\substack{k\equiv 1(m)\\ k\geq m+1}}^{m^{\sigma/2}} k^{-1} + \sum_{\substack{k\equiv -1(m)\\ k\geq m-1}}^{m^{\sigma/2}} k^{-1} \\
&\ll \frac{\log(m^{\sigma/2})}{m-2} + \frac{1}{m}\sum_{k}^{m^{\sigma/2}} k^{-1} + \frac{1}{m}\sum_{k}^{m^{\sigma/2}} k^{-1} + O\left(\frac{1}{m}\right) \\
&\ll \frac{\log m}{m} + \frac{\log m}{m} + \frac{\log m}{m}.
\end{aligned}$$

Dirichlet Characters: m Square-free

Fix an r and let $m_1, \ldots, m_r$ be distinct odd primes.

$$\begin{aligned} m &= m_1 m_2 \cdots m_r \\ M_1 &= (m_1 - 1)(m_2 - 1) \cdots (m_r - 1) = \phi(m) \\ M_2 &= (m_1 - 2)(m_2 - 2) \cdots (m_r - 2). \end{aligned}$$

M_2 is the number of primitive characters mod m, each of conductor m. A general primitive character mod m is given by $\chi(u) = \chi_{l_1}(u)\chi_{l_2}(u) \cdots \chi_{l_r}(u)$. Let $\mathcal{F} = \{\chi : \chi = \chi_{l_1}\chi_{l_2} \cdots \chi_{l_r}\}$.

$$\frac{1}{M_2} \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big) p^{-\frac{1}{2}} \sum_{\chi \in \mathcal{F}} [\chi(p) + \bar{\chi}(p)]$$

$$\frac{1}{M_2} \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\Big(2\frac{\log p}{\log(m/\pi)}\Big) p^{-1} \sum_{\chi \in \mathcal{F}} [\chi^2(p) + \bar{\chi}^2(p)]$$

Dirichlet Characters: m Square-free

$$\sum_{l_i=1}^{m_i-2} \chi_{l_i}(p) = \begin{cases} m_i - 1 - 1 & p \equiv 1(m_i) \\ -1 & \text{otherwise.} \end{cases}$$

Define

$$\delta_{m_i}(p,1) = \begin{cases} 1 & p \equiv 1(m_i) \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \sum_{\chi \in \mathcal{F}} \chi(p) &= \sum_{l_1=1}^{m_1-2} \cdots \sum_{l_r=1}^{m_r-2} \chi_{l_1}(p) \cdots \chi_{l_r}(p) \\ &= \prod_{i=1}^{r} \sum_{l_i=1}^{m_i-2} \chi_{l_i}(p) = \prod_{i=1}^{r} \Big(-1 + (m_i - 1)\delta_{m_i}(p,1) \Big). \end{aligned}$$

Expansion Preliminaries

$k(s)$ is an s-tuple $(k_1, k_2, \ldots, k_s)$ with $k_1 < k_2 < \cdots < k_s$.
This is just a subset of $(1, 2, \ldots, r)$, 2^r possible choices for $k(s)$.

$$\delta_{k(s)}(p,1) = \prod_{i=1}^{s} \delta_{m_{k_i}}(p,1).$$

If $s = 0$ we define $\delta_{k(0)}(p,1) = 1 \ \forall p$.
Then

$$\begin{aligned} & \prod_{i=1}^{r} \Big(-1 + (m_i - 1)\delta_{m_i}(p,1) \Big) \\ = \quad & \sum_{s=0}^{r} \sum_{k(s)} (-1)^{r-s} \delta_{k(s)}(p,1) \prod_{i=1}^{s} (m_{k_i} - 1). \end{aligned}$$

First Sum

$$\ll \sum_{p}^{m^\sigma} p^{-\frac{1}{2}} \frac{1}{M_2} \Big(1 + \sum_{s=1}^{r} \sum_{k(s)} \delta_{k(s)}(p,1) \prod_{i=1}^{s} (m_{k_i} - 1)\Big).$$

As $m/M_2 \leq 3^r$, $s = 0$ sum contributes

$$S_{1,0} = \frac{1}{M_2} \sum_{p}^{m^\sigma} p^{-\frac{1}{2}} \ll 3^r m^{\frac{1}{2}\sigma - 1},$$

hence negligible for $\sigma < 2$.

First Sum

$$\ll \sum_{p}^{m^\sigma} p^{-\frac{1}{2}} \frac{1}{M_2}\Big(1 + \sum_{s=1}^{r}\sum_{k(s)} \delta_{k(s)}(p,1) \prod_{i=1}^{s}(m_{k_i}-1)\Big).$$

Now we study

$$\begin{aligned} S_{1,k(s)} &= \frac{1}{M_2}\prod_{i=1}^{s}(m_{k_i}-1)\sum_{p}^{m^\sigma} p^{-\frac{1}{2}}\delta_{k(s)}(p,1) \\ &\ll \frac{1}{M_2}\prod_{i=1}^{s}(m_{k_i}-1)\sum_{n\equiv 1(m_{k(s)})}^{m^\sigma} n^{-\frac{1}{2}} \\ &\ll \frac{1}{M_2}\prod_{i=1}^{s}(m_{k_i}-1)\frac{1}{\prod_{i=1}^{s}(m_{k_i})}\sum_{n}^{m^\sigma} n^{-\frac{1}{2}} \ll 3^r m^{\frac{1}{2}\sigma-1}. \end{aligned}$$

First Sum

There are 2^r choices, yielding

$$S_1 \ll 6^r m^{\frac{1}{2}\sigma - 1},$$

which is negligible as m goes to infinity for fixed r if $\sigma < 2$.
Cannot let r go to infinity.
If m is the product of the first r primes,

$$\begin{aligned} \log m &= \sum_{k=1}^{r} \log p_k \\ &= \sum_{p \le r} \log p \approx r \end{aligned}$$

Therefore

$$6^r \approx m^{\log 6} \approx m^{1.79}.$$

Second Sum Expansions and Bounds

$$\sum_{l_i=1}^{m_i-2} \chi_{l_i}^2(p) = \begin{cases} m_i - 1 - 1 & p \equiv \pm 1 (m_i) \\ -1 & \text{otherwise} \end{cases}$$

$$\sum_{\chi \in \mathcal{F}} \chi^2(p) = \sum_{l_1=1}^{m_1-2} \cdots \sum_{l_r=1}^{m_r-2} \chi_{l_1}^2(p) \cdots \chi_{l_r}^2(p)$$

$$= \prod_{i=1}^{r} \sum_{l_i=1}^{m_i-2} \chi_{l_i}^2(p)$$

$$= \prod_{i=1}^{r} \Big(-1 + (m_i - 1)\delta_{m_i}(p,1) + (m_i - 1)\delta_{m_i}(p,-1) \Big).$$

Second Sum Expansions and Bounds

Handle similarly as before. Say

$$\begin{aligned} p &\equiv 1 \bmod m_{k_1}, \ldots, m_{k_a} \\ p &\equiv -1 \bmod m_{k_{a+1}}, \ldots, m_{k_b} \end{aligned}$$

How small can p be?
$+1$ congruences imply $p \geq m_{k_1} \cdots m_{k_a} + 1$.
-1 congruences imply $p \geq m_{k_{a+1}} \cdots m_{k_b} - 1$.
Since the product of these two lower bounds is greater than $\prod_{i=1}^{b}(m_{k_i} - 1)$, at least one must be greater than $\left(\prod_{i=1}^{b}(m_{k_i} - 1)\right)^{\frac{1}{2}}$.
There are 3^r pairs, yielding

$$\text{Second Sum} = \sum_{s=0}^{r} \sum_{k(s)} \sum_{j(s)} S_{2,k(s),j(s)} \ll 9^r m^{-\frac{1}{2}}.$$

Summary

Agrees with Unitary for $\sigma < 2$ for square-free $m \in [N, 2N]$ from:

Theorem

- *m square-free odd integer with $r = r(m)$ factors;*
- $m = \prod_{i=1}^{r} m_i$;
- $M_2 = \prod_{i=1}^{r}(m_i - 2)$.

Then family $\mathcal{F}_m$ of primitive characters mod m has

$$\begin{aligned} \textit{First Sum} &\ll \frac{1}{M_2} 2^r m^{\frac{1}{2}\sigma} \\ \textit{Second Sum} &\ll \frac{1}{M_2} 3^r m^{\frac{1}{2}}. \end{aligned}$$